ENTROPY RIGIDITY FOR CUSPED HITCHIN REPRESENTATIONS

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Abstract. We establish an entropy rigidity theorem for Hitchin representations of all geometrically finite Fuchsian groups which generalizes a theorem of Potrie and Sambarino for Hitchin representations of closed surface groups. In the process, we introduce the class of $(1,1,2)$-hypertransverse groups and show for such a group that the Hausdorff dimension of its conical limit set agrees with its (first) simple root entropy, providing a common generalization of results of Bishop and Jones, for Kleinian groups, and Pozzetti, Sambarino and Wienhard, for Anosov groups. We also introduce the theory of transverse representations of projectively visible groups as a tool for studying discrete subgroups of linear groups which are not necessarily Anosov or relatively Anosov.

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1. Introduction

Hitchin [30] discovered a component of the space of (conjugacy classes in $\mathsf{PGL}(d,\mathbb{R})$ of) representations of a closed surface group $\pi_1(S)$ into $\mathsf{PSL}(d,\mathbb{R})$ which is topologically a cell. Labourie [36] showed that the representations in this component, now known as Hitchin representations, are discrete and faithful and even quasi-isometric embeddings, so share many properties with classical Fuchsian surface groups. Fock and Goncharov [25] showed that these representations are exactly the representations which admit equivariant continuous positive maps of the Gromov boundary of $\pi_1(S)$, which one may identify with the limit set of any Fuchsian uniformization,

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into the space $\mathcal{F}$ of complete d-dimensional flags. One may then naturally define a representation ρ of a Fuchsian group $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ into $\mathsf{PSL}(d,\mathbb{R})$ to be Hitchin if there is a continuous positive ρ-equivariant map of the limit set of Γ into $\mathcal{F}$. Hitchin representations of geometrically finite Fuchsian groups share many of the same properties as classical Hitchin representations (see Canary-Zhang-Zimmer [19]).

The main result of this paper is an entropy rigidity theorem for Hitchin representations of geometrically finite Fuchsian groups which generalizes a result of Potrie and Sambarino [42] from the classical setting. Let $(\mathfrak{a}^*)^+$ denote the set of linear functionals which are strictly positive on the interior of the standard positive Weyl chamber for $\mathsf{PGL}(d,\mathbb{R})$. Each $\phi \in (\mathfrak{a}^*)^+$ can be written as a non-negative linear combination of the standard simple roots $\{\alpha_i\}_{i=1}^{d-1}$ which define the standard positive Weyl chamber. If $\phi \in (\mathfrak{a}^*)^+$, we obtain a length function on $\rho(\Gamma)$ given by $\ell^\phi(\rho(\gamma)) = \phi(\nu(\rho(\gamma)))$ where $\nu(\rho(\gamma))$ is the Jordan projection of $\rho(\gamma)$ (i.e. the logarithms of the moduli of generalized eigenvalues of $\rho(\gamma)$ in descending order). The ϕ-entropy $h^\phi(\rho)$ of a Hitchin representation is then the exponential growth rate of the number of conjugacy classes of hyperbolic elements whose images have ϕ-length at most T.

Given a subgroup $\Gamma \subset \mathsf{PSL}(d,\mathbb{R})$, the *Zariski closure* of Γ in $\mathsf{PSL}(d,\mathbb{R})$ is the intersection of its Zariski closure in $\mathsf{PGL}(d,\mathbb{R})$ with $\mathsf{PSL}(d,\mathbb{R})$. We recall that Sambarino [47] showed that the Zariski closure in $\mathsf{PSL}(d,\mathbb{R})$ of the image of a Hitchin representation is either all of $\mathsf{PSL}(d,\mathbb{R})$, an irreducible image of $\mathsf{PSL}(2,\mathbb{R})$ within $\mathsf{PSL}(d,\mathbb{R})$, or conjugate to either $\mathsf{PSO}(d,d-1) \subset \mathsf{PSL}(2d-1,\mathbb{R})$, $\mathsf{PSp}(2d,\mathbb{R}) \subset \mathsf{PSL}(2d,\mathbb{R})$, or the copy of G_2 in $\mathsf{PSL}(7,\mathbb{R})$.

Theorem 1.1 (see Theorem 11.8)**.** *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is geometrically finite, $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is Hitchin and $\phi = \sum c_j\alpha_j \in (\mathfrak{a}^*)^+$, then*

$$h^\phi(\rho) \le \frac{1}{c_1 + \cdots + c_{d-1}}.$$

Moreover, equality occurs if and only if Γ is a lattice and either

1. *$\phi = c_k\alpha_k$ for some k.*
2. *$\rho(\Gamma)$ lies in an irreducible image of $\mathsf{PSL}(2,\mathbb{R})$.*
3. *$d = 2n-1$, the Zariski closure of $\rho(\Gamma)$ is conjugate to $\mathsf{PSO}(n,n-1)$ and $\phi = c_k\alpha_k + c_{d-k}\alpha_{d-k}$ for some k.*
4. *$d = 2n$, the Zariski closure of $\rho(\Gamma)$ is conjugate to $\mathsf{PSp}(2n,\mathbb{R})$ and $\phi = c_k\alpha_k + c_{d-k}\alpha_{d-k}$ for some k.*
5. *$d = 7$, the Zariski closure of $\rho(\Gamma)$ is conjugate to G_2 and $\phi = c_1\alpha_1 + c_3\alpha_3 + c_4\alpha_4 + c_6\alpha_6$ or $\phi = c_2\alpha_2 + c_5\alpha_5$.*

In the process of establishing our main result, we introduce the class of transverse subgroups of $\mathsf{PGL}(d,\mathbb{R})$ which includes all Anosov subgroups, all images of Hitchin representations, all images of cusped Anosov representations of geometrically finite Fuchsian groups in the sense of [19], all images of relatively dominated representations in the sense of Zhu [56], all images of relatively Anosov representations in the sense of Kapovich-Leeb [32], all subgroups of these groups, all discrete subgroups of $\mathsf{PO}(d-1,1)$, and all discrete groups of projective automorphisms that preserve strictly convex domains with C^1 boundary in real projective space. We note that the definition of transverse groups and many of general results we establish about them do not assume finite generation.

We obtain upper bounds on the Hausdorff dimensions of conical limit sets of P_k-transverse groups, generalizing results of Glorieux-Montclair-Tholozan [27] and Pozzetti-Sambarino-Wienhard [43] from the Anosov setting.

We further introduce the class of $(1,1,q)$-hypertransverse subgroups which include images of the $(1,1,q)$-hyperconvex Anosov representations introduced by Pozzetti-Sambarino-Wienhard [43], their subgroups, Hitchin representations of Fuchsian groups, and all discrete subgroups of $\mathsf{PO}(d-1,1)$. We show that for such subgroups (with $\sigma_2(\gamma)=\sigma_q(\gamma)$ for all $\gamma\in\Gamma$), the Hausdorff dimension of the conical limit set agrees with the first simple root critical exponent. This result is a common generalization of results of Pozzetti-Sambarino-Wienhard [43], for Anosov groups, and Bishop-Jones [6], for Kleinian groups, and the proof makes use of techniques drawn from each source. We observe that the ϕ-entropy and the ϕ-critical exponent of a geometrically finite Hitchin representation agree. We conclude that the simple root entropies of Hitchin representations of geometically finite Fuchsian groups are at most 1, and are exactly 1 only for Hitchin representations of lattices. We combine this with convexity properties of the entropy functional to establish our main theorem.

We now give a few definitions which allow us to give a more detailed discussion of our work. We recall that the standard *Cartan subspace* for $\mathsf{PGL}(d,\mathbb{K})$, where $\mathbb{K}$ is either the real numbers or the complex numbers, is given by the set of real-valued diagonal matrices with trace zero:

$$\mathfrak{a}=\{\mathrm{diag}(A_1,\dots,A_d)\in\mathfrak{sl}(d,\mathbb{R})\mid A_1+\cdots+A_d=0\}.$$

The space $\mathfrak{a}^*$ of linear functionals on $\mathfrak{a}$ is generated by the simple roots $\{\alpha_i\}_{i=1}^{d-1}$ where $\alpha_i(A)=A_i-A_{i-1}$ and the standard *positive Weyl chamber* is the subset where all the simple roots are non-negative:

$$\mathfrak{a}^+=\{A\in\mathfrak{a}\mid A_1\geq A_2\geq\cdots\geq A_d\}.$$

We will be especially interested in the set $(\mathfrak{a}^*)^+$ of linear functionals which are strictly positive on the interior of the positive Weyl chamber, i.e.

$$(\mathfrak{a}^*)^+=\left\{\phi\in\mathfrak{a}^*-\{0\}\ :\ \phi=\sum_{j=1}^{d-1}c_j\alpha_j\ \text{ such that } c_j\geq 0\ \forall j\right\}.$$

Given $g\in\mathsf{PGL}(d,\mathbb{K})$, let $\bar g\in\mathsf{GL}(d,\mathbb{K})$ be a representative of g whose determinant has modulus 1. Then let

$$\lambda_1(g)\geq\cdots\geq\lambda_d(g)>0$$

denote the modulus of the generalized eigenvalues of $\bar g$, and let

$$\sigma_1(g)\geq\cdots\geq\sigma_d(g)>0$$

denote the singular values of $\bar g$. The *Jordan projection* and *Cartan projection*

$$\nu,\kappa:\mathsf{PGL}(d,\mathbb{K})\to\mathfrak{a}^+$$

are respectively given by

$$\nu(g)=\mathrm{diag}(\log\lambda_1(g),\dots,\log\lambda_d(g))\ \text{ and }\ \kappa(g)=\mathrm{diag}(\log\sigma_1(g),\dots,\log\sigma_d(g)).$$

We recall that if $g\in\mathsf{PGL}(d,\mathbb{K})$, then $g=\ell am$, where $\ell,m\in\mathsf{PO}(d,\mathbb{K})$ and $a\in\exp(\mathfrak{a}^+)$. If $\alpha_k(g)>0$, then $U_k(g)=\ell(\langle e_1,\dots,e_k\rangle)$ is well-defined, and is the image of the k-plane which is "stretched the most" by A.

If $\phi\in(\mathfrak{a}^*)^+$ and $\Gamma\subset\mathsf{PGL}(d,\mathbb{K})$ is discrete, we define its *ϕ-Poincaré series*

$$Q^\phi_\Gamma(s)=\sum_{\gamma\in\Gamma}e^{-s\phi(\kappa(\gamma))}$$

and its *ϕ-critical exponent*

$$\delta^\phi(\Gamma)=\inf\{s\ :\ Q^\phi_\Gamma(s)<+\infty\}.$$

If $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is a representation with discrete image and finite kernel, we define its ϕ-*critical exponent* by $\delta^\phi(\rho) = \delta^\phi(\rho(\Gamma))$. For all $\phi \in (\mathfrak{a}^*)^+$, the ϕ-critical exponent for any Hitchin representation is finite, see Corollary 1.7.

If $\Gamma \subset \mathsf{PGL}(d, \mathbb{K})$ is a subgroup, we say that it is P_k-*divergent* if $\alpha_k(\kappa(\gamma_n)) \to \infty$ for any sequence $\{\gamma_n\}$ in Γ of pairwise distinct elements. Notice that since $\alpha_k(\kappa(\gamma)) = \alpha_{d-k}(\kappa(\gamma^{-1}))$, Γ is P_k-divergent if and only if it is P_{d-k}-divergent. (Guichard-Wienhard [29] refer to P_k-divergent groups as α_k-divergent, while Kapovich-Leeb-Porti [33] call them τ_{mod}-regular.)

