

APPROXIMATIONS OF CONVEX BODIES BY POLYTOPES AND BY PROJECTIONS OF SPECTRAHEDRA

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ABSTRACT. We prove that for any compact set $B \subset \mathbb{R}^d$ and for any $\epsilon > 0$ there is a finite subset $X \subset B$ of $|X| = d^{O(1/\epsilon^2)}$ points such that the maximum absolute value of any linear function $\ell : \mathbb{R}^d \rightarrow \mathbb{R}$ on X approximates the maximum absolute value of ℓ on B within a factor of $\epsilon\sqrt{d}$. We also discuss approximations of convex bodies by projections of spectrahedra, that is, by projections of sections of the cone of positive semidefinite matrices by affine subspaces.

1. INTRODUCTION AND MAIN RESULTS

We present two results on approximating general convex bodies by efficiently computable convex bodies in the general spirit of [BV08]. Having fixed a compact set $B \subset \mathbb{R}^d$, we are interested in constructing an algorithm, preferably of a reasonably low complexity, which allows us to approximate the maximum of a given linear function $\ell : \mathbb{R}^d \rightarrow \mathbb{R}$ on B .

Our first result describes how well we can approximate B by a finite subset $X \subset B$ of a controlled size.

(1.1) Theorem. *Let $B \subset \mathbb{R}^d$ be a compact set. Then for any positive integer k there exists a set $X \subset B$ such that for the cardinality $|X|$ of X we have*

$$|X| \leq 1 + \frac{1}{2} \binom{d+k-1}{k} + \frac{1}{2} \binom{d+k-1}{k}^2$$

and such that

$$\max_{x \in B} |\ell(x)| \leq \binom{d+k-1}{k}^{\frac{1}{2k}} \max_{x \in X} |\ell(x)|$$

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for every linear function $\ell : \mathbb{R}^d \rightarrow \mathbb{R}$.

Let us fix k and let the dimension d grow. Then the cardinality $|X| = d^{O(k)}$ of X is polynomial in d while for the approximation factor we have

$$\binom{d+k-1}{k}^{\frac{1}{2k}} \approx \frac{\sqrt{d}}{(k!)^{1/2k}} \approx \sqrt{\frac{de}{k}}.$$

In particular, choosing a sufficiently large k we can replace B by a set X of polynomially many in d points such that the maximum absolute value any linear function on X approximates the maximum absolute value of the function on B within a factor $\epsilon\sqrt{d}$ for any $\epsilon > 0$, fixed in advance. To obtain a constant factor approximation we have to choose X consisting of exponentially many in d points.

Next, we consider approximations of convex bodies by more complicated sets.

In the space $\mathbb{R}^{r \times r}$ of $r \times r$ real matrices we consider the closed convex cone $\mathbb{R}_+^{r \times r}$ of symmetric positive semidefinite matrices. A section of $\mathbb{R}_+^{r \times r}$ by an affine subspace is called sometimes a *spectrahedron*, see, for example, [GN11]. The problem of optimizing a given linear function on the affine image (projection) of a spectrahedron is a problem of *semidefinite programming*, which, under some technical qualifications, can be solved in polynomial time, see, for example, [Tu10].

The following result was obtained by Gouveia, Parrilo and Thomas [G+10], [GT10] in the language of *theta bodies*. Nevertheless, we give a proof here as it connects the topic with the concept of *positive semidefinite rank* of a matrix introduced in [F+11] and [G+11] and raises some interesting questions. We note, however, that the proof uses the same idea as the proof from [GT10], only stated in a different language.

