Finding Maximum Length Tours Under Polyhedral Norms

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Abstract

We consider the traveling salesman problem when the cities are points in $\mathbb{R}^d$ for some fixed d and distances are computed according to a polyhedral norm. We show that for any such norm, the problem of finding a tour of *maximum* length can be solved in polynomial time. If arithmetic operations are assumed to take unit time, our algorithms run in time $O(n^{f+1})$, where f is the number of facets of the polyhedron determining the polyhedral norm. Thus for example we have $O(n^5)$ algorithms for the cases of points in the plane under the Rectilinear and Sup norms. This is in contrast to the fact that finding a *minimum* length tour in each case is NP-hard.

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1 Introduction

In the *Traveling Salesman Problem* (TSP), the input consists of a set $\{c_1, c_2, \ldots, c_n\}$ of *cities* together with the distances $d(c, c')$ between every pair of distinct cities c, c'. The goal is to find an ordering or *tour* of the cities that minimizes (Minimum TSP) or maximizes (Maximum TSP) the total tour length. Here the length of a tour $c_{\pi(1)}, c_{\pi(2)}, \ldots, c_{\pi(n)}$ is

$$\sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)}) + d(c_{\pi(n)}, c_{\pi(1)}).$$

Of particular interest are *geometric* instances of the TSP, in which cities correspond to points in $\mathbb{R}^d$ for some $d \geq 1$, and distances are computed according to some geometric norm. Perhaps the most popular norms are the Rectilinear, Euclidean, and Sup norms. These are examples of what is known as an "L^p norm" for $p = 1$, 2, and ∞. In general, the distance between two points $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_d)$ under the L^p norm, $p \geq 1$, is

$$d(\mathbf{x}, \boldsymbol{y}) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{1/p}$$

with the natural asymptotic interpretation that under the L^∞ norm the distance is $\max_{i=1}^{d} |x_i - y_i|$.

This paper concentrates on a second class of norms which also includes the Rectilinear and Sup norms, but can only approximate the Euclidean and other L^p norms. This is the class of *polyhedral norms*. Each polyhedral norm is determined by a *unit ball* which is a centrally-symmetric polyhedron $\boldsymbol{P}$ with the origin at its center. To determine $d(\mathbf{x}, \boldsymbol{y})$ under such a norm, first translate the space so that one of the points, say $\mathbf{x}$, is at the origin. Then determine the unique factor α by which one must rescale $\boldsymbol{P}$ (expanding if $\alpha > 1$, shrinking if $\alpha < 1$) so that the other point ($\boldsymbol{y}$) is on the boundary of the polyhedron. We then have $d(\mathbf{x}, \boldsymbol{y}) = \alpha$.

Alternatively, and more usefully for our purposes, we can view a polyhedral norm as follows. If $\boldsymbol{P}$ is a polyhedron as described above and has F facets, then F is divisible by 2 and there is a set $H_{\boldsymbol{P}} = \{\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{F/2}\}$ of points in $\mathbb{R}^d$ such that $\boldsymbol{P}$ is the intersection of a collection of half-spaces determined by $H_{\boldsymbol{P}}$:

$$\boldsymbol{P} = \left(\bigcap_{i=1}^{F/2} \{\mathbf{x} : \mathbf{x} \cdot \boldsymbol{h}_i \leq 1\} \right) \cap \left(\bigcap_{i=1}^{F/2} \{\mathbf{x} : \mathbf{x} \cdot \boldsymbol{h}_i \geq -1\} \right)$$

Then we have

$$d(\mathbf{x}, \boldsymbol{y}) = \max \left\{ \left| (\mathbf{x} - \boldsymbol{y}) \cdot \boldsymbol{h}_i \right| : 1 \leq i \leq F/2 \right\}$$

Note that for the Rectilinear norm in the plane we can take $H_{\boldsymbol{P}} = \{(1,1), (-1,1)\}$ and for the Sup norm in the plane we can take $H_{\boldsymbol{P}} = \{(1,0), (0,1)\}$.

For the Minimum TSP on geometric instances, all of the key complexity questions have been answered. As follows from results of Itai, Papadimitriou, and Swarcfiter [5], the Minimum TSP is NP-hard for any fixed dimension d and any L^p or polyhedral norm. On the other hand, recent results of Arora [1] [2] and Mitchell [7] imply that in all these cases a polynomial-time approximation scheme (PTAS) exists, i.e., a sequence of polynomial-time algorithms A_k, $1 \leq k < \infty$, where A_k is guaranteed to find a tour whose length is within a ratio of $1 + (1/k)$ of optimal.