Suppose that $\theta = \{k_1 < k_2 < \cdots < k_r\}$ is a symmetric subset of $\Delta = \{1, \ldots, d-1\}$ (i.e. $k \in \theta$ if and only if $d - k \in \theta$). The set of θ-flags is the set of partial flags with subspaces in all dimensions contained in θ, i.e.

$$\mathcal{F}_\theta(\mathbb{K}^d) = \{F^{k_1} \subset \cdots \subset F^{k_r} \subset \mathbb{K}^d \ : \ \dim(F^{k_i}) = k_i\}.$$

In particular, $\mathcal{F} = \mathcal{F}_\Delta(\mathbb{K}^d)$. When the context is clear, we write $\mathcal{F}_\theta = \mathcal{F}_\theta(\mathbb{K}^d)$.

We say that a subset X of $\mathcal{F}_\theta$ is *transverse* if whenever $k \in \theta$ and $F, G \in X$ are distinct, then F^k and G^{d-k} are transverse. If Γ is P_k-divergent for all $k \in \theta$, then

$$U_\theta(\gamma) = (U_k(\gamma))_{k\in\theta}$$

is well-defined for all but finitely many γ in Γ and we can define the θ-limit set $\Lambda_\theta(\Gamma)$ to be the set of accumulations points of $\{U_\theta(\gamma)\}_{\gamma\in\Gamma}$. Equivalently,

$$\Lambda_\theta(\Gamma) = \{\lim U_\theta(\gamma_n) \ : \ \{\gamma_n\} \subset \Gamma, \ \gamma_n \to \infty\}.$$

We then say that Γ is P_θ-*transverse* if it is P_k-divergent for all $k \in \theta$ and $\Lambda_\theta(\Gamma)$ is a transverse subset of $\mathcal{F}_\theta$. (In Kapovich-Leeb-Porti [33], P_k-transverse subgroups are called τ_{mod}-regular and τ_{mod}-antipodal.)

Motivated by results of Danciger-Guéritaud-Kassel [23] and Zimmer [57], to study of transverse groups we introduce and develop a theory of projectively visible groups. We say that a discrete subgroup Γ_0 of $\mathsf{PGL}(d, \mathbb{R})$ is *projectively visible* if it preserves a properly convex domain Ω in $\mathbb{P}(\mathbb{R}^d)$, every point in its full orbital limit set

$$\Lambda_\Omega(\Gamma_0) = \{z \in \partial\Omega \mid z = \lim \gamma_n(x) \ \text{ for some } x \in \Omega \text{ and some } \{\gamma_n\} \subset \Gamma_0\}$$

has a unique supporting hyperplane to Ω, and any two points in $\Lambda_\Omega(\Gamma_0)$ are joined by a projective line segment in Ω. As a key tool in our work, we show that every P_θ-transverse subgroup is the image of a projectively visible group Γ_0.

Theorem 1.2 (see Theorem 4.2)**.** *If $\Gamma \subset \mathsf{PGL}(d, \mathbb{K})$ is P_θ-transverse, then there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ (for some $d_0 \in \mathbb{N}$), a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$, a faithful representation $\rho : \Gamma_0 \to \mathsf{PGL}(d, \mathbb{K})$ and a ρ-equivariant continuous map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$ so that $\rho(\Gamma_0) = \Gamma$ and $\xi(\Lambda_\Omega(\Gamma_0)) = \Lambda_\theta(\Gamma)$.*

The domain Ω comes equipped with a natural, projectively invariant Finsler metric d_Ω, called the Hilbert metric. In general, $(\Omega, \mathrm{d}_\Omega)$ will not be Gromov hyperbolic, but it has enough hyperbolicity to play the role that the Cayley graph does when studying Anosov groups. Similarly, the restriction of the Hilbert geodesic flow to the convex hull of $\Lambda_\Omega(\Gamma_0)$ plays the role of the Gromov geodesic flow of a hyperbolic group.

We will observe, in Lemma 3.3, that if Γ is P_θ-transverse, then it acts on its θ-limit set as a convergence group, i.e. if $\{\gamma_n\}$ is a sequence of distinct elements in Γ, then there are points $x, y \in \Lambda_\theta(\Gamma)$ and a subsequence, still called $\{\gamma_n\}$, so that $\gamma_n(z) \to x$ for all $z \in \Lambda_\theta(\Gamma) \setminus \{y\}$. We recall that a point $x \in \Lambda_\theta(\Gamma)$ is a *conical limit point* if there exists $a, b \in \Lambda_\theta(\Gamma)$ and a sequence $\{\gamma_n\}$ in Γ so that $\gamma_n(x) \to a$ and $\gamma_n(y) \to b$ for all $y \in \Lambda_\theta(\Gamma) \setminus \{x\}$. The set of conical limit points for the action of Γ on $\Lambda_\theta(\Gamma)$ is called the θ-*conical limit set* and is denoted $\Lambda_{\theta,c}(\Gamma)$.

In many situations, the complement of the conical limit set is countable and consists of fixed points of "weakly unipotent" elements. Most classically, Beardon and Maskit [2] proved that this characterized geometrically finite Kleinian groups, see also Bishop [5]. This property also holds for P_θ-Anosov images of geometrically finite Fuchsian groups, and images of relatively dominated and relatively Anosov representations. Our first main result is an upper bound for the Hausdorff dimension of the conical limit set of a transverse group in terms of the simple root critical exponent. If $k \in \theta$ and Γ is P_θ-transverse, we define $\Lambda_{k,c}(\Gamma) = \pi_k(\Lambda_{\theta,c}(\Gamma))$, where $\pi_k : \mathcal{F}_\theta \to \mathrm{Gr}_k(\mathbb{R}^d)$ is the projection map. Our result generalizes work of Glorieux-Montclair-Tholozan [27] and Pozzetti-Sambarino-Wienhard [43] in the Anosov setting.

Theorem 1.3 (see Corollary 5.2)**.** *If $\Gamma \subset \mathsf{PGL}(d,\mathbb{K})$ is $P_{k,d-k}$-transverse, then*

$$\dim_H\left(\Lambda_{k,d-k,c}(\Gamma)\right) \le \delta^{\alpha_k}(\Gamma).$$

In particular, $\dim_H\left(\Lambda_{k,c}(\Gamma)\right) \le \delta^{\alpha_k}(\Gamma)$.

As a consequence we obtain the following generalization of results of Burger [16], Glorieux-Montclair-Tholozan [27] and Kim-Minsky-Oh [34] for pairs of convex cocompact representations.

Theorem 1.4 (see Theorem 5.3)**.** *Let $\rho_1 : \Gamma \to \mathsf{SO}(d_1-1,1)$ and $\rho_2 : \Gamma \to \mathsf{SO}(d_2-1,1)$ be geometrically finite representations so that $\rho_1(\alpha)$ is parabolic if and only if $\rho_2(\alpha)$ is parabolic. If we regard $\rho = \rho_1 \oplus \rho_2$ as a representation into $\mathsf{PSL}(d_1+d_2,\mathbb{R})$, then*

$$\dim_H\left(\Lambda_2(\rho(\Gamma))\right) = \max\left\{\dim_H\left(\Lambda_1(\rho_1(\Gamma))\right), \dim_H\left(\Lambda_1(\rho_2(\Gamma))\right)\right\}.$$

Notice that our assumptions imply that there is a Hölder homeomorphism $\xi : \Lambda_1(\rho_1(\Gamma)) \to \Lambda_1(\rho_2(\Gamma))$ which is typically not a diffeomorphism and that $\Lambda_2(\rho(\Gamma))$ can be smoothly identified with the graph of ξ. So, this is another instance, surprisingly common in Higher Teichmüller theory, where a Hölder homeomorphism fails to change Hausdorff dimension.

Following Pozzetti-Sambarino-Wienhard [43] we say a group Γ is $(1,1,q)$-*hypertransverse* if it is P_θ-transverse for some θ containing 1 and q, and

$$F^1 + G^1 + H^{d-q}$$

is a direct sum for all pairwise distinct $F, G, H \in \Lambda_\theta(\Gamma)$.

Examples of $(1,1,2)$-hypertransverse groups include all (images of) exterior powers of Hitchin representations and examples of $(1,1,d-1)$-hypertransverse groups include all discrete subgroups of $\mathsf{PO}(d-1,1)$. Further, by definition, images of $(1,1,q)$-hyperconvex representations, in the sense of Pozzetti-Sambarino-Wienhard [43], and their subgroups are also $(1,1,q)$-hypertransverse groups, so the $(1,1,q)$-hyperconvex representations into $\mathsf{SU}(n,1)$, $\mathsf{Sp}(n,1)$ and $\mathsf{SO}(p,q)$ constructed in [43] can be used to construct $(1,1,q)$-hypertransverse groups.

The following result generalizes results of both Pozzetti-Sambarino-Wienhard [43] and Bishop-Jones [6] and uses ideas from both of their proofs.

Theorem 1.5 (see Theorem 8.1)**.** *Suppose that $\Gamma \subset \mathsf{PGL}(d,\mathbb{K})$ is $(1,1,q)$-hypertransverse and*

$$\sigma_2(\gamma) = \sigma_q(\gamma)$$

for all $\gamma \in \Gamma$. Then

$$\dim_H(\Lambda_{1,c}(\Gamma)) = \delta^{\alpha_1}(\Gamma).$$

In order to apply this theorem, we first observe that the image of a Hitchin representation has limit set of Hausdorff dimension at most 1.

Proposition 1.6 (see Proposition 11.1)**.** *If Γ is a Fuchsian group and $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation, then*

$$\dim_H(\Lambda_\Delta(\rho(\Gamma))) \le 1.$$

After we observe that exterior powers of Hitchin representations are $(1,1,2)$-hypertransverse, Theorem 1.5 and Propositions 1.6 have the following consequence for the simple root entropies of Hitchin representations.

Corollary 1.7 (see Corollary 11.3)**.** *If Γ is a Fuchsian group and $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation, then*

$$\delta^{\alpha_k}(\rho) \le 1$$

for all $k \in \Delta$, and equality holds if Γ is a lattice. Furthermore, if $\phi = \sum_{k\in\Delta} c_k\alpha_k \in (\mathfrak{a}^*)^+$ is non-zero, then

$$\delta^\phi(\rho) \le \frac{1}{c_1 + \cdots + c_{d-1}}.$$

In the case when Γ is a uniform lattice, Corollary 1.7 was previously established by Potrie and Sambarino [42] by quite different methods, and in the convex co-compact case can also be deduced from the work of Pozzetti, Sambarino, and Wienhard [43]. Corollary 1.7 plays a central role in the construction of the (first) simple root pressure metric for Hitchin components of Fuchsian lattices, see Bray-Canary-Kao-Martone [12].

If $\phi \in (\mathfrak{a}^*)^+$, then the ϕ-critical exponent and the ϕ-entropy of a Hitchin representation of a geometrically finite Fuchsian group agree. We define the *ϕ-length* of an element of $g \in \mathsf{PGL}(d,\mathbb{R})$ as

$$\ell^\phi(g) = \phi(\nu(g)).$$

If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is Fuchsian and $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{R})$ is a representation, we define the *ϕ-entropy* as

$$h^\phi(\rho) = \limsup_{T\to\infty} \frac{\log \# R_T^\phi(\rho)}{T} \qquad \text{where} \qquad R_T^\phi(\rho) = \Big\{[\gamma] \in [\Gamma_{hyp}] \ : \ \ell^\phi(\rho(\gamma)) \le T\Big\}$$

where $[\Gamma_{hyp}]$ is the set of conjugacy classes of hyperbolic elements of Γ. When ρ is a Hitchin representation of a geometrically finite Fuchsian group, the $\limsup$ in the definition of $h^\phi(\rho)$ holds as a limit, and is always positive and finite (see [11]).

