RENORMALIZED VOLUME AND THE VOLUME OF THE CONVEX CORE

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Abstract. We obtain upper and lower bounds on the difference between the renormalized volume and the volume of the convex core of a convex cocompact hyperbolic 3-manifold which depend on the injectivity radius of the boundary of the universal cover of the convex core and the Euler characteristic of the boundary of the convex core. These results generalize results of Schlenker obtained in the setting of quasifuchsian hyperbolic 3-manifolds.

1. Introduction

Krasnov and Schlenker [16, 17] studied the renormalized volume of a convex cocompact hyperbolic 3-manifold. Renormalized volume was introduced in the more general setting of infinite volume conformally compact Einstein manifolds as a way to assign a finite normalized volume in a natural way (see Graham-Witten [11]). Krasnov and Schlenker's renormalized volume generalizes earlier work of Krasnov [15] and Takhtajan-Teo [21] for special classes of hyperbolic 3-manifolds. In particular, it is closely related to the Liouville action functional studied by Tahktajan-Teo [21] and the renormalized volume gives rise to a Kähler potential for the Weil-Petersson metric (see Krasnov-Schlenker [16, Section 8]).

Schlenker [20] showed that there exists $K > 0$ such that if M is a quasifuchsian hyperbolic 3-manifold, then

$$V_C(M) - K|\chi(\partial M)| \leq V_R(M) \leq V_C(M)$$

where $V_R(M)$ is the renormalized volume of M and $V_C(M)$ is the volume of the convex core $C(M)$ of M. This inequality, along with a variational formula for the renormalized volume, was used by Kojima-McShane [14] and Brock-Bromberg [7] to give an upper bound on the volume of a hyperbolic 3-manifold fibering over the circle in terms of the entropy of its monodromy map.

In this paper, we use the work of the authors [3, 4, 5, 6, 8] to generalize Schlenker's result to the setting of all convex cocompact hyperbolic 3-manifolds. We exhibit bounds on the difference between $V_C(M)$ and $V_R(M)$ in terms of the injectivity radius of the boundary of the universal cover of the convex core and the Euler characteristic of the boundary of the convex core. We will see that, even if $|\chi(\partial C(M))|$ is bounded, this difference can be arbitrarily large.

The *convex core* $C(M)$ of a complete hyperbolic 3-manifold M (with non-abelian fundamental group) is the smallest convex submanifold of M whose inclusion into M is a homotopy equivalence. Its boundary $\partial C(M)$ is a hyperbolic surface in

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its intrinsic metric (see Epstein-Marden [10, Theorem II.1.12.1] and Thurston [23, Proposition 8.5.1]). A complete hyperbolic 3-manifold M (with non-abelian fundamental group) is said to be *convex cocompact* if $C(M)$ is compact.

Our results, and their proofs, naturally divide into two cases, depending on whether the boundary of the convex core is incompressible. We recall that $\partial C(M)$ is *incompressible* if whenever S is a component of $\partial C(M)$, then $\pi_1(S)$ injects into $\pi_1(M)$. Equivalently, the boundary of the convex core is incompressible if and only if $\pi_1(M)$ is freely indecomposable. In particular, if M is a quasifuchsian hyperbolic 3-manifold, the boundary of its convex core is incompressible. In this case, we get the following generalization of Schlenker's result.

Theorem 1.1. *If $M = \mathbb{H}^3/\Gamma$ is a convex cocompact hyperbolic 3-manifold and $\partial C(M)$ is incompressible, then*

$$V_C(M) - 6.89|\chi(\partial C(M))| \leq V_R(M) \leq V_C(M).$$

Moreover, $V_R(M) = V_C(M)$ if and only if $\partial C(M)$ is totally geodesic.

In Proposition 5.1 we construct examples demonstrating the necessity of a linear dependence on $|\chi(\partial C(M))|$ in Theorem 1.1.

If the boundary of the convex core is compressible, then the boundary of the universal cover $\widetilde{C(M)}$ of the convex core is not simply connected and it is natural to consider its injectivity radius η, in its intrinsic metric. Equivalently, η is half the length of the shortest homotopically non-trivial curve in $\partial C(M)$ which bounds a disk in $C(M)$.

