A REMARK ON APPROXIMATING PERMANENTS OF POSITIVE DEFINITE MATRICES

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June 20, 2020

Abstract. Let A be an $n \times n$ positive definite Hermitian matrix with all eigenvalues between 1 and 2. We represent the permanent of A as the integral of some explicit log-concave function on $\mathbb{R}^{2n}$. Consequently, there is a fully polynomial randomized approximation scheme (FPRAS) for $\operatorname{per} A$.

1. Introduction and main results

Let $A = (a_{ij})$ be an $n \times n$ complex matrix. The *permanent* of A is defined as

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{k=1}^{n} a_{k\sigma(k)},$$

where S_n is the symmetric group of all $n!$ permutations of the set $\{1, \dots, n\}$. Recently, in particular because of connections with quantum optics, there was some interest in efficient computing (approximating) $\operatorname{per} A$, when A is a positive semidefinite Hermitian matrix, see [A+17], [GS18] and references therein. As is known, in that case $\operatorname{per} A$ is real and non-negative, see, for example, Chapter 2 of [Mi78]. In [A+17], Anari, Gurvits, Oveis Gharan and Saberi constructed a deterministic polynomial time algorithm approximating the permanent of a positive semidefinite $n \times n$ Hermitian matrix A within a multiplicative factor of c^n for $c = e^{1+\gamma} \approx 4.84$, where $\gamma \approx 0.577$ is the Euler constant. Similarly to the case of a non-negative real matrix A, the problem of exact computation of $\operatorname{per} A$ for a positive semidefinite matrix A is #P-hard [GS18].

If A is a non-negative real matrix, a fully polynomial randomized approximation scheme (FPRAS) for $\operatorname{per} A$ was constructed by Jerrum, Sinclair and Vigoda [J+04]. Given an $n \times n$ matrix non-negative A and a real $0 < \epsilon < 1$, the algorithm of [J+04]

1991 *Mathematics Subject Classification.* 15A15, 15A57, 68W20, 60J22, 26B25.
Key words and phrases. permanent, positive definite matrices, log-concave measures.
This research was partially supported by NSF Grant DMS 1855428.

produces in $(n/\epsilon)^{O(1)}$ time a number α approximating $\operatorname{per} A$ within relative error ϵ. The algorithm is randomized, meaning that the number α satisfies the desired condition with a sufficiently large probability p, for example, with $p = 0.9$ (then by running m independent copies of the algorithm and taking the median of the computed αs, one can make the probability of error exponentially small in m). No such algorithm is known in the case of a positive semidefinite Hermitian A, and the question of existence of an FPRAS in that case was asked in [A+17] and [GS18].

In this note, we show that that there is a fully polynomial randomized approximation scheme (FPRAS) for permanents of positive definite matrices with the eigenvalues between 1 and 2. Namely, we represent $\operatorname{per} A$ for such an $n \times n$ matrix A as the integral of an explicitly constructed log-concave function $f_A : \mathbb{R}^{2n} \longrightarrow \mathbb{R}_+$, so that

$$\int_{\mathbb{R}^{2n}} f_A(t) \ dt = \operatorname{per} A.$$

There is an FPRAS for integrating log-concave functions, see [LV07] for the detailed analysis and history of the Markov Chain Monte Carlo approach to the problem of integrating log-concave functions and a closely related problem of approximating volumes of convex bodies. Hence the above integral representation and an integration algorithm from [LV07] instantly produce an FPRAS for computing the permanent of a positive definite Hermitian matrix with all eigenvalues between 1 and 2. We note that a standard interpolation argument implies that the problem of computing $\operatorname{per} A$ exactly remains #P-hard, when restricted to positive definite matrices with eigenvalues between 1 and 2. Indeed, the set X_n of such $n \times n$ matrices has a non-empty interior in the vector space of all $n \times n$ Hermitian matrices. Given an arbitrary $n \times n$ Hermitian matrix B, one can draw a line L through B and an interior point of X_n. Since the restriction of the permanent onto that line is a univariate polynomial of degree at most n, by computing the permanent $\operatorname{per} A_i$ for $n+1$ distinct matrices $A_i \in (L \cap X_n)$, we would be able to compute $\operatorname{per} B$ exactly by interpolation, which is a #P-hard problem, cf. [GS18].