(1.2) Theorem. *Let $B \subset \mathbb{Z}^d$ be a finite set of integer vectors. For any positive integer k there exists a convex set $C \subset \mathbb{R}^d$ such that the following holds:*

- (1) *We can write C as a Minkowski sum $C = C' + \mathcal{L}^\perp$, where $\mathcal{L} \subset \mathbb{R}^d$ is a subspace and $C' \subset \mathcal{L}$ is an affine image of a section of the cone of $r \times r$ symmetric positive semidefinite matrices by an affine subspace with*

$$r \leq \binom{d+k+2}{k},$$

- (2) *We have*

$$B \subset C$$

and

- (3) *For any linear function*

$$\ell(x) = a_1x_1 + \dots + a_dx_d \quad \text{for } x = (x_1, \dots, x_d)$$

with integer coefficients a_1, \dots, a_d such that

$$\max_{x \in B} \ell(x) - \min_{x \in B} \ell(x) \leq k$$

we have

$$\max_{x \in B} \ell(x) = \max_{x \in C} \ell(x).$$

If k is fixed in advance then $r = d^{O(k)}$ is bounded by a polynomial in the dimension and C approximates the convex hull $\text{conv}(B)$ precisely with respect to any lattice direction for which the width of the convex hull is bounded by k . If we allow the lattice width k to be linear in d , the dimension r of the ambient space for the spectrahedron becomes exponentially large in d .

The paper is organized as follows.

In Section 2, we discuss some preliminaries concerning (symmetric) tensor powers of spaces.

In Section 3, we prove Theorem 1.1.

In Section 4, we prove Theorem 1.2.

In Section 5, we present a related result on approximating non-negative matrices by matrices with a small *positive semidefinite rank*, studied in [F+11] and [G+11].

2. PRELIMINARIES

We consider Euclidean space \mathbb{R}^d with scalar product

$$\langle x, y \rangle = \sum_{i=1}^d x_i y_i \quad \text{for } x = (x_1, \dots, x_d) \quad \text{and} \quad y = (y_1, \dots, y_d)$$

and the corresponding Euclidean norm

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

For a positive integer k we interpret the tensor product

$$(\mathbb{R}^d)^{\otimes k} = \underbrace{\mathbb{R}^d \otimes \dots \otimes \mathbb{R}^d}_{k \text{ times}}$$

as d^k -dimensional Euclidean space of arrays $X = (x_{i_1 \dots i_k})$, $1 \leq i_1, \dots, i_k \leq d$, with scalar product

$$\langle X, Y \rangle = \sum_{1 \leq i_1, \dots, i_k \leq d} x_{i_1 \dots i_k} y_{i_1 \dots i_k} \quad \text{for } X = (x_{i_1 \dots i_k}) \quad \text{and} \quad Y = (y_{i_1 \dots i_k}).$$

For a vector $x \in \mathbb{R}^d$, we define $x^{\otimes k} \in (\mathbb{R}^d)^{\otimes k}$ by

$$(x^{\otimes k})_{i_1 \dots i_k} = x_{i_1} \cdots x_{i_k} \quad \text{for } 1 \leq i_1, \dots, i_k \leq d \quad \text{and} \quad x = (x_1, \dots, x_d).$$

We have

$$\langle x^{\otimes k}, y^{\otimes k} \rangle = \langle x, y \rangle^k \quad \text{for all } x, y \in \mathbb{R}^d.$$

We observe that $x^{\otimes k}$ lies in the subspace $\text{Sym}(\mathbb{R}^d)^{\otimes k}$ consisting of the tensors $X = (x_{i_1 \dots i_k})$ for which values of the coordinates do not change when indices i_1, \dots, i_k are permuted. We have

$$\dim \text{Sym}(\mathbb{R}^d)^{\otimes k} = \binom{d+k-1}{k}.$$

Finally, the space $(\mathbb{R}^d)^{\otimes 2}$ is naturally identified with the space $\mathbb{R}^{d \times d}$ of $d \times d$ matrices while subspace $\text{Sym}(\mathbb{R}^d)^{\otimes 2}$ is identified with the subspace of symmetric matrices. For $x \in \mathbb{R}^d$ the matrix $x^{\otimes 2}$ is positive semidefinite. As is well-known, the cone $\mathbb{R}_+^{d \times d}$ of symmetric positive semidefinite matrices is spanned by matrices $x^{\otimes 2}$. We have $\langle X, Y \rangle \geq 0$ for any $d \times d$ symmetric positive semidefinite matrices.

3. PROOF THEOREM 1.1

We start with a lemma.