The situation for geometric versions of the Maximum TSP is less completely resolved. Barvinok [3] has shown that once again polynomial-time approximation schemes exist for all fixed dimensions d and all L^p or polyhedral norms (and in a sense for *any* fixed norm; see [3]). Until now, however, the complexity of the optimization problems themselves when d is fixed has remained open. For no fixed dimension d and L^p or polyhedral norm was the problem of determining the maximum tour length known either to be NP-hard or to be polynomial-time solvable. In this paper, we resolve the question for polyhedral norms, showing that, in contrast to the case for the Minimum TSP, the Maximum TSP is solvable in polynomial time for any fixed dimension d and any polyhedral norm:

Theorem 1 *Let dimension d be fixed and let $\|\cdot\|$ be a fixed polyhedral norm in $\mathbb{R}^d$ whose unit ball is a polyhedron $\boldsymbol{P}$ determined by a set $H_{\boldsymbol{P}}$ of k vectors. Then for any set of n points in $\mathbb{R}^d$, one can construct a traveling salesman tour of maximum length with respect to $\|\cdot\|$ in time $O(n^{2k+1})$, assuming arithmetic operations take unit time.*

As an immediate consequence of Theorem 1, we get relatively efficient algorithms for the Maximum TSP in the plane under Rectilinear and Sup norms:

Corollary 1.1 *The Maximum TSP for points in the plane under the L^1 and L^∞ norms can be solved in $O(n^5)$ time, assuming arithmetic operations take unit time.*

The restriction to unit cost arithmetic operations in Theorem 1 and Corollary 1.1 is made primarily to simplify the statements of the conclusions, although it does reflect that fact that our results hold for the *real number RAM* computational model. Suppose on the other hand that one assumes, as one typically must for complexity theory results, that the components of the vectors in $H_{\boldsymbol{P}}$ and the coordinates of the cities are all rationals. Let U denote the maximum absolute value of any of the corresponding numerators and denominators. Then the conclusions of the Theorem and Corollary hold with running times multipled by $n \log(U)$. If the components/coordinates are all integers with maximum absolute value U, the running times need only be multiplied by $\log(nU)$. For simplicity in the remainder of this paper, we shall stick to the model in which numbers can be arbitrary reals and arithmetic operations take unit time. The reader should have no trouble deriving the above variants.

The paper is organized as follows. Section 2 introduces a new special case of the TSP, the *Tunneling TSP*, and shows how the Maximum TSP under a polyhedral norm can be reduced to the Tunneling TSP with a bounded number of tunnels. Section 3 then shows how the latter problem can be solved in polynomial time. Section 4 then concludes by describing some related results and open problems.

2 The Tunneling TSP

The *Tunneling TSP* is a special case of the Maximum TSP in which distances are determined by what we shall call a *tunnel system* distance function. In such a distance function we are given a set $T = \{t_1, t_2, \ldots, t_k\}$ of auxiliary objects that we shall call *tunnels*. Each tunnel is viewed as a bidirectional passage having a front and a back end. For each pair c, t of a city and a tunnel we are given real-valued *access distances* $F(c,t)$ and $B(c,t)$ from the city to the front and back ends of the tunnel respectively. Each potential tour edge $\{c, c'\}$ must pass through some tunnel t, either by entering the front end and leaving the back (for a distance of $F(c,t) + B(c',t)$), or by entering the back end and leaving the front (for a distance of $B(c,t) + F(c',t)$). Since we are looking for a tour of maximum length, we can thus define the distance between cities c and c' to be

$$d(c,c') = \max\left\{F(c,t_i) + B(c',t_i),\ B(c,t_i) + F(c',t_i) : 1 \le i \le k\right\}$$

Note that this distance function, like our geometric norms, is symmetric.

It is easy to see that Maximum TSP remains NP-hard when distances are determined by arbitrary tunnel system distance functions. However, for the case where $k = |T|$ is fixed and not part of the input, we will show in the next section that Maximum TSP can be solved in $O(n^{2k+1})$ time. We are interested in this special case because of the following lemma.

