A REMARK ON THE RANK OF POSITIVE SEMIDEFINITE MATRICES SUBJECT TO AFFINE CONSTRAINTS

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Abstract. Let $\mathcal{K}_n$ be the cone of positive semidefinite $n \times n$ matrices and let $\mathcal{A}$ be an affine subspace of the space of symmetric matrices such that the intersection $\mathcal{K}_n \cap \mathcal{A}$ is non-empty and bounded. Suppose that $n \geq 3$ and that $\operatorname{codim} \mathcal{A} = \binom{r+2}{2}$ for some $1 \leq r \leq n-2$. Then there is a matrix $X \in \mathcal{K}_n \cap \mathcal{A}$ such that $\operatorname{rank} X \leq r$. We give a short geometric proof of this result, use it to improve a bound on realizability of weighted graphs as graphs of distances between points in Euclidean space, and describe its relation to theorems of Bohnenblust, Friedland and Loewy, and Au-Yeung and Poon.

1. Introduction

Let Sym_n be the space of $n \times n$ symmetric matrices. Thus Sym_n is a real vector space of dimension $\binom{n+1}{2}$. Let $\mathcal{K}_n \subset Sym_n$ be the convex cone of positive semidefinite matrices. The following result is well known, see for example [Barvinok 1995], Section 31.5 of [Deza and Laurent 1997] and [Pataki 1996].

(1.1) Theorem. *Let $\mathcal{A} \subset Sym_n$ be an affine subspace such that the intersection $\mathcal{K}_n \cap \mathcal{A}$ is non-empty and* $\operatorname{codim} \mathcal{A} \leq \binom{r+2}{2} - 1$ *for some non-negative integer r. Then there is a matrix $X \in \mathcal{K}_n \cap \mathcal{A}$ such that* $\operatorname{rank} X \leq r$.

The bound in Theorem 1.1 is sharp, meaning that for any n and for any $r < n$ one can construct an affine subspace $\mathcal{A}$ such that $\mathcal{A} \cap \mathcal{K}_n \neq \emptyset$, $\operatorname{codim} \mathcal{A} = \binom{r+2}{2}$ and for every $X \in \mathcal{K}_n \cap \mathcal{A}$ one has $\operatorname{rank} X > r$ (see Section 2).

The problem of finding a low-rank positive semidefinite matrix subject to affine constraints has attracted some attention recently. For example, it turned out to be relevant to problems of chemistry, statistics, archaeology, genetics, and geography (see [Alfakih and Wolkowicz 1998] for references and discussion) and to semidefinite programming in general, currently a very active of research (see, for example, [Vandenberghe and Boyd 1996] for a survey). Therefore, it is worth mentioning that in one particular case the bound can be sharpened.

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(1.2) Theorem. *Suppose that $r > 0$ and $n \geq r+2$. Let $\mathcal{A} \subset Sym_n$ be an affine subspace such that* $\operatorname{codim} \mathcal{A} = \binom{r+2}{2}$. *Suppose that the intersection $\mathcal{K}_n \cap \mathcal{A}$ is non-empty and bounded. Then there is a matrix $X \in \mathcal{K}_n \cap \mathcal{A}$ such that* $\operatorname{rank} X \leq r$.

Theorem 1.2 is equivalent to a host of known results in Matrix Theory (see Section 4), yet it appears that it hasn't been stated explicitly. The purpose of this note is to provide a direct proof of the theorem based solely on the convex geometry of $\mathcal{K}_n$ (Section 3), to compare Theorems 1.1 and 1.2 from the structural (this Section) and algorithmic points of view (Section 2), and to use Theorem 1.2 for a problem of Distance Geometry (Section 2).

