Math 669 Problems, Set no. III

Due Thursday, April 15

1. Problem. Let $A \subset \mathbb{R}^n$ be a closed convex set. Prove that if A has finitely many faces then A is a polyhedron.

2. Problem. Let Q_n be the polytope of $n \times n$ symmetric doubly stochastic matrices and let $X = (\xi_{ij})$ be a vertex of Q_n. Prove that $\xi_{ij} \in \{0, 1, 1/2\}$ for all i, j.

3. Problem. Let $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ be real numbers. Let $x = (\lambda_1, \ldots, \lambda_n)$ and let consider the permutation polytope P that is the convex hull of the $n!$ points $x_\sigma = \big(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}\big)$, where $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is a permutation. Let us split the integer interval $(1, 2, \ldots, n)$ into k subintervals $(1, \ldots, a_1)$, $(a_1+1, \ldots, a_2)$, $\ldots$, $(a_{k-1}+1, \ldots, n)$, where subintervals may consist of a single point. Prove that the convex hull of the points x_σ where σ permutes the numbers inside each subinterval is a face of P containing x and that every face of P containing x is constructed this way.

4. Problem. Let $T(G; b) \neq \emptyset$ be a transportation polyhedron. Prove that $T(G, b)$ is bounded if and only if the graph G does not contain directed cycles $v_1 \to v_2 \to \ldots \to v_k \to v_1$.

5. Problem. Let $P_1, P_2 \subset \mathbb{R}^n$ be polyhedra. Prove that $Q = P_1 + P_2$ is a polyhedron and that every face F of Q can be written as $F = F_1 + F_2$, where F_1 is a face of P_1 and F_2 is a face of P_2.

6. Problem. Let $K_1, K_2 \subset \mathbb{R}^n$ be closed convex cones. Prove that $(K_1 + K_2)^\circ = K_1^\circ \cap K_2^\circ$ and that $(K_1 \cap K_2)^\circ = K_1^\circ + K_2^\circ$, provided the cone $K_1^\circ + K_2^\circ$ is closed.

7. Problem. Let $P \subset \mathbb{R}^n$ be a non-empty polyhedron. Prove that $P^\circ \subset \mathbb{R}^n$ is a polyhedron.

8. Problem. Let $\mathcal{D} : \mathcal{C}(\mathbb{R}^n) \longrightarrow \mathcal{C}(\mathbb{R}^n)$ be the polarity valuation such that $\mathcal{D}([A]) = [A^\circ]$ for non-empty closed convex sets $A \subset \mathbb{R}^n$. Let

$$K = \Big\{(\xi_1, \ldots, \xi_n) : \ \xi_i > 0 \quad \text{for} \quad i = 1, \ldots, n\Big\}.$$

Prove that $\mathcal{D}([K]) = (-1)^n [K]$.

9. Problem. Let $A \subset \mathbb{R}^n$ be a convex compact set containing 0 in its interior. We define the support function $h_A : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$h_A(c) = \max_{x \in A} \langle c, x \rangle.$$

Prove that

$$\operatorname{vol} A^\circ = \frac{1}{n!} \int_{\mathbb{R}^n} e^{-h_A(c)} \, dc.$$

10. Problem. We consider the space V_n of $n \times n$ symmetric matrices as Euclidean space with inner product

$$\langle A, B \rangle = \operatorname{trace} AB.$$

Let

$$K = \{A \in V_n : \quad A \text{ is positive semi-definite}\}.$$

Prove that

$$K^\circ = -K.$$