We consider the space $\mathbb{C}^n$ with the standard norm

$$\|z\|^2 = |z_1|^2 + \ldots + |z_n|^2, \quad \text{where} \quad z = (z_1, \ldots, z_n).$$

We identify $\mathbb{C}^n = \mathbb{R}^{2n}$ by identifying $z = x + iy$ with (x, y). For a complex matrix $L = (l_{jk})$, we denote by $L^* = \left(l^*_{jk}\right)$ its conjugate, so that

$$l^*_{jk} = \overline{l_{kj}} \quad \text{for all} \quad j, k.$$

We prove the following main result.

(1.1) Theorem. *Let A be an $n \times n$ positive definite matrix with all eigenvalues between 1 and 2. Let us write $A = I + B$, where I is the $n \times n$ identity matrix and B is an $n \times n$ positive semidefinite Hermitian matrix with eigenvalues between 0* *and* 1. *Further, we write* $B = LL^*$, *where* $L = (l_{jk})$ *is an* $n \times n$ *complex matrix. We define linear functions* $\ell_1, \dots, \ell_n : \mathbb{C}^n \longrightarrow \mathbb{C}$ *by*

$$\ell_j(z) = \sum_{k=1}^{n} l_{jk} z_k \quad \text{for} \quad z = (z_1, \dots, z_n).$$

Let us define $f_A : \mathbb{C}^n \longrightarrow \mathbb{R}_+$ *by*

$$f_A(z) = \frac{1}{\pi^n} e^{-\|z\|^2} \prod_{j=1}^{n} \left(1 + |\ell_j(z)|^2\right).$$

(1) *Identifying* $\mathbb{C}^n = \mathbb{R}^{2n}$, *we have*

$$\operatorname{per} A = \int_{\mathbb{R}^{2n}} f_A(x, y)\ dxdy.$$

(2) *The function* $f_A : \mathbb{R}^{2n} \longrightarrow \mathbb{R}_+$ *is log-concave, that is, if* $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^{2n}$ *and if*

$$x = \alpha x_1 + (1 - \alpha) x_2 \quad \text{and} \quad y = \alpha y_1 + (1 - \alpha) y_2 \quad \text{for some} \quad 0 \leq \alpha \leq 1$$

then

$$f_A(x, y) \ \geq \ f_A^{\alpha}(x_1, y_1) f_A^{1-\alpha}(x_2, y_2).$$

2. Proofs

We start with a known integral representation of the permanent of a positive semidefinite matrix.

(2.1) The integral formula. Let μ be the Gaussian probability measure in $\mathbb{C}^n$ with density

$$\frac{1}{\pi^n} e^{-\|z\|^2} \quad \text{where} \quad \|z\|^2 = |z_1|^2 + \ldots + |z_n|^2 \quad \text{for} \quad z = (z_1, \dots, z_n).$$

For the expectations of products of coordinates, we have

$$\mathbf{E}\, z_i \overline{z_j} = \int_{\mathbb{C}^n} z_i \overline{z_j}\ d\mu(z) = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\ell_1, \dots, \ell_n : \mathbb{C}^n \longrightarrow \mathbb{C}$ be linear functions and let $B = (b_{jk})$ be the $n \times n$ matrix,

$$b_{jk} = \mathbf{E}\, \ell_j \overline{\ell_k} = \int_{\mathbb{C}^n} \ell_j(z) \overline{\ell_k(z)}\ d\mu(z) \quad \text{for} \quad j, k = 1, \dots, n.$$

Hence B is a positive semidefinite Hermitian matrix and the Wick formula (see, for example, Section 3.1.4 of [Ba16]) implies that

$$\operatorname{per} B = \mathbf{E}\ \left(|\ell_1|^2 \cdots |\ell_n|^2\right) = \int_{\mathbb{C}^n} |\ell_1(z)|^2 \cdots |\ell_n(z)|^2\ d\mu(z). \tag{2.1.1}$$

Next, we need a simple lemma.

(2.2) Lemma. *Let $q : \mathbb{R}^m \longrightarrow \mathbb{R}_+$ be a positive semidefinite quadratic form. Then the function*

$$h(x) = \ln\bigl(1 + q(x)\bigr) - q(x)$$

is concave.