Polyhedral analogy. Theorem 1.1 is proved by pointing out that every extreme point of the intersection $\mathcal{K}_n \cap \mathcal{A}$ will have the desired rank (see Section 31.5 of [Deza and Laurent 1997]). The proof can also be obtained via complementarity conditions for semidefinite programming (cf. [Barvinok 1995] and [Alizadeh, Haeberly and Overton 1997]), which constitute a particular case of the complementary conditions for general linear programs, see [Anderson and Nash 1987]. Thus Theorem 1.1 is somewhat analogous to the statement that the intersection of an affine subspace of codimension r with a non-negative orthant in $\mathbb{R}^n$, if non-empty, will contain a point where some $n-r$ coordinates are zero (see, for example, [Pataki 1998] for a discussion of this "polyhedral analogy" and a based on it semidefinite version of the simplex method). Furthermore, for a generic subspace the bound $n-r$ can not be improved. In the context of Theorem 1.2, it is not true that every extreme point of the intersection will have the desired rank, but rather that the rank of some extreme point will satisfy the bound. Theorem 1.2 demonstrates a point where the polyhedral analogy breaks down. Essentially, it asserts that there is no positive semidefinite analogue of a simple bounded polyhedron (simple polytope) as there must always be a degenerate vertex.

2. An Application to Distance Geometry

Let us fix an undirected weighted graph $G = (V, E; \rho)$ with the set $V = \{1, \ldots, n\}$ of vertices, a set $E \subset \binom{V}{2}$ of edges and non-negative weights ρ_{ij} on the edges. Let r be a positive integer. We say that G is *r-realizable* if G can be realized as the graph of distances between points in Euclidean space $\mathbb{R}^r$, that is, if we can find n (not necessarily distinct) points $v_1, \ldots, v_n$ in $\mathbb{R}^r$ such that $\|v_i - v_j\| = \rho_{ij}$ whenever $\{i, j\} \in E$. The problem of finding such a realization is known as the Euclidean distance matrix completion problem, see [Laurent 1998] for a survey. We use two reformulations of the problem, the first one is more "straightforward" while the second is more "economical".

(2.1) First reformulation. Let $G = (V, E; \rho)$ be a weighted graph on n nodes and let $v_1, \ldots, v_n \in \mathbb{R}^r$ be its realization. Let us consider the Gram matrix $X = (\xi_{ij})$ of $v_1, \ldots, v_n$, so $\xi_{ij} = \langle v_i, v_j \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. Then X is a positive semidefinite matrix such that $\operatorname{rank} X \leq r$. Furthermore, the distance conditions

$$\|v_i - v_j\| = \rho_{ij} \qquad \text{for} \quad \{i,j\} \in E$$

can be written as $|E|$ affine constraints

$$\xi_{ii} - 2\xi_{ij} + \xi_{jj} = \rho_{ij}^2 \qquad \text{for} \quad \{i,j\} \in E. \tag{2.1.1}$$

Conversely, if $X = (\xi_{ij})$ is an $n \times n$ positive semidefinite matrix satisfying (2.1.1) and such that $\operatorname{rank} X \leq r$, then X can be written in the form $\xi_{ij} = \langle v_i, v_j \rangle$, where $v_1, \ldots, v_n \in \mathbb{R}^r$ is an r-realization of G. Let $\mathcal{A}_n \subset Sym_n$ be the affine subspace of matrices satisfying (2.1.1). We conclude that G is r-realizable if and only if there exists a matrix $X \in \mathcal{K}_n \cap \mathcal{A}_n$ such that $\operatorname{rank} X \leq r$.

(2.2) Second reformulation. We observe that we can always translate the vertices $v_1, \ldots, v_n$ of a realization of G so that $v_n = 0$. Let $X = (\xi_{ij})$ be the $(n-1) \times (n-1)$ Gram matrix of the vectors $v_1, \ldots, v_{n-1}$, so $\xi_{ij} = \langle v_i, v_j \rangle$. Then the equations (2.1.1) can be replaced by $|E|$ affine constraints

$$\begin{aligned} \xi_{ii} &= \rho_{in}^2 \quad \text{if} \quad \{i,n\} \in E \quad \text{and} \\ \xi_{ii} - 2\xi_{ij} + \xi_{jj} &= \rho_{ij}^2 \quad \text{if} \quad 1 \leq i,j \leq n-1 \quad \text{and} \quad \{i,j\} \in E, \end{aligned} \tag{2.2.1}$$

so that G is r-realizable if and only if there is an $(n-1)\times(n-1)$ positive semidefinite matrix X satisfying (2.2.1) and such that $\operatorname{rank} X \leq r$. Let $\mathcal{A}_{n-1} \subset Sym_{n-1}$ be the affine subspace of matrices satisfying (2.2.1). We conclude that G is r-realizable if and only if there exists a matrix $X \in \mathcal{K}_{n-1} \cap \mathcal{A}_{n-1}$ such that $\operatorname{rank} X \leq r$.