Theorem 1.2. *If M is a convex cocompact hyperbolic 3-manifold, $\partial C(M)$ is compressible and $\eta > 0$ is the injectivity radius of the intrinsic metric on $\partial\widetilde{C(M)}$, then*

$$V_C(M) - |\chi(\partial M)|\left(45\log\left(\frac{1}{\min\{1,\eta\}}\right) + 67\right) \leq V_R(M) < V_C(M)$$

Furthermore, if $\eta \leq \sinh^{-1}(1)$, then

$$V_R(M) \leq V_C(M) - \pi\log\left(\frac{1}{\eta}\right) - 1.79.$$

If $M = \mathbb{H}^3/\Gamma$, then the *domain of discontinuity* $\Omega(\Gamma)$ is the largest open subset of $\hat{\mathbb{C}} = \partial\mathbb{H}^3$ which Γ acts properly discontinuously on. The quotient $\partial_c M = \Omega(\Gamma)/\Gamma$ is called the *conformal boundary* of M. The manifold M is convex cocompact if and only if

$$\widehat{M} = M \cup \partial_c M = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$$

is compact. $\Omega(\Gamma)$ admits a unique conformal metric of curvature -1, called the *Poincaré metric*. Since the Poincaré metric is conformally natural, it descends to a hyperbolic metric on the conformal boundary. We also obtain a version of our theorem where the bounds depend on the injectivity radius of the Poincaré metric on $\Omega(\Gamma)$.

Theorem 1.3. *If $M = \mathbb{H}^3/\Gamma$ is a convex cocompact hyperbolic 3-manifold, $\partial C(M)$ is compressible and $\nu > 0$ is the injectivity radius of the Poincare metric on $\Omega(\Gamma)$, then*

$$V_C(M) - |\chi(\partial C(M))|\left(\frac{205}{\nu} + 202\right) \leq V_R(M) < V_C(M).$$

Furthermore, if $\nu \leq \frac{1}{2}$, then

$$V_R(M) \leq V_C(M) - \left(\frac{9}{\nu} - 9\right)$$

One may loosely reformulate Theorem 1.2 as saying that $V_C(M) - V_R(M)$ is comparable to $\log \frac{1}{\eta(M)}$ when $\eta(M)$ is small, where $\eta(M)$ is the injectivity radius of $\widetilde{\partial C(M)}$. Similiarly, one may reformulate Theorem 1.3 as saying that $V_C(M) - V_R(M)$ is comparable to $\frac{1}{\nu(M)}$ when $\nu(M)$ is small, where $\nu(M)$ is the injectivity radius of $\Omega(\Gamma)$ in the Poincaré metric.

We note that one may obtain slightly more precise forms of our results by giving exact forms for the constants involved, but the expressions for the constants would be rather unpleasant and it seems unlikely that the constants obtained by our techniques are sharp. However, our estimates are of roughly the correct asymptotic form as ν or η approach 0.

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2. Renormalized Volume

In this section, we recall the work of Krasnov-Schlenker ([16, 17]) and Schlenker ([20]) on renormalized volume for convex cocompact hyperbolic 3-manifolds. We will assume for the remainder of the paper that $M = \mathbb{H}^3/\Gamma$ is convex cocompact.

If N is a compact, $C^{1,1}$ strictly convex submanifold such that the inclusion of N into M is a homotopy equivalence, the W*-volume* of N is given by

$$W(N) = V(N) - \frac{1}{2}\int_{\partial N} H dA$$

where H is the mean curvature function.[1] (We recall that a submanifold N is strictly convex if the interior of any geodesic in M joining two points in N lies in the interior of N.)

Notice that if N is $C^{1,1}$, then the curvature and mean curvature of ∂N are defined almost everywhere and the integral of mean curvature is well-defined and well-behaved. This is the natural regularity assumption, since a metric neighborhood of the convex core is $C^{1,1}$ (see Epstein-Marden [10, Lemma II.1.3.6]), but need not be C^2.

If $r > 0$ and N_r is the closed r-neighborhood of N, then N_r is $C^{1,1}$ and strictly convex, and $\{S_r = \partial N_r\}_{r>0}$ is a family of equidistant surfaces foliating the end of M. In particular, N_r is homeomorphic to $\widehat{M}$ for all $r > 0$. The following fundamental lemma relates $W(N_r)$ to $W(N)$.