Proof. It suffices to check that the restriction of h onto any affine line $x(\tau) = \tau a + b$ with $a, b \in \mathbb{R}^m$ is concave. Thus we need to check that the univariate function

$$G(\tau) = \ln\bigl(1 + (\alpha\tau + \beta)^2 + \gamma^2\bigr) - (\alpha\tau + \beta)^2 - \gamma^2 \quad \text{for} \quad \tau \in \mathbb{R},$$

where $\alpha \neq 0$, is concave, for which it suffices to check that $G''(\tau) \leq 0$ for all τ. Via the affine substitution $\tau := (\tau - \beta)/\alpha$, it suffices to check that $g''(\tau) \leq 0$, where

$$g(\tau) = \ln\left(1 + \tau^2 + \gamma^2\right) - \left(\tau^2 + \gamma^2\right).$$

We have

$$g'(\tau) = \frac{2\tau}{1 + \tau^2 + \gamma^2} - 2\tau$$

and

$$\begin{aligned} g''(\tau) =& \frac{2(1 + \tau^2 + \gamma^2) - 4\tau^2}{(1 + \tau^2 + \gamma^2)^2} - 2 \\ =& \frac{2(1 + \tau^2 + \gamma^2) - 4\tau^2 - 2\left(1 + \tau^2 + \gamma^2\right)^2}{(1 + \tau^2 + \gamma^2)^2} \\ =& \frac{2 + 2\tau^2 + 2\gamma^2 - 4\tau^2 - 2 - 2\tau^4 - 2\gamma^4 - 4\tau^2 - 4\gamma^2 - 4\tau^2\gamma^2}{(1 + \tau^2 + \gamma^2)^2} \\ =& -\frac{6\tau^2 + 2\gamma^2 + 2\tau^4 + 2\gamma^4 + 4\tau^2\gamma^2}{(1 + \tau^2 + \gamma^2)^2} \leq 0 \end{aligned}$$

and the proof follows. □

(2.3) Proof of Theorem 1.1. We have

$$\operatorname{per} A = \operatorname{per}(I + B) = \sum_{J \subset \{1, \ldots, n\}} \operatorname{per} B_J,$$

where B_J is the principal $|J| \times |J|$ submatrix of B with row and column indices in J and where we agree that $\operatorname{per} B_\emptyset = 1$. Let us consider the Gaussian probability measure in $\mathbb{C}^n$ with density $\pi^{-n} e^{-\|z\|^2}$. By (2.1.1), we have

$$\operatorname{per} B_J = \mathbf{E} \prod_{j \in J} |\ell_j(z)|^2$$

and hence

$$\operatorname{per} A = \mathbf{E} \prod_{j=1}^{n} \left(1 + |\ell_j(z)|^2\right) = \int_{\mathbb{R}^{2n}} f_A(x, y)\, dxdy,$$

and the proof of Part (1) follows.

We write

$$e^{-\|z\|^2} \prod_{j=1}^{n} \left(1 + |\ell_j(z)|^2\right) = e^{-q(z)} \prod_{j=1}^{n} \left(1 + |\ell_j(z)|^2\right) e^{-|\ell_j(z)|^2},$$

$$\text{where} \quad q(z) = \|z\|^2 - \sum_{j=1}^{n} |\ell_j(z)|^2.$$

By Lemma 2.2 each function $(1 + |\ell_j(z)|^2)e^{-|\ell_j(z)|^2}$ is log-concave on $\mathbb{R}^{2n} = \mathbb{C}^n$ and hence to complete the proof of Part (2) it suffices to show that q is a positive semidefinite Hermitian form. To this end, we consider the Hermitian form

$$p(z) = \sum_{j=1}^{n} |\ell_j(z)|^2 = \sum_{j=1}^{n} \left| \sum_{k=1}^{n} l_{jk} z_k \right|^2 = \sum_{j=1}^{n} \sum_{1 \le k_1, k_2 \le n} l_{jk_1} \overline{l_{jk_2}} z_{k_1} \overline{z_{k_2}}$$
$$= \sum_{1 \le k_1, k_2 \le n} c_{k_1 k_2} z_{k_1} \overline{z_{k_2}},$$

where

$$c_{k_1 k_2} = \sum_{j=1}^{n} l_{jk_1} \overline{l_{jk_2}} \quad \text{for} \quad 1 \le k_1, k_2 \le n.$$

Hence for the matrix $C = (c_{k_1 k_2})$ of p, we have $C = \overline{L^* L}$. We note that $B = LL^*$ and that the eigenvalues of B lie between 0 and 1. Therefore, the eigenvalues of $L^* L$ lie between 0 and 1 (in the generic case, when L is invertible, the matrices LL^* and $L^* L$ are similar). Consequently, the eigenvalues of C lie between 0 and 1 and hence the Hermitian form $q(z)$ with matrix $I - C$ is positive semidefinite, which completes the proof of Part (2). □

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