The following result was proved in [Barvinok 1995].

(2.3) Theorem. *Suppose that the number $|E|$ of edges of a weighted graph $G = (V, E; \rho)$ satisfies the inequality $|E| \leq \binom{r+2}{2} - 1$. Then G is realizable in $\mathbb{R}^r$ if and only if G is realizable in some Euclidean space.*

Proof. Follows by Reformulation 2.1 and Theorem 1.1. □

The following example shows that the bound in Theorem 2.3 and hence in Theorem 1.1 is indeed sharp.

(2.4) Example. Suppose that G consists of a complete graph on the first $r+2$ vertices (1-skeleton of a $(r+1)$-dimensional simplex) and $n-r-2$ isolated vertices. Let us assign weight 1 to every of $\binom{r+2}{2}$ edges of G. Clearly, any realization of G as a graph of distances in some space $\mathbb{R}^d$ amounts to placing the first $r+2$ vertices as the vertices of a regular $(r+1)$-dimensional simplex and then placing arbitrarily the remaining $(n-r-2)$ vertices. Thus G is realized in $\mathbb{R}^{r+1}$ but not realized in $\mathbb{R}^r$.

Theorem 1.2 allows us to prove that in the critical case of $|E| = \binom{r+2}{2}$, the graph from Example 2.4 is the only graph that defies the conclusion of Theorem 2.3.

(2.5) Theorem. *Suppose that $|E| = \binom{r+2}{2}$ for some $r > 0$ and that G is not a union of a complete graph with $r+2$ vertices and 0 or more isolated vertices. Then G is realizable in $\mathbb{R}^r$ if and only if G is realizable in some Euclidean space.*

Proof. Since G is r-realizable if and only if its connected components are realizable, without loss of generality we may assume that G is connected. Since G is not a complete graph, for the number n of vertices we have $n \geq r+3$. We use now Reformulation 2.2 to conclude that the r-realizability of G is equivalent to the existence of a matrix $X \in \mathcal{K}_{n-1} \cap \mathcal{A}_{n-1}$ such that $\operatorname{rank} X \leq r$, where $\mathcal{A}_{n-1} \subset Sym_{n-1}$ is the affine subspace defined by (2.2.1). Let us prove that the intersection $\mathcal{A}_{n-1} \cap \mathcal{K}_{n-1}$ is necessarily bounded. Since G is connected, each vertex i of G can be connected to the n-th vertex by a path in the graph. Therefore, for any realization of G with $v_n = 0$, the length $\|v_i\|$ is bounded by the sum of the lengths of the edges of the path, so we can write $\|v_i\| \leq R$, where R is the sum of all weights ρ_{ij}. Hence $|\xi_{ii}| \leq R^2$. Now we observe that the set of positive semidefinite matrices with a uniform bound on the diagonal entries is bounded and apply Theorem 1.2. □

Example 2.4 provides an evidence that both conditions $\mathcal{K}_n \cap \mathcal{A} \neq \emptyset$ and $n \geq r+2$ in Theorem 1.2 are necessary. Indeed, consider Reformulation 2.2 of the r-realizability problem for the graph G of Example 2.4. If $n = r+2$ (there are no isolated vertices) then the condition on the size of the matrix is violated (remember that we pass to $(n-1) \times (n-1)$ matrices). If $n > r+2$ and hence there are isolated vertices, then the "realization space" $\mathcal{K}_{n-1} \cap \mathcal{A}_{n-1}$ is not bounded, because an isolated vertex can move freely.