Proposition 1.8 (see Proposition 9.1)**.** *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is geometrically finite, $\phi \in (\mathfrak{a}^*)^+$ and $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation, then*

$$\delta^\phi(\rho) = h^\phi(\rho).$$

If Γ is geometrically finite, but not a lattice, then Γ is contained in a lattice Γ^D such than any Hitchin representation $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ extends to a Hitchin representation $\rho^D : \Gamma^D \to \mathsf{PSL}(d,\mathbb{R})$ (see Proposition A.1). We may then apply classical arguments, which go back to Furusawa [26] to establish that the ϕ-critical exponent of $\rho(\Gamma)$ is strictly less than the critical exponent of $\rho^D(\Gamma)$. (One may also view the proof as a concrete version of an argument in Dal'bo-Otal-Peigné [22, Thm. A] who make use of Patterson-Sullivan measure instead of working directly with the Poincaré series.)

Proposition 1.9 (see Proposition 11.5)**.** *Suppose that $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation of a Fuchsian group Γ, G is an infinite index, finitely generated subgroup of Γ, and $\phi \in (\mathfrak{a}^*)^+$, then*

$$\delta^\phi(\rho|_G) < \delta^\phi(\rho).$$

As an immediate consequence we obtain:

Corollary 1.10. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is geometrically finite, but not a lattice, and $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation, then*

$$\delta^{\alpha_k}(\rho) < 1$$

for all $k \in \Delta$.

Once we have established Corollaries 1.7 and 1.10 we may follow a similar outline of proof as in [42] to establish our main theorem. The key step is a convexity result for the behavior of entropy over the space of functionals, which generalizes [46, Cor. 4.9], see also [48].

Theorem 1.11 (see Theorem 10.1)**.** *Suppose that Γ is a geometrically finite Fuchsian group and $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation. Then*

$$\mathcal{Q}_\theta(\rho) = \{\phi \in (\mathfrak{a}^*)^+ \mid h^\phi(\rho) = 1\}$$

is a closed subset of a concave, analytic submanifold of $\mathfrak{a}^*$. Moreover, if $\phi_1, \phi_2 \in \mathcal{Q}(\rho)$, then the line segment in $\mathfrak{a}^*$ between ϕ_1 and ϕ_2 lies in $\mathcal{Q}(\rho)$ if and only if

$$\ell^{\phi_1}(\rho(\gamma)) = \ell^{\phi_2}(\rho(\gamma))$$

for all $\gamma \in \Gamma$.

As in [42], Theorem 1.1 also implies a rigidity result for the symmetric space critical exponent. Given a discrete subgroup $\Gamma \subset \mathsf{PSL}(d,\mathbb{R})$, the *symmetric space critical exponent*, denoted $\delta_X(\Gamma) \in [0,\infty)$, is the critical exponent of the series

$$Q^X_{\Gamma,x_0}(s) := \sum_{\gamma \in \Gamma} e^{-s\mathrm{d}_X(\gamma(x_0),x_0)}$$

(which is independent of the choice of $x_0 \in X$) where d_X is the symmetric space distance on $X = \mathsf{PSL}(d,\mathbb{R})/\mathsf{PSO}(d)$ scaled so that the embedding $\mathbb{H}^2 \hookrightarrow X$ induced by some (hence any) irreducible representation $\mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$ is isometric.

Corollary 1.12 (see Corollary 12.1)**.** *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is geometrically finite and $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is Hitchin, then*

$$\delta_X(\rho) \leq 1.$$

Moreover, $\delta_X(\rho) = 1$ if and only if Γ is a lattice and $\rho(\Gamma)$ lies in the image of an irreducible representation $\mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$.

2. Background

2.1. **Linear Algebra.** We recall some basic notation and terminology from the Lie theory of $\mathsf{PGL}(d,\mathbb{K})$, where $\mathbb{K}$ is either the real numbers or the complex numbers. We begin by discussing a subspace of the Cartan subspace naturally associated to a symmetric subset θ of $\Delta = \{1,\ldots,d-1\}$. Specifically, let

$$\mathfrak{a}_\theta = \{A \in \mathfrak{a} \mid \alpha_k(A) = 0 \;\text{ if }\; k \notin \theta\}$$

and let $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ be the projection map such that $\omega_k \circ p_\theta = \omega_k$ for all $k \in \theta$ where $\omega_k \in \mathfrak{a}^*$ is the k^{th} *fundamental weight* given by

$$\omega_k(A) = A_1 + \cdots + A_k.$$

Then $p_\theta^*: \mathfrak{a}_\theta^* \to \mathfrak{a}^*$ identifies $\mathfrak{a}_\theta^*$ as the subspace of $\mathfrak{a}^*$ spanned by $\{\omega_k\}_{k\in\theta}$. In particular, every $A \in \mathfrak{a}_\theta$ is determined by the tuple $(\omega_k(A))_{k\in\theta}$. We will be interested in the set $(\mathfrak{a}_\theta^*)^+$ of vectors in $\mathfrak{a}_\theta^*$ that are strictly positive on the interior of the θ-positive Weyl chamber $p_\theta(\mathfrak{a}^+)$. Explicitly,

$$(\mathfrak{a}_\theta^*)^+ = \left\{ \phi = \sum_{k\in\theta} c_k\omega_k \ : \ c_k \geq 0 \ \forall k \in \theta \text{ and } \phi \neq 0 \right\}.$$

We say that $g \in \mathsf{PGL}(d,\mathbb{K})$ is *θ-proximal* if $\alpha_k(\nu(\rho(\gamma))) > 0$ for all $k \in \theta$. We then define the *θ-Benoist limit cone* of a discrete subgroup Γ of $\mathsf{PGL}(d,\mathbb{K})$ to be the closure of the set of rays determined by ϕ-Jordan projections of θ-proximal elements, i.e.

$$\mathcal{B}_\theta(\Gamma) = \overline{\{\mathbb{R}^+ p_\theta(\nu(\gamma)) \ : \ \gamma \in \Gamma \ \text{ is } \theta\text{-proximal}\}} \subset \mathfrak{a}_\theta.$$

We will then be interested in the open set of linear functionals which are strictly positive on the θ-Benoist limit cone (except at 0),

$$\mathcal{B}_\theta^+(\Gamma) = \{\phi \in \mathfrak{a}_\theta^* \ : \ \phi(A) > 0 \ \forall \ A \in \mathcal{B}_\theta(\Gamma) - \{0\}\}.$$

When $\theta = \Delta$, we will use the standard notation $\mathcal{B}(\Gamma) = \mathcal{B}_\Delta(\Gamma)$ and $\mathcal{B}^+(\Gamma) = \mathcal{B}_\Delta^+(\Gamma)$. Also, if $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{R})$ is a representation, we denote $\mathcal{B}_\theta(\rho) = \mathcal{B}_\theta(\rho(\Gamma))$, $\mathcal{B}_\theta^+(\rho) = \mathcal{B}_\theta^+(\rho(\Gamma))$, $\mathcal{B}(\rho) = \mathcal{B}(\rho(\Gamma))$ and $\mathcal{B}^+(\rho) = \mathcal{B}^+(\rho(\Gamma))$.

Recall that the angle $\theta \in [0,\pi/2]$ between two lines $L_1, L_2 \in \mathbb{P}(\mathbb{K}^d)$ is defined by

$$\cos(\theta) = \frac{|\langle v_1, v_2\rangle|}{\|v_1\|\,\|v_2\|}$$

where $v_1 \in L_1, v_2 \in L_2$ are some (any) non-zero vectors. Further, this angle defines a distance, denoted $\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}$, on $\mathbb{P}(\mathbb{K}^d)$ which is induced by a Riemannian metric.

There is a natural smooth embedding of $\mathrm{Gr}_k(\mathbb{K}^d)$ into $\mathbb{P}(\bigwedge^k \mathbb{K}^d)$ which takes a k-subspace with basis $\{b_1,\dots,b_k\}$ to the line spanned by $b_1 \wedge \cdots \wedge b_k$. We then endow $\mathrm{Gr}_k(\mathbb{K}^d)$ with the distance $\mathrm{d}_{\mathrm{Gr}_k(\mathbb{K}^d)}$ obtained by pulling back the angle metric on $\mathbb{P}(\bigwedge^k \mathbb{K}^d)$. We then give $\prod_{k\in\theta}\mathrm{Gr}_k(\mathbb{K}^d)$ the product metric and give $\mathcal{F}_\theta$ the metric, denoted $\mathrm{d}_{\mathcal{F}_\theta}$, it inherits as a subset of $\prod_{k\in\theta}\mathrm{Gr}_k(\mathbb{K}^d)$.

The following lemma is an immediate consequence of the Cartan decomposition.

Lemma 2.1. *Let $\theta \subset \Delta$ be symmetric, $F^+, F^- \in \mathcal{F}_\theta$, and $\{g_n\}$ a sequence in $\mathsf{PGL}(d,\mathbb{K})$. The following are equivalent:*

(1) $\alpha_k(\kappa(g_n)) \to \infty$ for all $k \in \theta$, $U_\theta(g_n) \to F^+$, and $U_\theta(g_n^{-1}) \to F^-$,
(2) $g_n(F) \to F^+$ for all $F \in \mathcal{F}_\theta$ which are transverse to F^-.

2.2. **Cusped Anosov representations of geometrically finite Fuchsian groups.** Cusped Anosov representations of geometrically finite Fuchsian groups were introduced in [19] as natural generalizations of Anosov representations which take parabolic elements to elements whose (generalized) eigenvalues all have modulus 1. These representations are also relatively Anosov in the sense of Kapovich-Leeb [32] and relatively dominated in the sense of Zhu [56]. If Γ is a geometrically finite Fuchsian group with limit set $\Lambda(\Gamma) \subset \partial\mathbb{H}^2$, then a representation $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{K})$ is said to be *P_θ-Anosov* if there exists a continuous ρ-equivariant map $\xi_\rho : \Lambda(\Gamma) \to \mathcal{F}_\theta$ such that

(1) ξ_ρ is *transverse*, i.e. if x, y are distinct points in $\Lambda(\Gamma)$, then $\xi_\rho^k(x) \oplus \xi_\rho^{d-k}(y) = \mathbb{K}^d$ for all $k \in \theta$.
(2) ξ_ρ is *strongly dynamics preserving*, i.e. if $b_0 \in \mathbb{H}^2$ and $\{\gamma_n\}$ is a sequence in Γ such that $\gamma_n(b_0) \to x \in \Lambda(\Gamma)$ and $\gamma_n^{-1}(b_0) \to y \in \Lambda(\Gamma)$, then for all $k \in \theta$ and $V \in \mathrm{Gr}_k(\mathbb{K}^d)$ that is transverse to $\xi_\rho^{d-k}(y)$, we have $\rho(\gamma_n)(V) \to \xi_\rho^k(x)$.

(To be precise, in [19] Anosov representations were defined in terms of the exponential contraction of a linear flow on a vector bundle associated to a representation and this was shown to be equivalent to the definition above.)

If Γ contains a parabolic element, we refer to such representations as cusped Anosov when we want to distinguish them from traditional Anosov representations (which cannot contain unipotent elements in their image). We recall the properties of cusped Anosov representations we will need in our work. For any $x, y \in \mathbb{H}^2$, let $\mathrm{d}(x, y)$ denote the hyperbolic distance between x and y, and for any $\gamma \in \mathsf{PSL}(2,\mathbb{R})$, let $\ell(\gamma)$ denote the minimal translation distance of γ acting on $\mathbb{H}^2$.