Lemma 2.1. (Krasnov-Schlenker [16, Lemma 4.2], Schlenker [20, Lemma 3.6]) *If M is a convex cocompact hyperbolic 3-manifold and N is a strictly convex, $C^{1,1}$,*

[1] We are using the convention that the mean curvature H is the average of the principal curvatures, while Krasnov and Schlenker [16, 17] use the convention that H is the sum of the principal curvatures, so our definition, although apparently different, agrees with theirs.

compact submanifold such that the inclusion of N into M is a homotopy equivalence, then

$$W(N_r) = W(N) - r\pi\chi(\partial C(M)).$$

Lemma 2.1 follows from the fact that

$$\dot{W}_t = \frac{d}{dt}W(N_t) = \frac{d}{dt}V(N_t) - \frac{1}{2}\frac{d}{dt}\left(\int_{S_t} H_t dA_t\right) = A(t) - \frac{1}{4}A''(t)$$

where $A(t)$ is the area of S_t. The general solution to the equation $y'' - 4y = 0$ is $ae^{2t} + be^{-2t}$. Therefore as the exponential terms in $A(t)$ are of this form, they do not contribute to a change in W-volume. Further analysis shows that the remaining terms give $\dot{W}_t = -\pi\chi(\partial N)$.

If I_r is the intrinsic metric on S_r, the normal map identifies S_r with the conformal boundary $\partial_c M$ and one may define the limiting conformal metric I^* on $\partial_c M$ by

$$I^* = \lim_{r\to\infty} 4e^{-2r} I_r.$$

C. Epstein [9] showed that given any conformal $C^{1,1}$ metric h on $\partial_c M$, there exists an (asymptotically) unique family of equidistant submanifolds $N_r(h)$, called the *Epstein submanifolds* whose limiting conformal structure is h. Explicitly, let $\Omega \subseteq \hat{\mathbb{C}}$ be a hyperbolic domain in the Riemann sphere and let g be a $C^{1,1}$ conformal metric on Ω. Given $z \in \Omega$, let $H(z, g)$ be the horoball bounded by the horosphere

$$h(z, g) = \left\{x \in \mathbb{H}^3 \mid v_x(z) = g(z)\right\}$$

where v_x is the visual metric on $\hat{\mathbb{C}}$ obtained by identifying $\hat{\mathbb{C}}$ with $T^1_x\mathbb{H}^3$. Then

$$\Sigma(g) = \partial\left(\bigcup_{z\in\Omega} H(z, g)\right).$$

is the outer envelope of the collection of horospheres $\{h(z, g)\}_{z\in\Omega}$.

If h is a conformal metric on $\partial_c M$, then h lifts to a metric $\tilde{h}$ on $\Omega(\Gamma)$. For all sufficiently large r, $\Sigma(e^r\tilde{h})$ descends to a $C^{1,1}$ surface S_r bounding a strictly convex submanifold $N_r(h)$ of M. Lemma 2.1 indicates that it is natural to define the W-volume of h as

$$W(h) = W(N_r(h)) + r\pi\chi(\partial N_r(h))$$

for any r large enough that $N_r(h)$ is well-defined, strictly convex and $C^{1,1}$.

The *renormalized volume* $V_R(M) = W(\rho)$ where ρ is the Poincaré metric on the conformal boundary $\partial_c(M)$. Krasnov and Schlenker [16, Section 7] showed that the renormalized volume is the maximum of $W(h)$ as h varies over all smooth conformal metrics on $\partial_c M$ with area $2\pi|\chi(\partial_c M)|$.

The W-volume satisfies the following linearity and monotonicity properties, which will be very useful in establishing our bounds.

Lemma 2.2. (Schlenker [20, Proposition 3.11,Corollary 3.8][2]) *Let M be a convex cocompact hyperbolic manifold. Then*

(1) *(Linearity) If $s \in \mathbb{R}$ and h is a $C^{1,1}$ conformal metric on $\partial_c M$, then*

$$W(e^s h) = W(h) - s\pi\chi(\partial M).$$

[2] The references here and elsewhere in the paper are to the revised version of [20] which appears at arXiv:1109.6663. In particular, the assumption that g and h are non-positively curved is omitted from the published version.