Lemma 2.1 *If $\|\cdot\|$ is a polyhedral norm determined by a set of k vectors $H_{\boldsymbol{P}}$ in $\mathbb{R}^d$, then for any set C of points in $\mathbb{R}^d$ one can in time $O(dk|C|)$ construct a tunnel system distance function with k tunnels that yields $d(\mathbf{c}, \mathbf{c}') = \|\mathbf{c} - \mathbf{c}'\|$ for all $\mathbf{c}, \mathbf{c}' \in C$.*

Proof. The polyhedral distance between two cities $\mathbf{c}, \mathbf{c}' \in \mathbb{R}^d$ is

$$\begin{aligned} \|\boldsymbol{c} - \boldsymbol{c}'\| &= \max\left\{\left|(\boldsymbol{c} - \boldsymbol{c}') \cdot \boldsymbol{h}_i\right| : 1 \le i \le k\right\} \\ &= \max\left\{(\boldsymbol{c} - \boldsymbol{c}') \cdot \boldsymbol{h}_i,\ (\boldsymbol{c}' - \boldsymbol{c}) \cdot \boldsymbol{h}_i : 1 \le i \le k\right\} \end{aligned}$$

Thus we can view the distance function determined by $\|\cdot\|$ as a tunnel system distance function with set of tunnels $T = H_{\boldsymbol{P}}$ and $F(\boldsymbol{c}, \boldsymbol{h}) = \boldsymbol{c} \cdot \boldsymbol{h}$, $B(\boldsymbol{c}, \boldsymbol{h}) = -\boldsymbol{c} \cdot \boldsymbol{h}$ for all cities $\boldsymbol{c}$ and tunnels $\boldsymbol{h}$. ∎

3 An Algorithm for Bounded Tunnel Systems

This section is devoted to the proof of the following lemma, which together with Lemma 2.1 implies Theorem 1.

Lemma 3.1 *If the number of tunnels is fixed at k, the Tunneling TSP can be solved in time $O(n^{2k+1})$, assuming arithmetic operations take unit time.*

Proof. Suppose we are given an instance of the Tunneling TSP with sets $C = \{c_1, \ldots, c_n\}$ and $T = \{t_1, \ldots, t_k\}$ of cities and tunnels, and access distances $F(c,t)$, $B(c,t)$ for all $c \in C$ and $t \in T$. We begin by transforming the problem to one about subset construction.

Let $G = (C \cup T, E)$ be an edge-weighted, bipartite multigraph with two edges between each city c and tunnel t, denoted by $e[c,t,F]$ and $e[c,t,B]$. The weights of these two edges are $w(e[c,t,F]) = F(c,t)$ and $w(e[c,t,B]) = B(c,t)$. For notational convenience, let us partition the edges in E into sets $E[t,F] = \{e[c,t,F] : c \in C\}$ and $E[t,B] = \{e[c,t,B] : c \in C\}$, $t \in T$. Each tour for the TSP instance then corresponds to a subset E' of E that has $\sum_{e \in E'} w(e)$ equal to the tour length and satisfies

- (T1) Every city is incident to exactly two edges in E'.
- (T2) For each tunnel $t \in T$, $|E' \cap E[t,F]| = |E' \cap E[t,B]|$.
- (T3) The set E' is connected.

To construct E' we simply represent each tour edge $\{c, c'\}$ by a pair of edges from E that connect to the tunnel that determines $d(c,c')$. For instance, if $d(c,c') = F(c,t) + B(c',t)$, the tour edge $\{c, c'\}$ can be represented by the two edges $e[c,t,F]$ and $e[c',t,B]$. If $d(c,c')$ can be realized in more than one way, the set E' will not be unique. However, any set E' constructed in this fashion will still have $\sum_{e \in E'} w(e)$ equal to the tour length.

On the other hand, any set E' satisfying (T1) – (T3) corresponds to one (or more) tours having length at least $\sum_{e \in E'} w(e)$: Let $T' \in T$ be the set of tunnels t with $|E' \cap E[t,F]| > 0$. Then $G' = (C \cup T', E')$ is a connected graph all of whose vertex degrees are even by (T1) – (T3). By an easy result from graph theory, this means that G' contains an Euler tour that by (T1) passes through each city exactly once, thus inducing a TSP tour for C. Moreover, by (T2) one can construct such an Euler tour with the additional property that if $e[c,t,x]$ and $e[c',t,y]$ are consecutive edges in this tour, then $x \neq y$, i.e, either $x = F, y = B$ or $x = B, y = F$. Thus we will have $w(e[c,t,x]) + w(e[c',t,y]) \leq d(c,c')$, and hence the length of the TSP tour will be at least $\sum_{e \in E'} w(e)$, as claimed.