Algorithmic issues and mechanical analogies. It is relatively easy to find, at least approximately, a matrix X satisfying conditions of Theorem 1.1. In [Barvinok 1995], the following approach was suggested. Let us choose a positive definite matrix F and let us find a positive semidefinite matrix X from the intersection $\mathcal{K}_n \cap \mathcal{A}$ which minimizes $\operatorname{tr}(FX)$ (this is a problem of semidefinite programming, see [Vandenberghe and Boyd 1996]). It is shown in [Barvinok 1995] that if F is sufficiently generic, then the matrix X satisfies the required rank condition. This method has an especially transparent interpretation for realizability problems discussed in this section. Given a higher-dimensional realization of a weighted graph, we attach "springs" (with generic elasticities) to all pairs of vertices that are not connected by an edge. Then the graph will fold itself down onto the minimal energy configuration of an appropriate dimension. In [Alfakih and Wolkowicz 1998] it is shown how to avoid a non-constructive genericity condition on F. With any positive definite F, let us find a matrix X as above. If X is an extreme point of $\mathcal{K}_n \cap \mathcal{A}$, it satisfies the desired rank condition. Otherwise, let $\mathcal{F}$ be a face of $\mathcal{K}_n \cap \mathcal{A}$ containing X. Then [Alfakih and Wolkowicz 1998] show how to determine a direction in which to move from X in $\mathcal{F}$ until reaching a point Y of a smaller rank on the boundary of $\mathcal{F}$. Iterating this process, we get a matrix of the required rank.

In Theorem 1.2, the situation seems to be quite different. It is not clear how to construct the desired matrix X efficiently. Neither the proof of this paper, nor other approaches deducing Theorem 1.2 from known results in matrix theory (see Section 4) are constructive. For instance, it is not clear what would be the right "spring attachment" to obtain the r-dimensional realization in Theorem 2.5.

We state the problem of finding the matrix X in Theorem 1.2 efficiently as an open problem.

Finally, let us mention that Theorems 1.1 and 1.2 can be applied in a similar way to other problems of geometric realizability involving constraints on lengths and angles, as long as those constraints can be recasted as affine constraints on a Gram matrix.

(2.6) Example. Suppose we want to find a hexagon in $\mathbb{R}^r$ with prescribed lengths of its sides, prescribed angles between the 3 pairs of opposite sides and prescribed sum of the squared norms of the vertices. These constraints can be written as 10 affine constraints on the 6×6 Gram matrix of vectors $v_1, \ldots, v_6 \in \mathbb{R}^r$. Theorem 1.2 implies that such a hexagon exists in $\mathbb{R}^r$ if and only if it exists in $\mathbb{R}^3$.

3. Proof

Our proof uses the convex geometry of the cone $\mathcal{K}_n$ of positive semidefinite matrices.

Let $K \subset \mathbb{R}^d$ be a convex set. An affine hyperplane H isolates K if K lies in one of the closed halfspaces bounded by H. A set $F \subset K$ is called a *face* of K if there is a hyperplane H which isolates K and such that $F = K \cap H$. In this case, H is called a *supporting hyperplane* of F. We also agree that K is a face of itself.

By the *interior* $\operatorname{int} K$ of a convex set K we always mean its relative interior, that is the interior with respect to the affine hull $\operatorname{aff} K$. The set $\partial K = K \setminus \operatorname{int} K$ is called the *boundary* of K. See [Webster 1994] as a general reference.

Thus $\mathcal{K}_n$ is a pointed convex closed cone. We need a description of the facial structure of $\mathcal{K}_n$ (see Section 31.5 of [Deza and Laurent 1997]).

(3.1) The facial structure of $\mathcal{K}_n$. The faces of $\mathcal{K}_n$ are parameterized by the subspaces of $\mathbb{R}^n$. For a subspace $\mathcal{L} \subset \mathbb{R}^n$, let

$$\mathcal{F}_{\mathcal{L}} = \Big\{ X \in \mathcal{K}_n : \mathcal{L} \subset \operatorname{Nul} X \Big\},$$

where $\operatorname{Nul} X$ is the null space of X consisting of vectors $y \in \mathbb{R}^n$ such that $Xy = \mathbf{0}$. One has $\dim \mathcal{F}_{\mathcal{L}} = \binom{r+1}{2}$, where $r = \operatorname{codim} \mathcal{L}$. A matrix $X \in \mathcal{K}_n$ of rank r lies in the interior of the $\binom{r+1}{2}$ dimensional face $\mathcal{F}_{\operatorname{Nul} X}$.