(2) *(Monotonicity) If g and h are non-positively curved, $C^{1,1}$, conformal metrics on $\partial_c M$ and $g(x) \le h(x)$ for all $x \in \partial_c M$, then*

$$W(g) \le W(h).$$

The proof of (1) follows nearly immediately from the definitions. We note that from the definition of $N_r(h)$ that $N_r(e^s h) = N_{r+s}(h)$. Therefore,

$$\begin{aligned} W(e^s h) &= W(N_r(e^s h)) + \pi r \chi(M) = W(N_{r+s}(h)) + \pi r \chi(\partial M) \\ &= (W(N_r(h)) - \pi s \chi(\partial M)) + \pi r \chi(\partial M) \\ &= (W(N_r(h)) + \pi r \chi(\partial M)) - \pi s \chi(\partial M) \\ &= W(h) - \pi s \chi(\partial M). \end{aligned}$$

The proof of (2) is more involved. One first observes that if $g \le h$ and r is large enough that $N_r(g)$ and $N_r(h)$ are both defined, then $N_r(g) \subseteq N_r(h)$. Schlenker then defines a relative W-volume of the region $N_r(h) - N_r(g)$, which agrees with $W(N_r(h)) - W(N_r(g))$, and uses a foliation of $N_r(h) - N_r(g)$ by strictly convex, $C^{1,1}$, non-positively curved surfaces to prove that this relative W-volume is non-negative.

3. The Thurston metric on the conformal boundary

The *Thurston metric* $\tau = \tau(z)|dz|$ on a hyperbolic domain $\Omega \subset \hat{\mathbb{C}}$ is defined by letting the length of a vector $v \in T_z(\Omega)$ be the infimum of the hyperbolic length of all vectors $v' \in \mathbb{H}^2$ such that there exists a Möbius transformation f such that $f(\mathbb{H}^2) \subset \Omega$ and $df(v') = v$. The Thurston metric is clearly conformally natural and conformal to the Euclidean metric. Therefore, if $M = \mathbb{H}^3/\Gamma$ is convex cocompact, then the Thurston metric τ on $\Omega(\Gamma)$ descends to a conformal metric on $\partial_c M$ which we will again denote τ and call the Thurston metric. Kulkarni and Pinkall [18, Theorem 5.9] proved that the Thurston metric is $C^{1,1}$ and non-positively curved (see also Herron-Ibragimov-Minda [12, Theorem C]).

We recall that the Poincaré metric $\rho = \rho(z)|dz|$ on Ω can be similarly defined by letting the length of a vector $v \in T_z(\Omega)$ be the infimum of the hyperbolic length over all vectors v' such that there exists a conformal map $f : \mathbb{H}^2 \to \hat{\mathbb{C}}$ such that $f(\mathbb{H}^2) \subset \Omega$ and $df(v') = v$. So, by definition,

$$\rho(z) \le \tau(z)$$

for all $z \in \Omega$. So, by the monotonicity lemma, Lemma 2.2,

$$V_R(M) = W(\rho) \le W(\tau).$$

One may combine estimates of Beardon-Pommerenke [2], Canary [8, Corollary 3.3] and Kulkarni-Pinkall [18, Theorem 7.2] to establish the following relationship between the Poincaré metric and the Thurston metric of a uniformly perfect hyperbolic domain (see Bridgeman-Canary [6, Section 3]). Notice that if $M = \mathbb{H}^3/\Gamma$ is convex cocompact, then Γ acts cocompactly by isometries on $\Omega(\Gamma)$, so there is a lower bound on the injectivity radius of $\Omega(\Gamma)$ in the Poincaré metric.

Theorem 3.1. *Let Ω be a hyperbolic domain in $\hat{\mathbb{C}}$ and let $\nu > 0$ be the injectivity radius of the Poincare metric ρ on Ω. If τ is the Thurston metric on Ω and*

$k = 4 + \log(3 + 2\sqrt{2}) \approx 5.76$, then

$$\frac{\tau(z)}{2\sqrt{2}(k + \frac{\pi^2}{2\nu})} \le \rho(z) \le \tau(z)$$

for all $z \in \Omega$. Moreover, $\rho = \tau$ if and only if Ω is a round disk.

If Ω is a simply connected hyperbolic domain, then the Thurston metric and the Poincaré metric are 2-bilipschitz.

Theorem 3.2. (Anderson [1, Thm. 4.2], Herron-Ma-Minda [13, Lemma 3.2]) *If Ω is a simply connected hyperbolic domain with Poincare metric ρ and Thurston metric τ, then*

$$\frac{\tau(z)}{2} \le \rho(z) \le \tau(z)$$

for all $z \in \Omega$.