Thus our problem is reduced to finding a maximum weight set of edges $E' \subseteq E$ satisfying (T1) – (T3). This can be partitioned into a collection of subproblems based on the possibilities for the topological structure of E'. First, any such E' must induce a degree sequence $\boldsymbol{d} = (d_1, \ldots, d_k)$ with $\sum_{i=1}^{k} d_i = n$, where $d_i = |E' \cap E[t_i, F]|$,

$1 \le i \le k$. Note that there are $O(n^{k-1})$ possible degree sequences, since $d_1, \ldots, d_{k-1}$ together determine d_k.

For any such degree sequence $\boldsymbol{d}$ we can further break into subcases depending on how the tunnels are linked together. Let $T' = \{t_i : d_i > 0\}$ and let $k' = |T'|$. Note that by (T3) it must be the case that for any partition $T' = T'' \cup T'''$ of T' into nonempty disjoint subsets, there must be some city c with edges to tunnels in both subsets. We will describe the connection pattern for the tunnels using a sequence

$$\boldsymbol{s} = \Big(t'[1], c_{\sigma(1)}, t''[1], g_1, h_1, \ldots, t'[k'-1], c_{\sigma(k'-1)}, t''[k'-1], g_{k'-1}, h_{k'-1}\Big),$$

where $t'[i], t''[i] \in T'$ and $g_i, h_i \in \{F, B\}$, $1 \le i \le k'-1$. The sequence will be unique for each set E' satisfying (T1) – (T3) and is constructed as follows: Let $t'[1]$ be the lowest-indexed tunnel in T'. The remaining elements of the sequence are determined inductively as follows, starting with $i = 1$ and $T_1 = \{t'[1]\}$:

(1) Let $c_{\sigma(i)}$ be the lowest-indexed city adjacent in E' to tunnels in both T_i and $T' - T_i$.

(2) Let $e[c_{\sigma(i)}, t', g]$ and $e[c_{\sigma(i)}, t'', h]$ be the two edges in E' involving $c_{\sigma(i)}$, where $t' \in T_i$.

(3) Set $t'[i] = t'$, $t''[i] = t''$, $g_i = g$, $h_i = h$, and $T_{i+1} = T_i \cup \{t''\}$.

(4) Set $i = i + 1$ and continue if $i < k'$.

Note that there are $O(n^{k-1}(2k)^{2k-2})$ potential sequences $\boldsymbol{s}$, i.e. $O(n^{k-1})$ for fixed k.

We now show that for each pair $(\boldsymbol{d}, \boldsymbol{s})$, the problem of finding the maximum weight subset $E' \subseteq E$ consistent with $(\boldsymbol{d}, \boldsymbol{s})$, i.e., having degree sequence d and connection pattern s, can be reduced to a weighted b-matching problem on a new auxiliary graph G'. Recall that in the weighted b-matching problem, one is given a weighted graph $G = (V, E)$ and a target degree $b(v)$ for each $v \in V$. One wishes to find a maximum weight subset $E' \subseteq E$ that meets the degree targets, i.e., that is such that $deg(v) \equiv |\{e \in E' : v \in e\}| = b(v)$ for all $v \in V$. This problem can be solved in time $O((|V| + \sum_{v \in V} b(v))^3)$ by a transformation to the standard weighted matching problem on an expanded graph [4] [8].