(3.1) Lemma. *Let $B \subset \mathbb{R}^d$ be a compact set. Then there exists a set $X \subset B$ of not more than $1 + \frac{d(d+1)}{2}$ points such that for any linear function $\ell : \mathbb{R}^d \rightarrow \mathbb{R}$ one has*

$$\max_{x \in B} |\ell(x)| \leq \sqrt{d} \max_{x \in X} |\ell(x)|.$$

Proof. Without loss of generality we assume that B spans \mathbb{R}^d . We consider the (necessarily unique) ellipsoid $E \subset \mathbb{R}^d$ centered at the origin and of the minimum volume among those which contain B , see, for example, [Ba97]. Applying an invertible linear transformation, if necessary, we assume that $E = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ is the unit ball. F. John's conditions, see [Ba97], state that there is a finite subset $X \subset B$ and numbers $\alpha_x \geq 0$ such that

$$(3.1.1) \quad \begin{aligned} \sum_{x \in X} \alpha_x x^{\otimes 2} &= \frac{1}{d} I \\ &\text{and} \\ \sum_{x \in X} \alpha_x &= 1 \end{aligned}$$

where I is the $d \times d$ identity matrix.

For completeness, we sketch a proof of (3.1.1). If matrix $d^{-1}I$ does not lie in the convex hull of the compact set $\{x^{\otimes 2} : x \in B\}$ then it can be separated from the set by an affine hyperplane, which implies that there is a quadratic form $q : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $q(x) \leq 1$ for all $x \in B$ and such that $\text{trace } q > d$. Then for a sufficiently small $\epsilon > 0$ the ellipsoid \tilde{E} defined by the inequality

$$(1 - \epsilon)\|x\|^2 + \epsilon q(x) \leq 1$$

contains B and has a smaller volume, which is a contradiction.

Carathéodory's Theorem then implies that we can choose $|X| \leq 1 + \frac{d(d+1)}{2}$. The first equation of (3.1.1) can be also written as

$$\sum_{x \in X} \alpha_x \langle c, x \rangle^2 = \frac{1}{d} \|c\|^2 \quad \text{for all } c \in \mathbb{R}^d,$$

from which it follows that

$$\max_{x \in X} |\langle c, x \rangle| \geq \frac{1}{\sqrt{d}} \|c\| \geq \frac{1}{\sqrt{d}} \max_{x \in B} |\langle c, x \rangle|.$$

□

(3.2) Proof of Theorem 1.1. Let us consider

$$B_k = \{x^{\otimes k} : x \in B\}.$$

Thus B_k is a compact subset of a $\binom{d+k-1}{k}$ -dimensional space $\text{Sym}(\mathbb{R}^d)^{\otimes k}$. Applying Lemma 3.1 we conclude that there exists a subset $X \subset B$ such that

$$|X| \leq 1 + \frac{1}{2} \binom{d+k-1}{k} + \frac{1}{2} \binom{d+k-1}{k}^2$$

and such such that

$$\max_{x \in B} |\langle Y, x^{\otimes k} \rangle| \leq \binom{d+k-1}{k}^{1/2} \max_{x \in X} |\langle Y, x^{\otimes k} \rangle|$$

for any $Y \in \text{Sym}(\mathbb{R}^d)^{\otimes k}$. Choosing $Y = y^{\otimes k}$ we conclude that

$$\max_{x \in B} |\langle y, x \rangle| \leq \binom{d+k-1}{k}^{1/2k} \max_{x \in X} |\langle y, x \rangle|$$

for any $y \in \mathbb{R}^d$, which completes the proof. □

It follows from our proof that we can choose the set X among the contact points of the ellipsoid of the minimum volume centered at the origin and containing the set $B_k = \{x^{\otimes k} : x \in B\}$.

4. PROOF OF THEOREM 1.2

We deal with real matrices $A = (a_{ij})$ for $i \in I$ and $j \in J$, where I and J are possibly infinite sets of indices. We say that

$$\text{rank } A \leq n$$

if there exist vectors $u_i \in \mathbb{R}^n$ for $i \in I$ and $v_j \in \mathbb{R}^n$ for $j \in J$ such that

$$a_{ij} = \langle u_i, v_j \rangle \quad \text{for all } i \in I \quad \text{and all } j \in J.$$

We say that

$$\text{rank } A = n$$

if n is the smallest non-negative integer satisfying $\text{rank } A \leq n$. We can define $\text{rank } A = \infty$ if there is no such n , but we will only deal with matrices of a finite rank. Our definition agrees with the usual definition of the rank of a matrix, when I and J are finite.