The face $\mathcal{F}_{\mathcal{L}}$ "looks like" the cone of positive semidefinite matrices of size r, where $r = \operatorname{codim} \mathcal{L}$. Decomposing $\mathbb{R}^n = \mathcal{L} \oplus \mathcal{L}^{\perp}$, we see that in an appropriate basis, every matrix $X \in \mathcal{F}_{\mathcal{L}}$ has the structure

$$X = \begin{pmatrix} \mathbf{0}_{(n-r)\times(n-r)} & \mathbf{0}_{(n-r)\times r} \\ \mathbf{0}_{r\times(n-r)} & Y_{r \times r} \end{pmatrix},$$

where $\mathbf{0}$ is the zero matrix and $Y_{r\times r}$ is an $r \times r$ positive semidefinite matrix. In particular, identifying $\mathbb{R}^r$ with the orthogonal complement $\mathcal{L}^\perp$ of $\mathcal{L}$, we identify $\mathcal{F}_\mathcal{L}$ with $\mathcal{K}_r$. In other words, there is a rank-preserving isometry $\mathcal{K}_r \longrightarrow \mathcal{F}_\mathcal{L}$, where $\operatorname{codim} \mathcal{L} = r$.

The boundary $\partial \mathcal{F}_\mathcal{L}$ is the union of all faces $\mathcal{F}_\mathcal{M}$, where $\mathcal{M} \subset \mathbb{R}^n$ is a subspace strictly containing $\mathcal{L}$.

The following Lemma provides the key ingredient of the proof of Theorem 1.2.

(3.2) Lemma. *Let $r \geq 1$ and let $\mathcal{A} \subset Sym_{r+2}$ be an affine subspace such that* $\dim \mathcal{A} = r+2$ *and so* $\operatorname{codim} \mathcal{A} = \binom{r+2}{2}$. *Suppose that the intersection $\mathcal{A} \cap \mathcal{K}_{r+2}$ is non-empty and bounded. Then there is an $X \in \mathcal{K}_{r+2} \cap \mathcal{A}$ such that* $\operatorname{rank} X \leq r$.

Proof. We observe that if $\mathcal{A} \cap \operatorname{int} \mathcal{K}_{r+2} = \emptyset$ then the separation theorem (see [Webster 1994]) implies that $\mathcal{A}$ lies in the supporting hyperplane of a proper face $\mathcal{F}$ of $\mathcal{K}_{r+2}$. Since there is a rank-preserving isometry between $\mathcal{F}$ and $\mathcal{K}_s$ for some $s < r+2$ (see Section 3.1), we deduce the result from Theorem 1.1.

Hence without loss of generality we can assume that $\mathcal{A}$ contains an interior point of $\mathcal{K}_{r+2}$. Let $\mathcal{B} = \mathcal{A} \cap \mathcal{K}_{r+2}$. Suppose that $\operatorname{rank} X > r$ for all $X \in \mathcal{B}$.

For every $X \in \partial\mathcal{B}$ we have also $X \in \partial\mathcal{K}_{r+2}$ and so $\operatorname{rank} X < r+2$ (cf. Section 3.1). Since we assumed that $\operatorname{rank} X > r$ for all $X \in \mathcal{B}$, we must have $\operatorname{rank} X = r+1$ for each $X \in \partial\mathcal{B}$, so $\dim(\operatorname{Nul} X) = 1$. The correspondence $X \longmapsto \operatorname{Nul} X$ defines a map $\psi : \partial\mathcal{B} \longrightarrow \mathbb{P}(\mathbb{R}^{r+2})$, where $\mathbb{P}(\mathbb{R}^{r+2}) = \mathbb{RP}^{r+1}$ is the projective space of all lines through the origin in $\mathbb{R}^{r+2}$. Clearly, the map ψ is continuous.