It will be useful to be able to pass back and forth between lower bounds on the injectivity radius of the boundary $\partial\widetilde{C(M)}$ of the universal cover of the convex core, in the intrinsic metric, and lower bounds on the injectivity radius bound of the Poincaré metric on the domain of discontinuity.

Proposition 3.3. *Suppose that $M = \mathbb{H}^3/\Gamma$ is a convex cocompact hyperbolic 3-manifold and that $\partial C(M)$ is non-empty.*

(1) (Bridgeman-Canary [4, Lemma 8.1]) *If $\nu > 0$ is a lower bound for the injectivity radius of $\Omega(\Gamma)$ in the Poincaré metric, then*

$$\frac{e^{-m}e^{\frac{-\pi^2}{2\nu}}}{2}$$

is a lower bound for the injectivity radius of $\partial\widetilde{C(M)}$ in its intrinsic metric, where $m = \cosh^{-1}(e^2) \simeq 2.68854$.

(2) (Canary [8, Theorem 5.1]) *If $\eta > 0$ is a lower bound for the injectivity radius of $\partial\widetilde{C(M)}$ in its intrinsic metric, then*

$$\min\left\{\frac{1}{2}, \frac{\eta}{.153}\right\}$$

is a lower bound for the injectivity radius of $\Omega(\Gamma)$ in the Poincaré metric.

Remark: The Thurston metric is also known as the projective (or grafting) metric, as it arises from regarding Ω as a complex projective surface and giving it the metric Thurston described on such surfaces (see Tanigawa [22, Section 2] or McMullen [19, Section 3] for further details). Kulkarni and Pinkall [18] defined and studied a generalization of this metric in all dimensions and it is also sometimes called the Kulkarni-Pinkall metric.

4. The bending lamination and renormalized volume

The boundary of the convex core of a convex cocompact hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is a hyperbolic surface in its intrinsic metric. It is totally geodesic except along a lamination β_M, called the *bending lamination*. The bending lamination inherits a transverse measure which records the degree to which the surface is bent along the lamination. The length $L(\beta_M)$ of the bending lamination then records the total amount of bending of the convex core (see Epstein-Marden [10, Section II.1.11] for details on the bending lamination).

If N_r is the closed r-neighborhood of $C(M)$ for all $r > 0$, then one can easily check that $\{\tilde{S}_r = \partial \tilde{N}_r\}_{r>0}$ is a family of Epstein surfaces for the Thurston metric on $\Omega(\Gamma)$ (see Bridgeman-Canary [6, Lemma 3.5] for example). Using this observation, one may establish the following equality:

Lemma 4.1. (Schlenker [20, Lemma 4.1]) *If M is a convex cocompact hyperbolic 3-manifold, $\partial C(M)$ is non-empty and β_M is the bending lamination, then*

$$W(\tau) = V_C(M) - \frac{1}{4} L(\beta_M).$$

where τ is the Thurston metric on $\partial_c M$.

Furthermore, we have the following bounds on the length of the bending lamination of the convex core in terms of the injectivity radius of the Poincaré metric on the domain of discontinuity.

Theorem 4.2. (Bridgeman-Canary [5, Theorem 1′, Theorem 2′]) *If $M = \mathbb{H}^3/\Gamma$ is a convex cocompact hyperbolic 3-manifold and $\nu > 0$ is the injectivity radius of the Poincare metric on $\Omega(\Gamma)$, then*

$$L(\beta_M) \le |\chi(\partial M)| \left(\frac{807}{\nu} + 771 \right).$$

Furthermore, if $\nu \le 1/2$, then

$$L(\beta_M) \ge \frac{37}{\nu} - 36.$$

We also have a bounds on $L(\beta_M)$ in terms of the injectivity radius of $\partial \widetilde{C(M)}$ in its intrinsic metric.