Theorem 2.2 (Canary-Zhang-Zimmer [19])**.** *If Γ is a geometrically finite Fuchsian group, $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{K})$ is P_θ-Anosov and $b_0 \in \mathbb{H}^2$, then*

(1) There exist $A, a > 0$ so that if $\gamma \in \Gamma$, then

$$Ae^{a\mathrm{d}(b_0,\gamma(b_0))} \geq e^{\alpha_k(\kappa(\rho(\gamma)))} \geq \frac{1}{A} e^{\frac{\mathrm{d}(b_0,\gamma(b_0))}{a}}$$

for all $k \in \theta$.

(2) There exist $B, b > 0$ so that if $\gamma \in \Gamma$, then

$$Be^{b\ell(\gamma)} \geq e^{\alpha_k(\nu(\rho(\gamma)))} \geq \frac{1}{B} e^{\frac{\ell(\gamma)}{b}}$$

for all $k \in \theta$. In particular, if $\gamma \in \Gamma$ is hyperbolic, then $\rho(\gamma)$ is θ-proximal.

*(3) ρ has the P_θ-*Cartan property*, i.e. whenever $\{\gamma_n\}$ is a sequence of distinct elements of Γ such that $\gamma_n(b_0)$ converges to $z \in \Lambda(\Gamma)$, then $\xi_\rho(z) = \lim U_\theta(\rho(\gamma_n))$.*

Notice that part (3) is an immediate consequence of the definition and Lemma 2.1.

Bray, Canary, Kao and Martone [11] established counting and equidistribution results for cusped Anosov representations. We will need the following counting result, which generalizes a result of Sambarino [45, Thm. B], see also [48], which applies in the convex cocompact case. Notice that Theorem 2.2 implies that $(\mathfrak{a}_\theta^*)^+ \subset \mathcal{B}_\theta^+(\rho)$ whenever ρ is P_θ-Anosov.

Theorem 2.3 (Bray-Canary-Kao-Martone [11, Corollary 11.1])**.** *Suppose that Γ is a torsion-free, geometrically finite Fuchsian group and $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{K})$ is P_θ-Anosov. If $\phi \in \mathcal{B}_\theta^+(\rho)$, then*

$$\lim_{t\to\infty} R_t^\phi(\rho) \frac{t\delta_\phi(\rho)}{e^{t\delta_\phi(\rho)}} = 1$$

where $R_t^\phi(\rho) = \#\{[\gamma] \in [\Gamma_{hyp}] \ : \ \phi(\nu(\rho(\gamma))) \leq t\}$ and $[\Gamma_{hyp}]$ is the set of conjugacy classes of hyperbolic elements in Γ.

Remark. To be precise, Corollary 11.1 in [11] was stated for representations into $\mathsf{SL}(d,\mathbb{R})$, but the same argument taken verbatim works for representations into $\mathsf{PGL}(d,\mathbb{K})$ since the construction of the roof functions only involve the Cartan projection.

2.3. **Cusped Hitchin representations of Fuchsian groups.** In order to define Hitchin representations, we must first recall the definition of a positive map. Given a transverse pair of flags $F_1, F_2 \in \mathcal{F} = \mathcal{F}_\Delta(\mathbb{R}^d)$, an ordered basis $\mathcal{B} = (b_1, \ldots, b_d)$ for $\mathbb{R}^d$ is *compatible* with (F_1, F_2) if $b_i \in F_1^i \cap F_k^{d-i+1}$ for all $i \in \{1, \ldots, d\}$. Given a basis $\mathcal{B}$, let $U_{>0}(\mathcal{B}) \subset \mathsf{SL}(d,\mathbb{R})$ denote the set of unipotent elements that, when written in the basis $\mathcal{B}$, are upper triangular and all minors (which are not forced to be 0 by the fact that the matrix is upper triangular) are strictly positive. Following Fock and Goncharov [25], we say that an ordered k-tuple

$$(F_1, F_2, \ldots, F_k)$$

of flags in $\mathcal{F}$ is *positive* if there exists an ordered basis $\mathcal{B}$ compatible with (F_1, F_k), and elements $u_2, \dots, u_{k-1} \in U_{>0}(\mathcal{B})$ so that $F_i = u_{k-1} \cdots u_i F_k$ for all $i = 2, \dots, k-1$.

If X is a subset of $\partial \mathbb{H}^2$, then a map $\xi : X \to \mathcal{F}$ is *positive* if $(\xi(x_1), \dots, \xi(x_n))$ is a positive whenever $(x_1, \dots, x_n)$ is a cyclically ordered subset of distinct points in X. If Γ is a Fuchsian group, we say that a representation $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ is a *Hitchin representation* if there exists a continuous, positive, ρ-equivariant map $\xi : \Lambda(\Gamma) \to \mathcal{F}$. When $\Gamma \subset \mathsf{PSL}(2, \mathbb{R})$ is a cocompact torsion-free lattice, they agree with the representations introduced by Hitchin [30] and studied by Labourie [36]. If Γ is torsion-free and convex cocompact, but not a lattice, they were studied by Labourie and McShane [37].

We recall several well-known properties of positive tuples of flags that were observed by Fock-Goncharov [25] (see also [54, Appendix A] and [35, Section 3.1–3.3].)

Lemma 2.4. *If $(F_1, \dots, F_k)$ is a positive tuple of flags in $\mathcal{F}$, then*

(1) $(F_2, \dots, F_k, F_1)$ is positive.
(2) $(F_k, \dots, F_1)$ is positive.
(3) $(F_{i_1}, \dots, F_{i_\ell})$ is positive for all $1 \le i_1 < \cdots < i_\ell \le k$.
(4) $F_1^{n_1} \oplus \cdots \oplus F_k^{n_k} = \mathbb{R}^d$ for any integers $n_1, \dots, n_k$ that sum to d.
(5) $(F_1, \dots, F_n)$ is positive for all flags $F_{k+1}, \dots, F_n \in \mathcal{F}$ such that $(F_1, F_i, F_k, F_{k+1}, \dots, F_n)$ is positive for some $i \in \{2, \dots, k-1\}$.

If Γ is geometrically finite, then the following generalization of the main result in [36] is established in [19].

Theorem 2.5. (Canary-Zhang-Zimmer [19, Theorem 7.1]) *If $\Gamma \subset \mathsf{PSL}(2, \mathbb{R})$ is geometrically finite and $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ is a Hitchin representation, then ρ is P_Δ-Anosov.*

Remark. To be precise, Theorem 7.1 in [19] was stated for representations into $\mathsf{SL}(d, \mathbb{R})$, but the same argument taken verbatim works for representations into $\mathsf{PSL}(d, \mathbb{R})$, since the entire argument takes place in $\mathcal{F}(\mathbb{R}^d)$.

2.4. **Properly convex domains.** We briefly recall some standard facts about properly convex domains in projective space. A domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is *properly convex* if it is a bounded convex subset of some affine chart A for $\mathbb{P}(\mathbb{R}^d)$. If $x, y \in \overline{\Omega}$, let $[x, y]_\Omega$ denote the closed projective line segment in $\overline{\Omega}$ with endpoints x and y. We also define $(x, y)_\Omega = [x, y]_\Omega - \{x, y\}$, $[x, y)_\Omega = [x, y]_\Omega - \{y\}$, and $(x, y]_\Omega = [y, x)_\Omega$.

With this notation we can define *radial projection maps*: given $b_0 \in \Omega$, define

$$\iota_{b_0} : \overline{\Omega} - \{b_0\} \to \partial\Omega$$

by letting $\iota_{b_0}(z) \in \partial\Omega$ be the unique boundary point such that $z \in (b_0, \iota_{b_0}(z)]_\Omega$.

Every boundary point $x \in \partial\Omega$ of a properly convex domain is contained in a *supporting hyperplane* H, that is: $H = \mathbb{P}(V)$ for some codimension one linear subspace $V \subset \mathbb{R}^d$, $x \in H$ and $H \cap \Omega = \emptyset$. When x is contained in a unique supporting hyperplane, we say that x is a C^1 *point* of $\partial\Omega$ and denote this unique supporting hyperplane by $T_x\partial\Omega$.

A properly convex domain Ω has a natural projectively invariant Finsler metric d_Ω, called the *Hilbert metric*, which is defined in terms of the cross ratio. If $a, b \in \Omega$, then there is a projective line ℓ containing a and b which intersects $\partial\Omega$ at points a' and b' (ordered so that $\{a', a, b, b'\}$ appear monotonically along ℓ). Then

$$\mathrm{d}_\Omega(a, b) = \log \frac{|a' - b||b' - a|}{|a' - a||b' - b|},$$

where $|\cdot|$ denotes some (any) norm on some (any) affine chart containing a', a, b, b'. If $a, b \in \overline{\Omega}$, then the projective line segment $(a,b)_\Omega$ joining them is a geodesic in the Hilbert metric, although geodesics need not be unique. We also let

$$B_\Omega(p,r) \subset \Omega$$

denote the open ball of radius r centered at $p \in \Omega$ with respect to the Hilbert metric.

We will use the following basic estimate several times: if $p_1, p_2, q_1, q_2 \in \Omega$, then

$$\mathrm{d}_\Omega^{\mathrm{Haus}}([p_1,p_2]_\Omega, [q_1,q_2]_\Omega) \leqslant \max\{\mathrm{d}_\Omega(p_1,q_1), \mathrm{d}_\Omega(p_2,q_2)\}, \tag{1}$$

(see for instance [31, Proposition 5.3]). In Equation (1), $\mathrm{d}_\Omega^{\mathrm{Haus}}$ denotes the Hausdorff distance induced by the Hilbert distance.

It is often useful to consider the dual of a properly convex domain. Let $V = \mathbb{R}^d$. For any $k \in \Delta$, there is a natural identification $\mathrm{Gr}_k(V^*) \cong \mathrm{Gr}_{d-k}(V)$ given by

$$\mathrm{Span}_{\mathbb{R}}(\alpha_1, \dots, \alpha_k) \mapsto \bigcap_{i=1}^{k} \ker(\alpha_i).$$

Also, we may identify $\mathrm{Gr}_k(V)$ with the set of $(k-1)$-dimensional projective hyperplanes in $\mathbb{P}(V)$.

The *dual* of a properly convex domain $\Omega \subset \mathbb{P}(V)$ is the set

$$\Omega^* = \left\{[f] \in \mathbb{P}(V^*) : f(X) \neq 0 \text{ for all } [X] \in \overline{\Omega}\right\}.$$

We record the following standard fact for later use.

Lemma 2.6. *If $\Omega \subset \mathbb{P}(V)$ is a (non-empty) properly convex domain, then Ω^* is a (non-empty) properly convex domain in $\mathbb{P}(V^*)$. Furthermore, $(\Omega^*)^* = \Omega$ and $\mathrm{Aut}(\Omega) = \mathrm{Aut}(\Omega^*)$ under the canonical identification $\mathsf{PGL}(V) \simeq \mathsf{PGL}(V^*)$.*

We refer the reader to Marquis [40] for further discussion of the Hilbert metric on Ω and its automorphism group.

The following proposition describes the limiting behavior of divergent sequences in $\mathrm{Aut}(\Omega)$.