We need some concepts and results of [F+11] and [G+11].

(4.1) Definition. Let $A = (a_{ij})$, $i \in I$, $j \in J$ be a non-negative matrix. We say that

$$\text{rank}_{\text{psd}} A \leq r$$

if there exist $r \times r$ symmetric positive semidefinite matrices U_i for $i \in I$ and V_j for $j \in J$ such that

$$a_{ij} = \langle U_i, V_j \rangle \quad \text{for all } i, j.$$

The following result was proved in [F+11] and [G+11]. For completeness, we present its proof here.

(4.2) Lemma. Let $\{u_i : i \in I\} \subset \mathbb{R}^d$ and $\{v_j : j \in J\} \subset \mathbb{R}^d$ be sets of vectors such that

$$\langle u_i, v_j \rangle \leq 1 \quad \text{for all } i, j.$$

Suppose further that $\text{span}(v_j : j \in J) = \mathbb{R}^d$.

Let us define matrix $A = (a_{ij})$ by

$$a_{ij} = 1 - \langle u_i, v_j \rangle \quad \text{for all } i, j.$$

Suppose that

$$\text{rank}_{\text{psd}} A \leq r.$$

Then there exists a convex set $C \subset \mathbb{R}^d$ which is an affine image of a section of the cone of $r \times r$ symmetric positive semidefinite matrices by an affine subspace such that

$$u_i \in C \quad \text{for all } i \in I$$

and

$$\langle x, v_j \rangle \leq 1 \quad \text{for all } j \in J \quad \text{and all } x \in C.$$

Proof. Since $\text{rank}_{\text{psd}} A \leq r$ there exist $r \times r$ positive semidefinite matrices U_i and V_j such that

$$(4.2.1) \quad 1 - \langle u_i, v_j \rangle = \langle U_i, V_j \rangle \quad \text{for all } i, j.$$

Let us define an affine subspace $\mathcal{L} \subset \mathbb{R}^d \oplus \mathbb{R}^{r \times r}$ by the equations

$$(4.2.2) \quad 1 - \langle x, v_j \rangle = \langle X, V_j \rangle \quad \text{for } j \in J,$$

where $x \in \mathbb{R}^d$ and $X \in \mathbb{R}^{r \times r}$. The map

$$(x, X) \longmapsto X$$

projects \mathcal{L} onto an affine subspace $\mathcal{A} \subset \mathbb{R}^{r \times r}$. We claim that for every $X \in \mathcal{A}$ there is a unique $x \in \mathbb{R}^d$ such that $(x, X) \in \mathcal{L}$. Indeed, if $(x, X) \in \mathcal{L}$ and $(y, X) \in \mathcal{L}$, then

$$\langle x - y, v_j \rangle = 0 \quad \text{for } j \in J$$

and since the set $\{v_j : j \in J\}$ spans \mathbb{R}^d we conclude that $x = y$. This allows us to define an affine map (projection)

$$T : \mathcal{A} \longrightarrow \mathbb{R}^d$$

by letting

$$T(X) = x \quad \text{if } (x, X) \in \mathcal{L}.$$

We let

$$C = T(\mathcal{A} \cap \mathbb{R}_+^{r \times r}).$$

Since by (4.2.1)–(4.2.2) we have $(u_i, U_i) \in \mathcal{L}$, we conclude that $U_i \in \mathcal{A}$ and $T(U_i) = u_i$. Since $U_i \in \mathbb{R}_+^{r \times r}$, we have $u_i \in C$ for all $i \in I$.

Let us pick any $x \in C$. Then there exists an $X \in \mathbb{R}_+^{r \times r}$ such that (x, X) satisfies (4.4.2). Since $V_j \in \mathbb{R}_+^{r \times r}$ we have $\langle X, V_j \rangle \geq 0$ for all j and hence $\langle x, v_j \rangle \leq 1$ for all $j \in J$. \square

The following observation is also from [F+11] and [G+11].