Theorem 4.3. (Bridgeman-Canary [5, Theorem 1, Theorem 2]) *If $M = \mathbb{H}^3/\Gamma$ is a convex cocompact hyperbolic 3-manifold and $\eta > 0$ is the injectivity radius of the intrinsic metric on $\partial \widetilde{C(M)}$, then*

$$L(\beta_M) \le |\chi(\partial M)| \left(164 \log \left(\frac{1}{\min\{1, \eta\}} \right) + 218 \right).$$

Furthermore, if $\eta \le \sinh^{-1}(1)$, then

$$L(\beta_M) \ge 4\pi \log \left(\frac{2 \sinh^{-1}(1)}{\eta} \right).$$

If $C(M)$ has incompressible boundary, we obtain the following bound which improves on the bound obtained in Theorem 3 in [5]. (A similar argument is given in the proof of Theorem 6.7 in Anderson [1].)

Theorem 4.4. *If M is a convex cocompact hyperbolic 3-manifold, $\partial C(M)$ is incompressible, and β_M is the bending lamination, then*

$$L(\beta_M) \le 6\pi |\chi(\partial C(M))|.$$

Proof. Recall that, by Theorem 3.2, $\tau(z) \le 2\rho(z)$ for all $z \in \Omega(\Gamma)$, so

$$Area_\tau(\partial M) = \int_{\partial M} \tau^2 \le 4 \int_{\partial M} \rho^2 = 4 Area_\rho(\partial M) = 4(2\pi |\chi(\partial M)|).$$

A simple calculation shows that $Area_\tau(\partial M) = 2\pi|\chi(\partial M)| + L(\beta_M)$ (see Schlenker [20, Section 4.2]). Therefore,

$$2\pi|\chi(\partial M)| + L(\beta_M) \le 4(2\pi|\chi(\partial M)|),$$

which implies that

$$L(\beta_M) \le 6\pi|\chi(M)|.$$

□

Remark: One may use the proof of Theorem 4.4 and the estimate from Theorem 3.1 to bound the length of the bending locus in the compressible case. However, in this situation the argument gives that

$$L(\beta_M) \le \left(16\pi\left(k + \frac{\pi^2}{2\nu}\right)^2 - 2\pi\right)|\chi(\partial M)|$$

which is significantly worse than the bound obtained in Theorem 4.2.

5. Proofs of main results

We have now assembled the necessary ingredients to prove our main results. We begin by proving Theorem 1.1 which gives the bounds in the simplest case where the convex core has incompressible boundary.

Proof of Theorem 1.1: Suppose that M is a convex cocompact hyperbolic 3-manifold such that $\partial C(M)$ is incompressible. Let ρ be the Poincaré metric on $\partial_c M$ and let τ be the Thurston metric on $\partial_c M$. Theorem 3.2 implies that

$$\frac{\tau}{2} \le \rho \le \tau,$$

so the monotonicity lemma, Lemma 2.2, implies that

$$W(\tau) + \pi\log(2)\chi(\partial M) = W\left(\frac{\tau}{2}\right) \le W(\rho) \le W(\tau).$$

Theorem 4.4 implies that

$$L(\beta_M) \le 6\pi|\chi(\partial C(M))|$$

and Lemma 4.1 implies that

$$W(\tau) = V_C(M) - \frac{1}{4}L(\beta_M) \le V_C(M).$$

It follows that

$$V_C(M) - \left(\pi\log(2) + \frac{6\pi}{4}\right)|\chi(\partial C(M))| \le W(\rho) \le V_C(M).$$

Since $V_R(M) = W(\rho)$ and $\pi\log(2) + \frac{6\pi}{4} \le 6.89$, it follows that

$$V_C(M) - 6.89|\chi(\partial C(M))| \le V_R(M) \le V_C(M)$$

as claimed.

If $\partial C(M)$ is totally geodesic, then every component of $\Omega(\Gamma)$ is a round disk, so $\rho = \tau$, $L_\beta(M) = 0$ and $W(\tau) = V_C(M) = V_R(M) = W(\rho)$. On the other hand, if $V_C(M) = V_R(M)$, then $L(\beta_M) = 0$, so $\partial C(M)$ is totally geodesic. Therefore, $V_R(M) = V_C(M)$ if and only if $\partial C(M)$ is totally geodesic. □

Proposition 5.1. *There exists a sequence $\{M_n\}$ of quasifuchsian hyperbolic 3-manifolds such that*

$$\lim_{n\to\infty} V_C(M_n) - V_R(M_n) = +\infty$$

and there exists $D > 0$ such that

$$\frac{V_C(M_n) - V_R(M_n)}{|\chi(\partial M_n)|} \geq D$$

for all n.