Proposition 2.7. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $b_0 \in \Omega$. Let $\{\gamma_n\}$ is a sequence in $\mathrm{Aut}(\Omega)$ such that $\gamma_n(b_0) \to x \in \partial\Omega$ and $\gamma_n^{-1}(b_0) \to y \in \partial\Omega$.*

1. *If $\gamma_n \to S \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$, then $S(\Omega) \subset \partial\Omega$, $y \in \mathbb{P}(\ker S)$, and $\mathbb{P}(\ker S) \cap \Omega$ is empty.*
2. *If $\alpha_1(\kappa(\gamma_n)) \to \infty$ and $\gamma_n \to S \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$, then $S(\Omega) = x$ and the $\mathbb{P}(\ker S)$ is a supporting hyperplane to Ω at y. In particular, $\gamma_n(b) \to x$ for all $b \in \mathbb{P}(\mathbb{R}^d) - \mathbb{P}(\ker S)$, and this convergence is locally uniform.*
3. *If $\alpha_1(\kappa(\gamma_n)) \to \infty$ and y is a C^1-point of $\partial\Omega$, then $\gamma_n \to S \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ with the defining property $S(\Omega) = x$ and $\mathbb{P}(\ker(S)) = T_y\partial\Omega$.*

Proof. See Islam-Zimmer [31, Prop. 5.6] for a proof of (1). The assumption that $\alpha_1(\kappa(\gamma_n)) \to \infty$ implies that S is the projectivization of a rank 1 linear map, so (2) follows from (1). To prove (3), first observe that by taking a subsequence of $\{\gamma_n\}$, we may assume that $\gamma_n \to T \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. It then suffices to show that $T = S$. By (2), $T(\Omega) = x$ and $\mathbb{P}(\ker T)$ is a supporting hyperplane to Ω at y. Since y is a C^1-point of $\partial\Omega$, $\mathbb{P}(\ker T) = T_y\partial\Omega$. Thus, $S = T$. ☐

2.5. **Special representations.** We recall the skew-symmetric and symmetric tensor representations and their basic properties.

2.5.1. *Skew-symmetric tensors.* Given a $\mathbb{K}$-vector space V, let $W = \bigwedge^k V$ be the vector space of skew-symmetric tensors of order k. Let $d = \dim_{\mathbb{K}}(V)$ and $D = \dim_{\mathbb{K}}(W)$, and let

$$E^k = E^k_V : \mathsf{PGL}(V) \to \mathsf{PGL}(W)$$

denote the representation defined by

$$E^k(g)[v_1 \wedge \cdots \wedge v_k] = [(gv_1) \wedge \cdots \wedge (gv_k)].$$

It is straightforward to verify that E^k is faithful and irreducible. We may also define a continuous, transverse, E^k-equivariant map

$$\xi_{E^k} : \mathcal{F}_{k,d-k}(V) \to \mathcal{F}_{1,D-1}(W)$$

by

$$\begin{aligned} &\xi_{E^k}\left(\mathrm{Span}(v_1, \dots, v_k), \mathrm{Span}(v_1, \dots, v_{d-k})\right) \\ &\quad = \left([v_1 \wedge \cdots \wedge v_k], \ker\left(w \in W \mapsto w \wedge v_1 \wedge \cdots \wedge v_{d-k} \in \bigwedge^d V\right)\right). \end{aligned}$$

In the special case when $V = \mathbb{K}^d$, the standard basis $(e_1, \dots, e_d)$ of $\mathbb{K}^d$ induces a standard basis $(e_{i_1} \wedge \cdots \wedge e_{i_k})_{1 \le i_1 < \cdots < i_k \le d}$ of W, and thus gives an identification $W \simeq \mathbb{K}^D$. Under this identification, we have

$$\alpha_1(\kappa(E^k(g))) = \alpha_k(\kappa(g)) \quad \text{and} \quad \sigma_1(E^k(g)) = (\sigma_1 \cdots \sigma_k)(g) \tag{2}$$

for all $g \in \mathsf{PGL}(d, \mathbb{K})$.

2.5.2. *Hermitian symmetric tensors.* Given a $\mathbb{K}$-vector space V, fix an (Hermitian) inner product $\langle \cdot, \cdot \rangle$ on V and let $X \mapsto X^*$ denote the associated transpose on $\mathrm{End}(V)$, the space of $\mathbb{K}$-linear maps $V \to V$. Also, given $v \in V$ let $v^* \in V^*$ be the functional $v^* = \langle \cdot, v \rangle$.

Let $\mathrm{Her}(V)$ denote the **real** vector space

$$\mathrm{Her}(V) = \{X \in \mathrm{End}(V) : X^* = X\},$$

let $d = \dim_{\mathbb{K}}(V)$, and let $D = \dim_{\mathbb{R}}(\mathrm{Her}(V))$.

Next let

$$S_V : \mathsf{PGL}(V) \to \mathsf{PGL}(\mathrm{Her}(V))$$

denote the representation defined by $S_V(g)(X) = g \circ X \circ g^*$. It is straightforward to verify that S_V is faithful and irreducible. We may also define a continuous, transverse, S_V-equivariant map

$$\xi_{S_V} : \mathcal{F}_{1,d-1}(V) \to \mathcal{F}_{1,D-1}(\mathrm{Her}(V))$$

by

$$\xi_{S_V}([v], H) = \left([v \cdot v^*], \mathrm{Span}\{w \cdot v^* + v \cdot w^* : v \in V, w \in H\}\right).$$

An element $X \in \mathrm{Her}(V)$ is *positive definite* if $\langle X(v), v \rangle > 0$ for all non-zero $v \in V$. One can then verify that

$$\Omega_0 = \{[X] \in \mathbb{P}(\mathrm{Her}(V)) : X \text{ is positive definite}\}$$

is a $S_V(\mathsf{PGL}(V))$-invariant properly convex domain.

In the special case when $V = \mathbb{R}^d$, the standard basis $(e_1, \dots, e_d)$ of $\mathbb{R}^d$ induces a standard basis $(e_i \cdot e_j^* + e_j \cdot e_i^*)_{1 \le i \le j \le d}$ of $\mathrm{Her}(V)$ and in the special case when $V = \mathbb{C}^d$, the standard basis $(e_1, \dots, e_d)$ of $\mathbb{C}^d$ induces a standard basis

$$(e_i \cdot e_i^*)_{1 \le i \le d} \cup (e_i \cdot e_j^* + e_j \cdot e_i^*)_{1 \le i < j \le d} \cup (ie_i \cdot e_j^* - ie_j \cdot e_i^*)_{1 \le i < j \le d}$$

of $\mathrm{Her}(V)$ and thus gives an identification $\mathrm{Her}(V) \simeq \mathbb{R}^D$. Under these identifications, we have

$$\alpha_1(\kappa(S_V(g))) = \alpha_1(\kappa(g)) \tag{3}$$

for all $g \in \mathsf{PGL}(d, \mathbb{K})$.

3. Divergent and transverse subgroups

In this section, we study divergent and transverse subgroups and their limit sets. We exhibit examples and show that projectively visible subgroups are $P_{1,d-1}$-transverse.

3.1. Properties of the limit set.

Let $\theta \subset \Delta$ be symmetric.

Proposition 3.1. *If $\Gamma \subset \mathsf{PGL}(d, \mathbb{K})$ is P_θ-divergent, then $\Lambda_\theta(\Gamma)$ is Γ-invariant.*

Proof. Fix $F^+ \in \Lambda_\theta(\Gamma)$ and $\gamma \in \Gamma$. By Lemma 2.1 there exist $F^- \in \Lambda_\theta(\Gamma)$ and a sequence $\{\gamma_n\}$ in Γ such that $\gamma_n(F) \to F^+$ for all F transverse to F^-. Then

$$\gamma\gamma_n(F) \to \gamma(F^+)$$

for all F transverse to F^-. Then Lemma 2.1 implies that $U_\theta(\gamma\gamma_n) \to \gamma(F^+)$ and so $\gamma(F^+) \in \Lambda_\theta(\Gamma)$. □

The *P_θ-domain of discontinuity* for Γ, denoted $\Omega_\theta(\Gamma)$, is the set of flags in $\mathcal{F}_\theta$ that are transverse to every flag in $\Lambda_\theta(\Gamma)$. Since $\Lambda_\theta(\Gamma)$ is compact, observe that $\Omega_\theta(\Gamma)$ is a (possibly empty) open set. It also follows from Proposition 3.1 that $\Omega_\theta(\Gamma)$ is Γ-invariant. The following is a special case of a result from Guichard-Wienhard [29, Theorem 7.4].

Proposition 3.2. *If $\Gamma \subset \mathsf{PGL}(d, \mathbb{K})$ is P_θ-divergent, then the action of Γ on $\Omega_\theta(\Gamma)$ is properly discontinuous.*

Since a hyperbolic group acts on its Gromov boundary as a convergence group, Anosov groups act on their limit sets as convergence groups. We extend this property to transverse groups. Recall that if M is a compact metric space, a group Γ acts as a *convergence group* on M if for any infinite sequence $\{\gamma_n\}$ of distinct elements in Γ, there exist some $x, y \in M$ and a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_k}(z) \to x$ for all $z \in M - \{y\}$.

Proposition 3.3. *If $\Gamma \subset \mathsf{PGL}(d, \mathbb{K})$ is P_θ-transverse, then Γ acts on $\Lambda_\theta(\Gamma)$ as a convergence group.*

Proof. Suppose $\{\gamma_n\}$ is an infinite sequence of elements in Γ. By taking a subsequence, we may assume that there exists $F^+, F^- \in \mathcal{F}_\theta$ such that $U_\theta(\gamma_n) \to F^+$ and $U_\theta(\gamma_n^{-1}) \to F^-$. Since Γ is P_θ-divergent, $\alpha_k(\kappa(\gamma_n)) \to \infty$ for all $k \in \theta$. Since Γ is P_θ-transverse, by Lemma 2.1, $\gamma_n(F) \to F^+$ for all $F \in \Lambda_\theta(\Gamma) - \{F^-\}$. □

Next, we consider P_1-divergent subgroups Γ that leave invariant a properly convex domain in $\Omega \subset \mathbb{P}(\mathbb{R}^d)$. Recall that the the orbital limit set of Γ is

$$\Lambda_\Omega(\Gamma) = \{z \in \partial\Omega \mid z = \lim \gamma_n(x) \text{ for some } x \in \Omega \text{ and some } \{\gamma_n\} \subset \Gamma\}.$$

Also, recall that if $k \in \theta$, then $\pi_k : \mathcal{F}_\theta \to \mathrm{Gr}_k(\mathbb{K}^d)$ denotes the projection map. The next result relates the two limit sets of a $P_{1,d-1}$-divergent subgroup which preserves a properly convex domain Ω.

Proposition 3.4. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \subset \mathrm{Aut}(\Omega)$. If Γ is $P_{1,d-1}$-divergent, then*

$$\pi_1(\Lambda_{1,d-1}(\Gamma)) = \Lambda_\Omega(\Gamma),$$

and, identifying $\mathrm{Gr}_{d-1}(\mathbb{R}^d) = \mathbb{P}(\mathbb{R}^{d*})$,

$$\pi_{d-1}(\Lambda_{1,d-1}(\Gamma)) = \Lambda_{\Omega^*}(\Gamma).$$

In particular, if $F \in \Lambda_{1,d-1}(\Gamma)$*, then* F^{d-1} *is a supporting hyperplane to* Ω *at* F^1*.*

Proof. The statements that $\pi_1(\Lambda_{1,d-1}(\Gamma)) = \Lambda_\Omega(\Gamma)$ and $\pi_{d-1}(\Lambda_{1,d-1}(\Gamma)) = \Lambda_{\Omega^*}(\Gamma)$ are dual, so we only prove the former.