Now we claim that there exist two distinct points $X_1, X_2 \in \partial\mathcal{B}$ such that $\operatorname{Nul} X_1 = \operatorname{Nul} X_2 = \ell$ for some 1-dimensional subspace $\ell \subset \mathbb{R}^{r+2}$. Suppose that this is not the case. Then $\psi : \partial B \longrightarrow \mathbb{RP}^{r+1}$ is an embedding. We note that since $\mathcal{A} \cap \left(\operatorname{int} \mathcal{K}_{r+2}\right) \neq \emptyset$, the set $\mathcal{B}$ is an $(r+2)$-dimensional convex body and the boundary $\partial\mathcal{B}$ is homeomorphic to the sphere $\mathbb{S}^{r+1}$. However, $\mathbb{S}^{r+1}$ can not be embedded into $\mathbb{RP}^{r+1}$ for $r > 0$. Indeed, since $\mathbb{S}^{r+1}$ and $\mathbb{RP}^{r+1}$ are equidimensional manifolds, $\mathbb{S}^{r+1}$ is compact and $\mathbb{RP}^{r+1}$ is connected, the Invariance of Domain Theorem (see, for example, Chapter III, Section 6 of [Massey 1980]) implies that such an embedding would have been a homeomorphism between $\mathbb{S}^{r+1}$ and $\mathbb{RP}^{r+1}$, which is a contradiction, since $\mathbb{S}^{r+1}$ and $\mathbb{RP}^{r+1}$ are not homeomorphic for $r \geq 1$.

So, we proved that there exist two distinct points $X_1, X_2 \in \partial\mathcal{B}$ such that $\operatorname{Nul} X_1 = \operatorname{Nul} X_2 = \ell$ for some 1-dimensional subspace $\ell \subset \mathbb{R}^{r+2}$. Therefore, $X_1, X_2 \in \mathcal{F}_\ell$ for the face $\mathcal{F}_\ell$ of $\mathcal{K}_{r+2}$ (see Section 3.1). The intersection of the straight line $\left\{\tau X_1 + (1-\tau)X_2 : \tau \in \mathbb{R}\right\}$ passing through X_1 and X_2 with the face $\mathcal{F}_\ell$ is an interval $[Y_1, Y_2]$ with $Y_1, Y_2 \in \partial\mathcal{F}_\ell$. Hence $\operatorname{rank} Y_1, Y_2 \leq r$ (see Section 3.1) and $Y_1, Y_2 \in \mathcal{A} \cap \mathcal{K}_{r+2}$. The obtained contradiction shows that $\operatorname{rank} X \leq r$ for some $X \in \mathcal{K}_{r+2} \cap \mathcal{A}$.

□

Proof of Theorem 1.2. As in Section 31.5 of [Deza and Laurent 1997], the intersection $\mathcal{K}_n \cap \mathcal{A}$ has an extreme point Y and the rank of every such point is at most $r+1$, so $\operatorname{codim} \operatorname{Nul} Y \leq r+1$. Let us choose a subspace $\mathcal{L}$ such that $\mathcal{L} \subset \operatorname{Nul} Y$

and $\operatorname{codim} \mathcal{L} = r+2$. Let $\mathcal{F}_{\mathcal{L}}$ be the corresponding face of $\mathcal{K}_n$. Hence $Y \in \mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\mathcal{L}} \cap \mathcal{A} \neq \emptyset$. Since there is a rank-preserving isometry between $\mathcal{K}_{r+2}$ and $\mathcal{F}_{\mathcal{L}}$, by Lemma 3.2, it follows that there is an $X \in \mathcal{F}_{\mathcal{L}} \cap \mathcal{A}$ such that $\operatorname{rank} X \leq r$. □

Remark. Our proof exploited in a simple minded way the fact that, as described in Section 3.1, the poset of faces of $\mathcal{K}_n$ is isomorphic to the poset of all subspaces of $\mathbb{R}^n$. It is our belief, however, that this fundamental fact will lead to deeper consequences in the future.