Proof. Let M be a quasifuchsian hyperbolic 3-manifold such that $L(\beta_M) \neq 0$. Let $\{\pi_n : M_n \to M\}$ be a sequence of finite covers of M whose degrees $\{d_n\}$ tend to infinity. The convex core $C(M_n) = \pi_n^{-1}(C(M))$ and similarly the bending lamination β_{M_n} is the pre-image of β_M. It follows that $|\chi(\partial C(M_n)| = d_n|\chi(\partial C(M))|$ and $L(\beta_{M_n}) = d_n L(\beta_M)$ for all n. Since, as we saw in the above proof,

$$V_R(M_n) = W(\rho_n) \leq W(\tau_n) = V_C(M_n) - \frac{1}{4}L(\beta_{M_n})$$

where ρ_n is the Poincaré metric on $\partial_c M_n$ and τ_n is the Thurston metric on $\partial_c M_n$, it follows that

$$V_C(M_n) - V_R(M_n) \geq \frac{1}{4}L(\beta_{M_n}) = \frac{d_n}{4}L(\beta_M) = \frac{L(\beta_M)}{4|\chi(\partial C(M))|}|\chi(\partial C(M_n)|.$$

The result follows if we choose $D = \frac{L(\beta(M)}{4|\chi(\partial C(M)|}$. □

We now prove Theorem 1.3 which bounds $V_R(M)$ in terms of $\chi(\partial C(M))$ and the injectivity radius of the domain of discontinuity in its Poincaré metric.

Proof of Theorem 1.3: Suppose that $M = \mathbb{H}^3/\Gamma$ is a convex cocompact hyperbolic 3-manifold such that $\partial C(M)$ is compressible. Let $\nu > 0$ be the injectivity radius of $\Omega(\Gamma)$ in its Poincaré metric. Let ρ be the Poincaré metric on $\partial_c M$ and let τ be the Thurston metric on $\partial_c M$.

Theorem 3.1 implies that

$$\frac{\tau}{2\sqrt{2}(k + \frac{\pi^2}{2\nu})} \leq \rho \leq \tau$$

so the monotonicity lemma, Lemma 2.2, implies that

$$W(\tau) + \pi \log\left(2\sqrt{2}\left(k + \frac{\pi^2}{2\nu}\right)\right)\chi(\partial M) = W\left(\frac{\tau}{2\sqrt{2}(k + \frac{\pi^2}{2\nu})}\right) \leq W(\rho) \leq W(\tau).$$

Lemma 4.1 implies that

$$W(\tau) = V_C(M) - \frac{1}{4}L(\beta_M) < V_C(M)$$

while Theorem 4.2 implies that

$$L(\beta_M) \leq |\chi(\partial M)|\left(\frac{807}{\nu} + 771\right)$$

and, if $\nu \leq 1/2$, then

$$L(\beta_M) \geq \frac{37}{\nu} - 36.$$

Since $W(\rho) = V_R(M)$, we may combine the above estimates to see that

$$V_C(M) - K_1(\nu)|\chi(\partial M)| \leq V_R(M) < V_C(M)$$

where

$$K_1(\nu) = \pi \log\left(2\sqrt{2}\left(k + \frac{\pi^2}{2\nu}\right)\right) + \frac{1}{4}\left(\frac{807}{\nu} + 771\right)$$

As $\log(a+b) \le \log(a) + b/a$ if $a > 1$ and $b > 0$ we have

$$\begin{aligned} K_1(\nu) &\le \pi\left(\log(2k\sqrt{2}) + \frac{\pi^2}{2k\nu}\right) + \frac{202}{\nu} + 193 \\ &\le \left(202 + \frac{\pi^3}{2k}\right)\left(\frac{1}{\nu}\right) + \left(\pi\log(2k\sqrt{2}) + 193\right) \\ &\le \frac{205}{\nu} + 202. \end{aligned}$$

Moreover, if $\nu \le 1/2$, then

$$V_R(M) \le V_C(M) - \frac{1}{4}\left(\frac{37}{\nu} - 36\right) \le V_C(M) - \left(\frac{9}{\nu} - 9\right).$$

□

Remark: One may apply the technique of proof of Proposition 5.1 to produce a sequence $\{M_n = \mathbb{H}^3/\Gamma_n\}$ of Schottky hyperbolic 3-manifolds such that the injectivity radius $\nu(M_n)$ of $\Omega(\Gamma_n)$ is constant, yet

$$V_C(M_n) - V_R(M_n) \to \infty \quad \text{and} \quad \liminf \frac{V_C(M_n) - V_R(M_n)}{|\chi(\partial C(M_n))|} > 0.$$

Such a sequence demonstrates the dependence on $|\chi(\partial C(M))|$ is necessary in Theorem 1.3. We recall that a convex cocompact hyperbolic 3-manifold M is called Schottky if $\pi_1(M)$ is a free group.