(4.3) Lemma. *Let $A = (a_{ij})$ and $B = (b_{ij})$ for $i \in I$ and $j \in J$ be matrices such that*

$$a_{ij} = b_{ij}^2 \quad \text{for all } i, j.$$

Then

$$\text{rank}_{\text{psd}} A \leq \text{rank } B.$$

Proof. Let $\text{rank } B = d$. Then there exist vectors $u_i, v_j \in \mathbb{R}^d$ such that

$$b_{ij} = \langle u_i, v_j \rangle \quad \text{for all } i \in I \quad \text{and all } j \in J.$$

Then

$$a_{ij} = \langle u_i^{\otimes 2}, v_j^{\otimes 2} \rangle \quad \text{for all } i, j.$$

Since $u_i^{\otimes 2}$ and $v_j^{\otimes 2}$ are $d \times d$ positive semidefinite matrices, the result follows. \square

The following result is a standard linear algebra fact.

(4.4) Lemma. Let $A = (a_{ij})$ be a real matrix and let $p : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree k . Let us define a matrix $B = (b_{ij})$ by

$$b_{ij} = p(a_{ij}) \quad \text{for all } i \in I \quad \text{and all } j \in J.$$

Then

$$\text{rank } B \leq \binom{k + \text{rank } A}{k}.$$

Proof. We write

$$p(t) = \sum_{m=0}^k \alpha_m t^m$$

for some $\alpha_m \in \mathbb{R}$. Let $\text{rank } A = d$, so

$$a_{ij} = \langle u_i, v_j \rangle \quad \text{for all } i, j$$

and some vectors $u_i : i \in I$ and $v_j : j \in J$ in \mathbb{R}^d . Then we can write

$$b_{ij} = p(a_{ij}) = \alpha_0 + \sum_{m=1}^k \alpha_m \langle u_i, v_j \rangle^m = \alpha_0 + \sum_{m=1}^k \alpha_m \langle u_i^{\otimes m}, v_j^{\otimes m} \rangle.$$

Let us introduce vectors

$$U_i = \alpha_0 \oplus \alpha_1 u_i \oplus \alpha_2 u_i^{\otimes 2} \oplus \dots \oplus \alpha_k u_i^{\otimes k} \quad \text{and} \quad V_j = 1 \oplus v_j \oplus v_j^{\otimes 2} \oplus \dots \oplus v_j^{\otimes k}$$

in Euclidean space

$$\mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} \oplus \dots \oplus (\mathbb{R}^d)^{\otimes k}.$$

Hence we can write

$$b_{ij} = \langle U_i, V_j \rangle \quad \text{for all } i, j.$$

It remains to notice that the dimension of the space spanned by vectors U_i and V_j does not exceed

$$1 + \sum_{m=1}^k \dim \text{Sym} (\mathbb{R}^d)^{\otimes m} = 1 + \sum_{m=1}^k \binom{d + m - 1}{m} = \binom{d + k}{k}.$$

□

(4.5) Corollary. Let $A = (a_{ij})$ for $i \in I$ and $j \in J$ be a real matrix, let $S \subset \mathbb{R}$,

$$S = \{a_{ij} : i \in I, j \in J\},$$

be the set of all distinct values among the matrix entries a_{ij} and let $\phi : S \rightarrow \mathbb{R}$ be a function. Let us define a matrix $B = (b_{ij})$ by

$$b_{ij} = \phi(a_{ij}) \quad \text{for all } i, j.$$

If $|S| \leq k$ then

$$\text{rank } B \leq \binom{k-1 + \text{rank } A}{k-1}.$$

Proof. Since $|S| \leq k$ there is a polynomial $p : \mathbb{R} \rightarrow \mathbb{R}$ with $\deg p \leq k-1$ such that $\phi(t) = p(t)$ for all $t \in S$, so $b_{ij} = p(a_{ij})$. We write

$$p(t) = \sum_{m=0}^{k-1} \alpha_m t^m$$

for some $\alpha_m \in \mathbb{R}$. The proof follows by Lemma 4.4. □

(4.6) Lemma. Let $A = (a_{ij})$ be a real non-negative matrix such that the number of distinct values among the matrix entries a_{ij} does not exceed k . Then