Fix $F \in \Lambda_{1,d-1}(\Gamma)$ and let $\{\gamma_n\}$ be a sequence in Γ such that $U_{1,d-1}(\gamma_n) \to F$. Pass to a subsequence so that $\gamma_n(b_0) \to x$ and $\gamma_n^{-1}(b_0) \to y$ (for some $b_0 \in \Omega$) and $\{\gamma_n\}$ converges to $S \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Proposition 2.7 part (2) implies that S has rank 1 and $S(\Omega) = x$. Since $U_1(\gamma_n) \to F_1$, by Lemma 2.1 we must have $x = F_1$. Therefore, $\pi_1(\Lambda_{1,d-1}(\Gamma)) \subset \Lambda_\Omega(\Gamma)$.

It remains to show that $\Lambda_\Omega(\Gamma) \subset \pi_1(\Lambda_{1,d-1}(\Gamma))$. Fix $x \in \Lambda_\Omega(\Gamma)$ and let $\{\gamma_n\}$ be a sequence in Γ such that $\gamma_n(b_0) \to x$ for some $b_0 \in \Omega$. Passing to a subsequence we may assume that $U_{1,d-1}(\gamma_n) \to F \in \Lambda_{1,d-1}(\Gamma)$, $\gamma_n^{-1}(b_0) \to y \in \Lambda_\Omega(\Gamma)$, and $\gamma_n \to S \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Then by Proposition 2.7 part (2), S has rank 1 and $S(\Omega) = x$. Since $U_1(\gamma_n) \to F_1$, we again see that $x = F_1$.

Since x is arbitrary, $\Lambda_\Omega(\Gamma) \subset \pi_1(\Lambda_{1,d-1}(\Gamma))$. □

3.2. **Examples of transverse subgroups.** A large source of examples of P_θ-transverse subgroups of $\mathsf{PGL}(d,\mathbb{K})$ are the images of P_θ-Anosov representations, P_θ-relatively dominated representations in the sense of Zhu [56], P_θ-asymptotically embedded representations in the sense of Kapovich-Leeb [32] and any subgroup of one of these groups. Notice that these subgroups are not required to be finitely generated.

Another source of examples comes from the Klein-Beltrami model of real hyperbolic space. In particular, $\mathsf{PO}(d-1,1)$ preserves the properly convex domain

$$\mathbb{B} = \left\{ [x_1 : \cdots : x_{d-1} : 1] \in \mathbb{P}(\mathbb{R}^d) : \sum_{j=1}^{d-1} x_j^2 < 1 \right\}$$

whose boundary is smooth and contains no line segments. Then, any discrete subgroup of $\mathsf{PO}(d-1,1)$ is a $P_{1,d-1}$-transverse subgroup of $\mathsf{PGL}(d,\mathbb{R})$, since its $P_{1,d-1}$-limit set is contained in $\{(x, T_x\partial\mathbb{B}) : x \in \partial\mathbb{B}\}$ which is a transverse subset of $\mathcal{F}_{1,d-1}$.

Recall, that a discrete group $\Gamma \subset \mathsf{PGL}(d,\mathbb{R})$ is *projectively visible* if there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ preserved by Γ where the full orbital limit set $\Lambda_\Omega(\Gamma)$ satisfies:

(1) $(x,y)_\Omega \subset \Omega$ for all $x, y \in \Lambda_\Omega(\Gamma)$,
(2) every $x \in \Lambda_\Omega(\Gamma)$ is a C^1 point of $\partial\Omega$.

In this case we also say that Γ *is a projectively visible subgroup of* $\mathrm{Aut}(\Omega)$.

Notice that any discrete subgroup of $\mathsf{PO}(d-1,1)$ is a projectively visible subgroup of $\mathrm{Aut}(\mathbb{B})$. Our next result gives some basic properties of projectively visible subgroups.

Proposition 3.5. *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain and* $\Gamma \subset \mathrm{Aut}(\Omega)$ *is a projectively visible subgroup. Fix* $b_0 \in \Omega$*.*

(1) $\Gamma \subset \mathsf{PGL}(d,\mathbb{R})$ *is a* $P_{1,d-1}$*-transverse subgroup and*

$$\Lambda_{1,d-1}(\Gamma) = \{(x, T_x\partial\Omega) : x \in \Lambda_\Omega(\Gamma)\}.$$

(2) If $b_0 \in \Omega$ *and* $\{\gamma_n\}$ *is a sequence in* Γ *with* $\gamma_n(b_0) \to x \in \Lambda_\Omega(\Gamma)$ *and* $\gamma_n^{-1}(b_0) \to y \in \Lambda_\Omega(\Gamma)$*, then*

$$\gamma_n(F) \to (x, T_x\partial\Omega)$$

for all $F \in \mathcal{F}_{1,d-1}$ *transverse to* $(y, T_y\partial\Omega)$*. Moreover, the convergence is locally uniform.*

(3) Γ *acts as a convergence group on* $\Lambda_\Omega(\Gamma)$*.*

Proof. (1): First, we prove that Γ is $P_{1,d-1}$-divergent. Let $\{\gamma_n\}$ be a sequence in Γ of pairwise distinct elements. By taking a subsequence, we may assume that $\gamma_n(b_0) \to x \in \Lambda_\Omega(\Gamma)$, $\gamma_n^{-1}(b_0) \to y \in \Lambda_\Omega(\Gamma)$, and $\gamma_n \to S \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. By Proposition 2.7 part (1), $S(\Omega) \subset \partial\Omega$, $y \in \mathbb{P}(\ker S)$, and $\mathbb{P}(\ker S) \cap \Omega$ is empty. Thus, if we pick any $w \in \Omega$, then

$$S(w) = \lim_{n\to\infty} \gamma_n(w),$$

so $S(\Omega) \subset \Lambda_\Omega(\Gamma)$. It follows that

$$[S(w), x]_\Omega = [S(w), S(b_0)]_\Omega = S([w, b_0]_\Omega) \subset S(\Omega) \subset \Lambda_\Omega(\Gamma),$$

which implies that $S(w) = x$ because Γ is projectively visible. Since $w \in \Omega$ was arbitrary, $S(\Omega) = \{x\}$. Since Ω is open, $\mathrm{im}\, S = x$. Thus, S is the projectivization of a rank 1 linear map, so $\alpha_1(\kappa(\gamma_n)) \to \infty$. Since $\{\gamma_n\}$ was arbitrary, Γ is $P_{1,d-1}$-divergent.

Next, we prove that Γ is $P_{1,d-1}$-transverse. Since each $x \in \Lambda_\Omega(\Gamma)$ has a unique supporting hyperplane, namely $T_x\partial\Omega$, Proposition 3.4 implies that

$$\Lambda_{1,d-1}(\Gamma) = \{(x, T_x\partial\Omega) : x \in \Lambda_\Omega(\Gamma)\}.$$

If Γ is not $P_{1,d-1}$-transverse, then there is some $x, y \in \Lambda_\Omega(\Gamma)$ such that $x \in T_y\partial\Omega$. It follows that $[x,y]_\Omega \subset T_y\partial\Omega$, which implies $[x,y]_\Omega \subset \partial\Omega$. This contradicts the visibility of Ω and hence Γ is $P_{1,d-1}$-transverse.

(2): By part (1), Γ is $P_{1,d-1}$-divergent, so Proposition 2.7 part (3) implies that $\gamma_n \to S \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ given by $S(\Omega) = x$ and $\mathbb{P}(\ker(S)) = T_y\partial\Omega$, and $\gamma_n^{-1} \to T \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ given by $T(\Omega) = y$ and $\mathbb{P}(\ker T) = T_x\partial\Omega$. Thus, as a sequence in $\mathsf{PGL}(\mathbb{R}^{d*}) \simeq \mathsf{PGL}(d,\mathbb{R})$, $\gamma_n \to S^* \in \mathbb{P}(\mathrm{End}(\mathbb{R}^{d*}))$ with the defining property that $S^*(\Omega^*) = T_x\partial\Omega$ and $\mathbb{P}(\ker S^*) = y$.

Since F^1 does not lie in $T_y\partial\Omega = \ker S$,

$$\lim_{n\to\infty} \gamma_n(F^1) = S(F^1) = x.$$

Similarly, since F^{d-1} does not contain y,

$$\lim_{n\to\infty} \gamma_n(F^{d-1}) = S^*(F^{d-1}) = T_x\partial\Omega.$$

This proves (2).

(3): This is immediate from (1) and Propositions 3.3 and 3.4. $\square$

3.3. **Conical limit points.** Let Γ act as a convergence group on M. Recall that a point $x \in M$ is a *conical limit point* if there exist an infinite sequence $\{\gamma_n\}$ of distinct elements in Γ and distinct points $a, b \in M$ such that $\gamma_n(x) \to a$ and $\gamma_n(y) \to b$ for all $y \in M \setminus \{x\}$. When $\Gamma \subset \mathsf{PGL}(d,\mathbb{K})$ is a P_θ-transverse subgroup, we denote the set of conical limit points of the Γ action on $\Lambda_\theta(\Gamma)$ by $\Lambda_{\theta,c}(\Gamma)$. If $k \in \theta$ and $\pi_k : \mathcal{F}_\theta \to \mathrm{Gr}_k(\mathbb{K}^d)$ is the projection map, let

$$\Lambda_k(\Gamma) = \pi_k(\Lambda_\theta(\Gamma)) \quad \text{and} \quad \Lambda_{k,c}(\Gamma) = \pi_k(\Lambda_{\theta,c}(\Gamma)).$$

Similarly, when $\Gamma \subset \mathrm{Aut}(\Omega)$ is projectively visible, we denote the set of conical limit points of the Γ action on $\Lambda_\Omega(\Gamma)$ by $\Lambda_{\Omega,c}(\Gamma)$. The points in $\Lambda_{\Omega,c}(\Gamma)$ have a characterization very similar to the classical characterization/definition in hyperbolic space.

Lemma 3.6. *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain and* $\Gamma \subset \mathrm{Aut}(\Omega)$ *is a projectively visible subgroup. If* $x \in \mathbb{P}(\mathbb{R}^d)$*, then* $x \in \Lambda_{\Omega,c}(\Gamma)$ *if and only if there exist* $b_0 \in \Omega$ *and a sequence* $\{\gamma_n\}$ *in* Γ *such that* $\gamma_n(b_0) \to x$ *and*

$$\sup_{n\geq 1} \mathrm{d}_\Omega\left(\gamma_n(b_0), [b_0, x)_\Omega\right) < +\infty.$$

Proof. (⇐): Suppose that there exist an infinite sequence $\{\gamma_n\}$ of distinct elements in Γ and distinct points $a, b \in \Lambda_\Omega(\Gamma)$ such that $\gamma_n(x) \to a$ and $\gamma_n(y) \to b$ for all $y \in \Lambda_\Omega(\Gamma) \setminus \{x\}$. Fix some $b_0 \in \Omega$. Then Proposition 3.5 implies that $\gamma_n(b_0) \to b$ and $\gamma_n^{-1}(b_0) \to x$ (since $a \neq b$). Since $a \neq b$, the visibility property implies that $(a, b)_\Omega \subset \Omega$. Hence

$$\limsup_{n\to\infty} \mathrm{d}_\Omega\left(\gamma_n^{-1}(b_0), [b_0, x)_\Omega\right) = \limsup_{n\to\infty} \mathrm{d}_\Omega\left(b_0, [\gamma_n(b_0), \gamma_n(x))_\Omega\right) = \mathrm{d}_\Omega\left(b_0, (b, a)_\Omega\right) < +\infty.$$

(⇒): Suppose that $\gamma_n(b_0) \to x$ and

$$\sup_{n\geq 1} \mathrm{d}_\Omega(\gamma_n(b_0), [b_0, x)_\Omega) < +\infty$$

for some sequence $\{\gamma_n\}$ in Γ and $b_0 \in \Omega$. Pick $\{p_n\}$ in $[b_0, x)$ such that $\{\gamma_n^{-1}(p_n)\}$ is relatively compact in Ω. Passing to a subsequence we can suppose that $\gamma_n^{-1}(p_n) \to p$, $\gamma_n^{-1}(x) \to a$ and $\gamma_n^{-1}(b_0) \to b$. Then $a, b \in \Lambda_\Omega(\Gamma)$ and $p \in (a, b)_\Omega$, so $a \neq b$. Further, Proposition 3.5 implies that $\gamma_n^{-1}(y) \to b$ for all $y \in \Lambda_\Omega(\Gamma) \setminus \{x\}$. So $x \in \Lambda_{\Omega,c}(\Gamma)$. □

4. Transverse representations

In this section, we develop the basic theory of transverse representations which will be a crucial tool in our work. Our main result is that every P_θ-transverse subgroup is the image of P_θ-transverse representation of a projectively visible subgroup.