Our auxiliary graph G' has a vertex set consisting of the cities $c \in C$ plus two vertices t^F and t^B for each tunnel $t \in T$. The edge set consists of an edge of weight $F(c,t)$ connecting each pair $\{c, t^F\}$ and an edge of weight $B(c,t)$ connecting each pair $\{c, t^B\}$. Note that the problem of finding the maximum weight edge set $E' \subseteq E$ consistent with a given degree sequence $\boldsymbol{d}$ and connection pattern $\boldsymbol{s}$ can be recast as a weighted b-matching problem on G' with the values $b(v)$ specified as follows:

Let $C_{\boldsymbol{s}}$ be the set of cities c occurring in $\boldsymbol{s}$ and let $E_{\boldsymbol{s}}$ be the set of edges of the form $e[c_{\sigma(i)}, t'[i], g_i]$ and $e[c_{\sigma(i)}, t''[i], h_i]$ induced by $\boldsymbol{s}$. Next, for each $t \in T$ define $E_{\boldsymbol{s}}[t, F]$ to be the set of edges of the form $e[c, t, F]$ contained in $E_{\boldsymbol{s}}$ and $E_{\boldsymbol{s}}[t, B]$ to be the set of edges of the form $e[c,t,B]$ contained in E_s. The target vertex degrees in the b-matching problem are then

$$\begin{aligned} b(c) &= 0, \text{ if } c \in C_s \\ b(c) &= 2, \text{ if } c \in C - C_s \\ b(t_i^F) &= d_i - |E_s[t_i, F]|, \text{ for } 1 \leq i \leq k \\ b(t_i^B) &= d_i - |E_s[t_i, B]|, \text{ for } 1 \leq i \leq k \end{aligned}$$

The resulting b-matching problem can be solved in time $O((n + 2k + \sum_{i=1}^{k} d_i)^3) = O(n^3)$ for fixed k. Since there are $O(n^{k-1} \cdot n^{k-1})$ possible pairs $(\boldsymbol{d}, \boldsymbol{s})$ to consider, this yields an overall running time of $O(n^{2k+1})$ as claimed. ■

4 Conclusion

We have derived a polynomial time algorithm for the Maximum TSP when the cities are points in $\mathbb{R}^d$ for some fixed d and when the distances are measured according to some polyhedral norm. The complexity of the Maximum TSP with *Euclidean distances* and fixed d remains unsettled, however, even for $d = 2$. Although the Euclidean norm can be approximated arbitrarily closely by polyhedral norms (and hence Barvinok's result [3] that Maximum TSP has a PTAS), it is not itself a polyhedral norm.

A further difficulty with the Euclidean norm (one shared by both the Minimum and Maximum TSP) is that we still do not know whether the TSP is in NP under this norm. Even if all city coordinates are rationals, we do not know how to compare a tour length to a given rational target in less than exponential time. Such a comparison would appear to require us to evaluate a sum of n square roots to some precision, and currently the best upper bound known on the number of bits of precision needed to insure a correct answer remains exponential in n. Thus even if we were to produce an algorithm for the Euclidean Maximum TSP that ran in polynomial time when arithmetic operations (and comparisons) take unit time, it might not run in polynomial time on a standard Turing machine.

Another set of questions concerns the complexity of the Maximum TSP when d is *not* fixed. It is relatively easy to show that the problem is NP-hard for all L^p norms (the most natural norms that are defined for all $d > 0$). For the case of L^∞ one can use a transformation from Hamiltonian Circuit in which each edge is represented by a separate dimension. For the L^p norms, $1 \leq p < \infty$, one can use a transformation from the Hamiltonian Circuit problem for cubic graphs, with a dimension for each *non*-edge. However, this still leaves open the question of whether there might exist a PTAS for any such norm when d is not fixed. Trevisan [10] has shown that the Minimum TSP is Max SNP-hard for any such norm, and so cannot have such PTAS's unless P = NP. We can obtain a similar result for the Maximum TSP under L^∞ by modifying our NP-hardness transformation so that the source problem is the Minimum TSP with all edge lengths in $\{1, 2\}$, a special case that was proved Max SNP-hard by Papadimitriou and

Yannakakis [9]. The question remains open for L^p, $1 \leq p < \infty$, although we conjecture that these cases are Max SNP-hard as well.

Finally, we note that our results can be extended in several ways. For instance, one can get polynomial-time algorithms for asymmetric versions of the Maximum TSP in which distances are computed based on non-symmetric unit balls. Also, algorithmic approaches analogous to ours can be applied to geometric versions of other NP-hard maximization problems: For example, consider the *Weighted 3-Dimensional Matching Problem* that consists in partitioning a set of $3n$ elements into n triples of maximum total weight. The special case where the elements are points in $\mathbb{R}^d$ and where the weight of a triple equals the perimeter of the corresponding triangle measured according to some fixed polyhedral norm can be solved in polynomial time.

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