$$\text{rank}_{\text{psd}} A \leq \binom{k-1 + \text{rank } A}{k-1}.$$

Proof. Let us define a matrix $B = (b_{ij})$ by

$$b_{ij} = \sqrt{a_{ij}} \quad \text{for all } i, j.$$

By Corollary 4.5,

$$\text{rank } B \leq \binom{k-1 + \text{rank } A}{k-1}.$$

Since

$$a_{ij} = b_{ij}^2 \quad \text{for all } i, j,$$

the proof follows by Lemma 4.3. □

(4.7) Lemma. Let $\{u_i : i \in I\} \subset \mathbb{R}^d$ and $\{v_j : j \in J\} \subset \mathbb{R}^d$ be sets of vectors such that

$$\langle u_i, v_j \rangle \leq 1 \quad \text{for all } i, j.$$

Let

$$\mathcal{L} = \text{span}(v_j : j \in J).$$

Let us define a matrix $A = (a_{ij})$ by

$$a_{ij} = 1 - \langle u_i, v_j \rangle \quad \text{for all } i \in I \quad \text{and all } j \in J.$$

Suppose further, that the number of distinct values among the entries a_{ij} does not exceed some positive integer k .

Then there exists a convex set $C \subset \mathbb{R}^d$ such that the following holds:

- (1) We can write C as a Minkowski sum $C = C' + \mathcal{L}^\perp$, where $C' \subset \mathcal{L}$ is an affine image of a section of the cone of $r \times r$ symmetric positive semidefinite matrices by an affine subspace for

$$r \leq \binom{k+d}{k-1};$$

- (2) We have

$$u_i \in C \quad \text{for all } i \in I$$

and

- (3) We have

$$\langle x, v_j \rangle \leq 1 \quad \text{for all } j \in J \quad \text{and all } x \in C.$$

Proof. Since $\text{rank } A \leq d + 1$ it follows by Lemma 4.6 that

$$\text{rank}_{\text{psd}} A \leq \binom{k+d}{k-1}.$$

Let u'_i be the orthogonal projection of u_i onto \mathcal{L} . By Lemma 4.2, there exists a convex set $C' \subset \mathcal{L}$ which is an affine image of a section of the cone of $r \times r$ symmetric positive semidefinite matrices by an affine subspace for

$$r \leq \binom{k+d}{k-1},$$

such that

$$u'_i \in C' \quad \text{for all } i \in I$$

and such that

$$\langle x, v_j \rangle \leq 1 \quad \text{for all } x \in C' \quad \text{and all } j \in J.$$

Then $C = C' + \mathcal{L}^\perp$ satisfies the desired conditions. □

(4.8) Proof of Theorem 1.2. We consider \mathbb{R}^d as the coordinate hyperplane $x_{d+1} = 0$ of \mathbb{R}^{d+1} . For a vector $x \in \mathbb{R}^d$ and $a \in \mathbb{R}$ we denote by $(x, a) \in \mathbb{R}^{d+1}$ the vector obtained from x by appending the $(d+1)$ -st coordinate equal to a .

For a vector $u \in B$ let $\hat{u} = (u, 1) \in \mathbb{Z}^{d+1}$ and let $\{\hat{u}_i : i \in I\}$ be the set of vectors obtained this way.

For any vector $v \in \mathbb{Z}^d$ and any $m \in \mathbb{Z}$ such that

$$\max_{u \in B} \langle u, v \rangle = m \quad \text{and} \quad \min_{u \in B} \langle u, v \rangle \geq m - k$$

we let $\hat{v} = (v, k - m) \in \mathbb{Z}^{d+1}$ and let $\{\hat{v}_j : j \in J\}$ be the set of all vectors obtained this way. In particular,

$$\max_{i \in I} \langle \hat{u}_i, \hat{v}_j \rangle = k \quad \text{and} \quad \min_{i \in I} \langle \hat{u}_i, \hat{v}_j \rangle \geq 0 \quad \text{for all } j \in J.$$

We define matrix $A = (a_{ij})$ by

$$a_{ij} = 1 - \frac{1}{k} \langle \hat{u}_i, \hat{v}_j \rangle \quad \text{for all } i \in I \quad \text{and all } j \in J.$$

Hence we have $0 \leq a_{ij} \leq 1$ and $ka_{ij} \in \mathbb{Z}$ for all i and j . In particular, the number of distinct values among the entries a_{ij} does not exceed $k + 1$.