Definition 4.1. Suppose that $\theta \subset \{1, \ldots, d\}$ is a symmetric subset, $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is a properly convex domain and $\Gamma \subset \mathrm{Aut}(\Omega)$ is a projectively visible subgroup. A representation $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is *P_θ-transverse* if there exists a continuous embedding

$$\xi : \Lambda_\Omega(\Gamma) \to \mathcal{F}_\theta$$

with the following properties:

1. ξ is ρ-equivariant, i.e. $\rho(\gamma) \circ \xi = \xi \circ \gamma$ for all $\gamma \in \Gamma$,
2. $\xi(\Lambda_\Omega(\Gamma))$ is a transverse subset of $\mathcal{F}_\theta$,
3. if $\{\gamma_n\}$ is a sequence in Γ so that $\gamma_n(b_0) \to x \in \Lambda_\Omega(\Gamma)$ and $\gamma_n^{-1}(b_0) \to y \in \Lambda_\Omega(\Gamma)$ for some (any) $b_0 \in \Omega$, then $\rho(\gamma_n)(F) \to \xi(x)$ if $F \in \mathcal{F}_\theta$ is transverse to $\xi(y)$.

We refer to ξ as the *limit map* of ρ.

It follows from Lemma 2.1 that if $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is P_θ-transverse, then it is P_θ-divergent, so it has finite kernel and $\rho(\Gamma)$ is a P_θ-transverse subgroup.

By Proposition 3.5, if Γ is a projectively visible subgroup of $\mathrm{Aut}(\Omega)$ for some properly convex domain $\Omega \subset \mathsf{PGL}(d, \mathbb{R})$, then the inclusion representation $\Gamma \hookrightarrow \mathsf{PGL}(d, \mathbb{R})$ is $P_{1,d-1}$-transverse, and its limit map is given by $x \mapsto (x, T_x\partial\Omega)$. If, in addition Γ acts cocompactly on the convex hull of $\Lambda_\Omega(\Gamma)$ in Ω, then Γ is hyperbolic (see [23, Thm. 1.15] or [57, Thm. 5.1]) and it follows from [28, Thm. 1.3] that P_θ-transverse representations in this case coincide with P_θ-Anosov representations. Moreover, if Ω is the Klein-Beltrami model of the real hyperbolic 2-plane, Γ is finitely generated, and $\theta = \{k, d-k\}$, then P_θ-transverse representations coincide with the cusped P_k-Anosov representations introduced in [19].

Theorem 4.2. *If $\Gamma \subset \mathsf{PGL}(d, \mathbb{K})$ is P_θ-transverse, then for any $k \in \theta$ such that $k \leq d-k$, there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ for some $d_0 \in \mathbb{N}$, a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$, and a faithful P_θ-transverse representation $\rho : \Gamma_0 \to \mathsf{PGL}(d, \mathbb{K})$ with limit map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$ so that*

(1) $\rho(\Gamma_0) = \Gamma$.

(2) $\xi(\Lambda_\Omega(\Gamma_0)) = \Lambda_\theta(\Gamma)$.
(3) $\alpha_1(\kappa(\gamma)) = \alpha_k(\kappa(\rho(\gamma)))$ *for all* $\gamma \in \Gamma_0$.

The main content of the proof of Theorem 4.2 is the following two propositions, which are motivated by previous work of Zimmer [57] and Danciger-Gueritaud-Kassel [23] in the setting of Anosov representations.

The first proposition provides a representation ϕ, so that $\phi(\Gamma)$ preserves a properly convex domain. We say that a map $\xi : \mathcal{F}_\theta(\mathbb{K}^d) \to \mathcal{F}_{\theta'}(\mathbb{K}^{d_0})$ is *transverse* if it sends every transverse pair of flags in $\mathcal{F}_\theta(\mathbb{K}^d)$ to a transverse pair of flags in $\mathcal{F}_{\theta'}(\mathbb{K}^{d_0})$.

Proposition 4.3. *If* $1 \leqslant k \leqslant d/2$*, then there exists a faithful representation*

$$\phi : \mathsf{PGL}(d, \mathbb{K}) \to \mathsf{PGL}(d_0, \mathbb{R})$$

for some $d_0 \in \mathbb{N}$*, a* ϕ*-equivariant, continuous, transverse map*

$$\xi_\phi : \mathcal{F}_{k,d-k}(\mathbb{K}^d) \to \mathcal{F}_{1,d_0-1}(\mathbb{R}^{d_0}),$$

and a properly convex domain $\Omega_0 \subset \mathbb{P}(\mathbb{R}^{d_0})$ *such that:*

(1) $\phi(\mathsf{PGL}(d, \mathbb{K})) \subset \mathrm{Aut}(\Omega_0)$.
(2) $\alpha_1(\kappa(\phi(g))) = \alpha_k(\kappa(g))$ *for all* $g \in \mathsf{PGL}(d, \mathbb{K})$.
(3) If $\Gamma \subset \mathsf{PGL}(d, \mathbb{K})$ *is* $P_{k,d-k}$*-transverse, then* $\phi(\Gamma)$ *is* P_{1,d_0-1}*-transverse and* ξ_ϕ *induces a homeomorphism* $\Lambda_{k,d-k}(\Gamma) \to \Lambda_{1,d_0-1}(\phi(\Gamma))$.

The second proposition shows that one can enlarge the properly convex domain so that $\phi(\Gamma)$ acts as a projectively visible subgroup.

Proposition 4.4. *If* $\Gamma \subset \mathsf{PGL}(d_0, \mathbb{R})$ *is* P_{1,d_0-1}*-transverse and preserves a properly convex domain* $\Omega_0 \subset \mathbb{P}(\mathbb{R}^{d_0})$*, then there exists a properly convex domain* $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$*, containing* Ω_0*, such that* Γ *preserves* Ω *and is a projectively visible subgroup of* $\mathrm{Aut}(\Omega)$.

Assuming these two propositions, we give the proof of Theorem 4.2.

Proof of Theorem 4.2. For any $k \in \theta$, let ϕ, ξ_ϕ, d_0 and Ω_0 be given by Proposition 4.3 and let

$$\Gamma_0 = \rho(\Gamma).$$

Then Γ_0 is P_{1,d_0-1}-transverse and preserves Ω_0. So by Proposition 4.4 there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ where $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ is a projectively visible subgroup. Since ϕ is faithful, we may define

$$\rho = \phi|_\Gamma^{-1} : \Gamma_0 \to \mathsf{PGL}(d, \mathbb{K})$$

which is necessarily a faithful representation.

By Proposition 3.5 part (1), there is an obvious Γ_0-invariant homeomorphism

$$i : \Lambda_\Omega(\Gamma_0) \to \Lambda_{1,d_0-1}(\Gamma_0).$$

Also, since ρ is P_θ-transverse, there is an obvious Γ-invariant homeomorphism

$$i' : \Lambda_{k,d-k}(\Gamma) \to \Lambda_\theta(\Gamma).$$

Furthermore, by Proposition 4.3 part (3) and the fact that ξ_ϕ is ϕ-equivariant, we may define a ρ-equivariant homeomorphism

$$\bar{\xi} = \xi_\phi|_{\Lambda_{k,d-k}(\Gamma)}^{-1} : \Lambda_{1,d_0-1}(\Gamma_0) \to \Lambda_{k,d-k}(\Gamma).$$

Together, these give a ρ-equivariant homeomorphism

$$\xi = i' \circ \bar{\xi} \circ i : \Lambda_\Omega(\Gamma_0) \to \Lambda_\theta(\Gamma).$$

It is immediate that (1) and (2) hold, and (3) is a consequence of Proposition 4.3 part (2).

It remains to show that ρ is a P_θ-transverse representation whose limit map is ξ. To do so, it suffices to prove condition (3) of Definition 4.1; the other conditions are clear.

Let $\{\gamma_n\}$ be a sequence in Γ_0 so that $\gamma_n(b_0) \to x \in \Lambda_\Omega(\Gamma_0)$ and $\gamma_n^{-1}(b_0) \to y \in \Lambda_\Omega(\Gamma_0)$ for any $b_0 \in \Omega$. By taking a subsequence, we may assume that

$$U_\theta(\rho(\gamma_n)) \to F^+ \text{ and } U_\theta(\rho(\gamma_n)^{-1}) \to F^-$$

for some $F^+, F^- \in \mathcal{F}_\theta$. By Proposition 2.7 part (2), $\gamma_n(z) \to x$ for all $z \in \Lambda_\Omega(\Gamma_0) - \{y\}$, so

$$\rho(\gamma_n)(\xi(z)) = \xi(\gamma_n(z)) \to \xi(x)$$

for all $z \in \Lambda_\Omega(\Gamma_0) - \{y\}$. At the same time, Lemma 2.1 implies that $\rho(\gamma_n)(F) \to F^+$ for all $F \in \mathcal{F}_\theta$ transverse to F^-. Thus, $F^+ = \xi(x)$, so condition (3) of Definition 4.1 holds. □

We now prove Propositions 4.3 and 4.4.

Proof of Proposition 4.3: Using the notation in Section 2.5, let

$$\phi = S_V \circ E^k \quad \text{and} \quad \xi_\phi = \xi_{S_V} \circ \xi_{E^k}.$$

where $V = \bigwedge^k \mathbb{K}^d$. Then ϕ is faithful and ξ_ϕ is continuous, transverse, and ϕ-equivariant. Furthermore, by Equations (2) and (3) in Section 2.5, we can identify $\mathrm{Her}(V)$ with $\mathbb{R}^{d_0}$ in such a way that

$$\alpha_1(\kappa(\phi(g))) = \alpha_k(\kappa(g))$$

for all $g \in \mathsf{PGL}(d, \mathbb{K})$. Via this identification,

$$\phi : \mathsf{PGL}(d, \mathbb{K}) \to \mathsf{PGL}(d_0, \mathbb{R}) \quad \text{and} \quad \xi_\phi : \mathcal{F}_{k,d-k}(\mathbb{K}^d) \to \mathcal{F}_{1,d_0-1}(\mathbb{R}^{d_0}),$$

and condition (2) holds.

To verify condition (3), it is enough to prove that $\xi_\phi(\Lambda_{k,d-k}(\Gamma)) = \Lambda_{1,d_0-1}(\phi(\Gamma))$. This follows from the following lemma.