4. Related Results from Matrix Theory

Theorem 1.2 is equivalent to some results known in the literature. One can show that Theorem 1.2 is equivalent to the following result from [Au-Yeung and Poon 1979] and [Poon 1994].

(4.1) Theorem. *Let us fix a number $r \geq 1$, a number $k \leq \binom{r+2}{2} - 1$ and a number $n \geq r+2$. Let $q_1, \ldots, q_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ be quadratic forms and let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be the corresponding quadratic map, $\phi(x) = \big(q_1(x), \ldots, q_k(x)\big)$.*
Let $\mathbb{S}^{n-1} = \big\{x \in \mathbb{R}^n : \|x\| = 1\big\}$ be the unit sphere. Then every point from $\operatorname{conv} \phi(\mathbb{S}^{n-1})$ can be represented as a convex combination of r (not necessarily distinct) points from $\phi(\mathbb{S}^{n-1})$.

In turn, it is shown in [Au-Yeung and Poon 1979] that Theorem 4.1 is equivalent to the following (unpublished) result of Bohnenblust:

(4.2) Theorem. *Suppose that $r > 0$ and $n \geq r+2$. Let $\mathcal{U} \subset Sym_n$ be a subspace such that $\dim \mathcal{U} < \binom{r+2}{2} - 1$. Suppose further, that the following condition is satisfied: whenever for some vectors $x_1, \ldots, x_r \in \mathbb{R}^n$ one has*

$$\sum_{i=1}^{r} x_i^t A x_i = 0 \quad \textit{for all} \quad A \in \mathcal{U},$$

one must have $x_i = 0$ for $i = 1, \ldots, r$. Then $\mathcal{U}$ contains a positive definite matrix.

Finally, [Friedland and Loewy 1976] proved that Theorem 4.2 is equivalent to the following result.

(4.3) Theorem. *Suppose that $2 \leq r \leq n-1$. Let $\mathcal{W} \subset Sym_n$ be a subspace such that $\dim \mathcal{W} \geq (r-1)(2n-r+2)/2$. Then W contains a non-zero matrix whose largest eigenvalue is at least of multiplicity r.*

To complete the circle, we sketch the proof of equivalence of Theorem 1.2 and Theorem 4.1.

Theorem 1.2 $\Longrightarrow$ Theorem 4.1.

In the context of Theorem 1.4, let $y = (\eta_1, \ldots, \eta_k)$ be a point from the convex hull of $\phi(\mathbb{S}^{n-1})$. Hence $y = \alpha_1 \phi(x_1) + \ldots + \alpha_m \phi(x_m)$ for some vectors $x_1, \ldots, x_m \in \mathbb{S}^{n-1}$ and some non-negative $\alpha_1, \ldots, \alpha_m$ such that $\alpha_1 + \ldots + \alpha_m = 1$. Let $Q_1, \ldots, Q_k$ be the matrices of the forms $q_1, \ldots, q_k$ so that $q_i(x) = x^t Q_i x$. Let $X = \alpha_1 x_1 x_1^t + \ldots + \alpha_m x_m x_m^t$. Then X is a positive semidefinite matrix satisfying k affine constraints $\operatorname{tr}(Q_i X) = \eta_i$ for $i = 1, \ldots, k$ and one additional constraint $\operatorname{tr}(X) = 1$, which makes the set of all feasible positive semidefinite matrices bounded. Applying Theorem 1.2, we conclude that there exists a positive semidefinite matrix X' satisfying the same set of constraints and such that $\operatorname{rank} X' \leq r$. Such a matrix can be decomposed $X' = \alpha_1 x_1' {x_1'}^t + \ldots + \alpha_r x_r' {x_r'}^t$, which gives rise to a representation of y as a convex combination of $\phi(x_i')$, $i = 1, \ldots, r$.

Theorem 4.1 $\Longrightarrow$ Theorem 1.2.

Reversing the above reasoning, one can show that Theorem 1.2 holds provided the subspace $\mathcal{A}$ lies in the affine hyperplane $\operatorname{tr}(X) = 1$. A straightforward, but lengthy argument deduces the general case of Theorem 1.2 from there.

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