Let

$$\mathcal{L} = \text{span}(\hat{v}_j : j \in J).$$

Since vector $(0, k)$ is among vectors \hat{v}_j , we have

$$\mathcal{L}^\perp \subset \mathbb{R}^d.$$

By Lemma 4.7, there is an affine image $\hat{C}' \subset \mathcal{L}$ of a section of the cone of $r \times r$ symmetric positive semidefinite matrices with

$$r \leq \binom{k + d + 2}{k}$$

such that for $\hat{C} = \hat{C}' + \mathcal{L}^\perp$ we have

$$\hat{u}_i \in \hat{C} \quad \text{for all } i \in I \quad \text{and} \quad \langle x, \hat{v}_j \rangle \leq k \quad \text{for all } x \in \hat{C} \quad \text{and all } j \in J.$$

Let us define

$$C' = \left\{ x \in \mathbb{R}^d : (x, 1) \in \hat{C}' \right\} \quad \text{and} \quad C = C' + \mathcal{L}^\perp.$$

Since $\mathcal{L}^\perp \subset \mathbb{R}^d$ we have

$$C = \left\{ x \in \mathbb{R}^d : (x, 1) \in \hat{C} \right\}.$$

We observe that C' is an affine image of a section of the cone of $r \times r$ positive semidefinite matrices and that

$$B \subset C.$$

Moreover, if for some $v \in \mathbb{Z}^d$ and some integer m we have

$$\max_{u \in B} \langle u, v \rangle = m \quad \text{and} \quad \min_{u \in B} \langle u, v \rangle \geq m - k$$

then $(v, k - m) = \widehat{v}_j$ for some $j \in J$ and

$$\langle x, v \rangle \leq m \quad \text{for all } x \in C.$$

Since $B \subset C$ we necessarily have

$$\max_{x \in C} \langle x, v \rangle = m.$$

□

5. APPROXIMATING NON-NEGATIVE MATRICES BY MATRICES OF A SMALL POSITIVE SEMIDEFINITE RANK

As a by-product of our proof of Theorem 1.2 we obtain the following result.

(5.1) Theorem. *For any $\epsilon > 0$ there is a positive integer $d = d(\epsilon)$ such that the following holds. Let $A = (a_{ij})$, $i \in I$, $j \in J$ be a matrix such that*

$$0 \leq a_{ij} \leq 1 \quad \text{for all } i, j.$$

Then there exists a non-negative $m \times n$ matrix $A' = (a'_{ij})$ such that

$$|a_{ij} - a'_{ij}| \leq \epsilon \quad \text{for all } i, j$$

and

$$\text{rank}_{\text{psd}} A' \leq (\text{rank } A)^d.$$

Proof. Without loss of generality, we assume that $\text{rank } A \geq 2$ and that $\epsilon < 1/2$. There is a univariate polynomial $p(t)$ of some degree $k = k(\epsilon)$ such that

$$(5.1.1) \quad \left| \sqrt{t} - p(t) \right| \leq \frac{\epsilon}{3} \quad \text{for all } 0 \leq t \leq 1.$$

We let $B = (b_{ij})$ by

$$b_{ij} = p(a_{ij}) \quad \text{for all } i, j.$$

By Lemma 4.4 that

$$\text{rank } B \leq \binom{k + \text{rank } A}{k}.$$

Let us define A' by

$$a'_{ij} = b_{ij}^2 \quad \text{for all } i, j.$$

It follows by (5.1.1) that a'_{ij} approximates a_{ij} within ϵ . By Lemma 4.3

$$\text{rank}_{\text{psd}} A' \leq \text{rank } B \leq \binom{k + \text{rank } A}{k}$$

and the proof follows. \square

It would be interesting to find out if Theorem 5.1 leads to any non-trivial approximations of general convex bodies by projections of spectrahedra. So far, the author was unable to beat the bounds established by Theorem 1.1.

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