One may derive a version of Theorem 1.2 directly from Theorem 1.3 and Proposition 3.3. However, we will obtain better estimates by giving a more direct proof.

Proof of Theorem 1.2: Suppose that $M = \mathbb{H}^3/\Gamma$ is a convex cocompact hyperbolic 3-manifold such that $\partial C(M)$ is compressible. Let $\eta > 0$ be the injectivity radius of $\widetilde{\partial C(M)}$ in its intrinsic metric. Let ρ be the Poincaré metric on $\partial_c M$ and let τ be the Thurston metric on $\partial_c M$. We will consider the two bounds separately. As before we have

$$V_R(M) \le W(\tau) = V_C(M) - \frac{1}{4}L(\beta_M) < V_C(M).$$

If $\eta < \sinh^{-1}(1)$, then Theorem 4.3 implies that

$$\begin{aligned} V_R(M) &\le V_C(M) - \pi\log\left(\frac{2\sinh^{-1}(1)}{\eta}\right) \\ &= V_C(M) - \pi\log\left(2\sinh^{-1}(1)\right) - \pi\log\left(\frac{1}{\eta}\right) \\ &\le V_C(M) - 1.79 - \pi\log\left(\frac{1}{\eta}\right) \end{aligned}$$

Proposition 3.3 implies that $\min\{1/2, \eta/.153\}$ is a lower bound for the injectivity radius of $\Omega(\Gamma)$ in the Poincaré metric. Theorem 3.1 then implies that

$$\frac{\tau}{2\sqrt{2}\left(k + \frac{\pi^2}{2\min\{1/2,\eta/.153\})}\right)} \le \rho \le \tau,$$

so Lemma 2.2 implies that

$$\begin{aligned} V_R(M) = V(\rho) &\ge W\left(\frac{\tau}{2\sqrt{2}\left(k + \frac{\pi^2}{\min\{1,\eta/.076\}}\right)}\right) \\ &= W(\tau) - \pi \log\left(2\sqrt{2}\left(k + \frac{\pi^2}{\min\{1,\eta/.076\}}\right)\right)|\chi(\partial M)|. \end{aligned}$$

Theorem 4.3 gives that

$$L(\beta_M) \le |\chi(\partial M)|\left(164\log\left(\frac{1}{\min\{1,\eta\}}\right) + 218\right),$$

so

$$V_R(M) \ge V_C(M) - K_1'(\eta)|\chi(\partial M)|$$

where

$$\begin{aligned} K_1'(\eta) &= \pi\log\left(2\sqrt{2}\left(k + \frac{\pi^2}{\min\{1,\eta/.076\}}\right)\right) + \frac{1}{4}\left(164\log\left(\frac{1}{\min\{1,\eta\}}\right) + 218\right) \\ &\le \pi\log\left(2\sqrt{2}\left(k + \frac{\pi^2}{\min\{1,\eta\}}\right)\right) + \frac{1}{4}\left(164\log\left(\frac{1}{\min\{1,\eta\}}\right) + 218\right) \\ &\le \pi\log\left(\frac{1}{\min\{1,\eta\}}\right) + \pi\log\left(2\sqrt{2}\left(k\min\{1,\eta\} + \pi^2\right)\right) + \\ &\quad \frac{1}{4}\left(164\log\left(\frac{1}{\min\{1,\eta\}}\right) + 218\right) \\ &\le \pi\left(\log\left(2\sqrt{2}(k+\pi^2)\right)\right) + \frac{218}{4} + \left(\pi + \frac{164}{4}\right)\log\left(\frac{1}{\min\{1,\eta\}}\right) \\ &\le 45\log\left(\frac{1}{\min\{1,\eta\}}\right) + 67. \end{aligned}$$

□

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