Lemma 4.5. *Suppose $\{g_n\}$ is a sequence in $\mathsf{PGL}(d, \mathbb{K})$. If there exist $F^\pm \in \mathcal{F}_{k,d-k}(\mathbb{K}^d)$ such that*

$$\lim_{n\to\infty} g_n(F) = F^+ \tag{4}$$

for all $F \in \mathcal{F}_{k,d-k}(\mathbb{K}^d)$ transverse to F^-, then

$$\lim_{n\to\infty} \phi(g_n)(F) = \xi_\phi(F^+)$$

for all $F \in \mathcal{F}_{1,d_0-1}(\mathbb{R}^{d_0})$ transverse to $\xi_\phi(F^-)$.

Proof. Let $e_1, \dots, e_d$ denote the standard basis for $\mathbb{K}^d$. Using the Cartan decomposition we can write $g_n = k_{1,n} a_n k_{2,n}$ where $k_{1,n}, k_{2,n} \in \mathsf{PO}(d)$ and $a_n \in \exp(\mathfrak{a}^+)$. Let

$$F_0^+ = (\mathrm{Span}(e_1, \dots, e_k), \mathrm{Span}(e_1, \dots, e_{d-k}))$$

and

$$F_0^- = (\mathrm{Span}(e_{d-k+1}, \dots, e_d), \mathrm{Span}(e_{k+1}, \dots, e_d)).$$

Then $U_{k,d-k}(g_n) = k_{1,n}(F_0^+)$ and $U_{k,d-k}(g_n^{-1}) = k_{2,n}^{-1}(F_0^-)$. Since $\phi(g_n) = \phi(k_{1,n})\phi(a_n)\phi(k_{2,n})$ is the Cartan decomposition of $\phi(g_n)$, the ϕ-equivariance of ξ_ϕ implies

$$U_{1,d_0-1}(\phi(g_n)) = \xi_\phi(k_{1,n}(F_0^+)) \quad \text{and} \quad U_{1,d_0-1}(\phi(g_n^{-1})) = \xi_\phi(k_{2,n}^{-1}(F_0^-)).$$

By Lemma 2.1 and Equation (4),

$$\alpha_k(\kappa(g_n)) \to \infty, \quad k_{1,n}(F_0^+) \to F^+ \quad \text{and} \quad k_{2,n}^{-1}(F_0^-) \to F^-.$$

Then by the continuity of ξ_ϕ,

$$\alpha_1(\kappa(\phi(g_n))) \to \infty, \quad U_{1,d_0-1}(\phi(g_n)) \to \xi_\phi(F^+) \quad \text{and} \quad U_{1,d_0-1}(\phi(g_n^{-1})) \to \xi_\phi(F^-),$$

so Lemma 2.1 implies that

$$\lim_{n\to\infty} \phi(g_n)(F) = \xi_\phi(F^+)$$

for all $F \in \mathcal{F}_{1,d_0-1}(\mathbb{R}^{d_0})$ transverse to $\xi_\phi(F^-)$. □

Also, we observed in Section 2.5.2 that

$$\Omega_0 = \{[X] \in \mathbb{P}(\mathrm{Her}(V)) : X \text{ is positive definite}\}.$$

is a $S_V(\mathsf{PGL}(V))$-invariant properly convex domain. By the identification of $\mathrm{Her}(V)$ with $\mathbb{R}^{d_0}$, $\Omega_0 \subset \mathbb{P}(\mathbb{R}^{d_0})$ is a $\phi(\mathsf{PGL}(d,\mathbb{K}))$-invariant properly convex domain. Thus, condition (1) holds. □

Proof of Proposition 4.4. Suppose that $\Gamma \subset \mathsf{PGL}(d_0,\mathbb{R})$ is P_{1,d_0-1}-transverse and preserves a properly convex domain $\Omega_0 \subset \mathbb{P}(\mathbb{R}^{d_0})$. We will enlarge Ω_0 to a properly convex domain Ω so that $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ is a projectively visible subgroup of $\mathrm{Aut}(\Omega)$. If Γ_0 were irreducible it would suffice to consider the convex domain obtained by intersecting all half-spaces containing Ω_0 and bounded by hyperplanes in $\pi_{d-1}(\Lambda_{1,d-1}(\Gamma))$. In general, we will construct a properly convex domain D in Ω_0^* and let $\Omega = D^*$.

Let $B \subset \Omega_0^*$ be an open set whose closure is contained in Ω_0^* and let D denote the convex hull of $\Gamma(B)$ in Ω_0^*. Notice that, by construction, D is a non-empty properly convex domain in $\mathbb{P}(\mathbb{R}^{d_0 *})$ such that $\Gamma \subset \mathrm{Aut}(D)$. Set $\Omega = D^*$. By Lemma 2.6, $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is a properly convex domain and $\Gamma \subset \mathrm{Aut}(\Omega)$.

We will show that Γ is a visible subgroup of $\mathrm{Aut}(\Omega)$. Since Γ is P_{1,d_0-1}-transverse, Proposition 3.4 implies that

$$\Lambda_\Omega(\Gamma) = \pi_1(\Lambda_{1,d_0-1}(\Gamma)) = \Lambda_{\Omega_0}(\Gamma) \tag{5}$$

Applying Proposition 3.4 to the action of Γ on Ω_0^* shows

$$\overline{\Gamma(B)} = \Gamma(B) \cup \pi_{d_0-1}(\Lambda_{1,d_0-1}(\Gamma)) \tag{6}$$

(where we identify $\mathrm{Gr}_{d_0-1}(\mathbb{R}^{d_0}) = \mathbb{P}(\mathbb{R}^{d_0 *})$).

Lemma 4.6.

(1) If $x, y \in \Lambda_\Omega(\Gamma)$ are distinct points, then $(x,y)_\Omega \subset \Omega$.
(2) If $x \in \Lambda_\Omega(\Gamma)$, then $\partial\Omega$ is C^1 at x.

Proof. Let $\mathcal{C}$ be a component of the cone over Ω in $\mathbb{R}^{d_0} \setminus \{0\}$. Since $\overline{D}$ is the convex hull of $\overline{\Gamma(B)}$, Equation (6) implies that if $f \in \overline{D}$, then we may write $f = \sum_{j=1}^{\ell} f_j$ where $f_j|_{\mathcal{C}} > 0$ and

$$[f_j] \in \overline{\Gamma(B)} = \Gamma(B) \cup \pi_{d_0-1}(\Lambda_{1,d_0-1}(\Gamma)).$$

for all $j = 1, \ldots, \ell$. Moreover, if $[f] \in \partial D$, then

$$[f_j] \in \partial D \cap \overline{\Gamma(B)} = \pi_{d_0-1}(\Lambda_{1,d_0-1}(\Gamma)).$$

(1) Fix $p \in (x,y)_\Omega$. Let $\tilde{x}, \tilde{y} \in \partial\mathcal{C}$ be lifts of x, y. Then let $\tilde{p}$ be the lift of p contained in the line segment joining $\tilde{x}$ to $\tilde{y}$. If p lies in $\partial\Omega$, then there exists $f \in (\mathbb{R}^{d_0})^*$ so that $f(\tilde{p}) = 0$ and $[f] \in \partial D$. Write $f = \sum_{j=1}^{\ell} f_j$ where $f_j|_{\mathcal{C}} > 0$ and

$$[f_j] \in \pi_{d_0-1}(\Lambda_{1,d_0-1}(\Gamma)).$$

Fix $j \in \{1, \dots, \ell\}$. Since $f_j|_{\mathcal{C}} > 0$, we have $f_j(\tilde{x}) \geq 0$ and $f_j(\tilde{y}) \geq 0$. Since $\Lambda_{1,d-1}(\Gamma)$ is transverse, either $f_j(\tilde{x}) > 0$ or $f_j(\tilde{y}) > 0$, which implies that $f_j(\tilde{p}) > 0$. Since this is true for all j, $f(\tilde{p}) > 0$, and we have a contradiction.

(2) By Equation (5), there exists H_x such that $(x, H_x) \in \Lambda_{1,d_0-1}(\Gamma)$. Let $H \in \mathrm{Gr}_{d_0-1}(\mathbb{R}^{d_0})$ be a supporting hyperplane at x, namely $x \in \mathbb{P}(H)$ and $\mathbb{P}(H) \cap \Omega = \emptyset$. Since $D = \Omega^*$, there exists $[f] \in \partial D$, so that $\ker f = H$. Write $f = \sum_{j=1}^{\ell} f_j$ where $f_j(\mathcal{C}) > 0$ and

$$[f_j] \in \pi_{d_0-1}(\Lambda_{1,d_0-1}(\Gamma)).$$

Since $f(x) = 0$, we must have $f_j(x) = 0$ for all $j \in \{1, \dots, \ell\}$. Then, by transversality, $\ker f_j = H_x$ for all j. Hence, $H = \ker f = H_x$, so x is a C^1 point of $\partial\Omega$ with $T_x\partial\Omega = H_x$. □

It follows from the above lemma that Γ is a projectively visible subgroup of $\mathrm{Aut}(\Omega)$. □

5. Upper bounds on shadows and Hausdorff dimension

Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is a properly convex domain and $\Gamma \subset \mathrm{Aut}(\Omega)$ is a projectively visible subgroup. Equip Ω with its Hilbert metric d_Ω. Given $b, z \in \Omega$ and $r > 0$, we define the *shadow*

$$\mathcal{O}_r(b, z) \subset \partial\Omega$$

to be the set of points x where the geodesic ray $[b, x)_\Omega$ intersects the closed ball $\overline{B_\Omega(z, r)}$ of radius r centered at z. Then let

$$\widehat{\mathcal{O}}_r(b, z) = \mathcal{O}_r(b, z) \cap \Lambda_\Omega(\Gamma)$$

be the intersection of the shadow with the limit set. Notice that both $\mathcal{O}_r(b, z)$ and $\widehat{\mathcal{O}}_r(b, z)$ are closed subsets of $\mathbb{P}(\mathbb{R}^{d_0})$. Let $B_{\mathrm{Gr}_k(\mathbb{K}^d)}(z, t)$ denote the open ball of radius $t > 0$ about $z \in \mathrm{Gr}_k(\mathbb{K}^d)$ with respect to the distance defined in Section 2.1.

Our first shadow lemma gives an upper bound on the diameter of the image of a shadow.

Theorem 5.1. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is a properly convex domain, $\Gamma \subset \mathrm{Aut}(\Omega)$ is a projectively visible subgroup and $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is a P_θ-transverse representation with limit map $\xi : \Lambda_\Omega(\Gamma) \to \mathcal{F}_\theta$. For any $k \in \theta$, $b_0 \in \Omega$ and $r > 0$ there exists $C > 1$ so that if $x \in \Lambda_\Omega(\Gamma)$, $z \in [b_0, x)_\Omega$, and $\gamma \in \Gamma$ satisfy $\mathrm{d}_\Omega(z, \gamma(b_0)) < r$, then*

$$\xi^k\left(\widehat{\mathcal{O}}_r(b_0, z)\right) \subset B_{\mathrm{Gr}_k(\mathbb{K}^d)}\left(\xi^k(x), C\frac{\sigma_{k+1}(\rho(\gamma))}{\sigma_k(\rho(\gamma))}\right).$$

Proof. We first prove the theorem assuming that $k = 1$. Assume for contradiction that the theorem does not hold. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\Lambda_\Omega(\Gamma)$, $\{z_n\}$ in Ω and $\{\gamma_n\}$ in Γ so that for all n, we have $z_n \in [b_0, x_n)_\Omega$, $\mathrm{d}_\Omega(z_n, \gamma_n(b_0)) < r$, $y_n \in \widehat{\mathcal{O}}_r(b_0, z_n)$ and

$$\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}(\xi^1(x_n), \xi^1(y_n)) \geq n\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}. \tag{7}$$

We first observe that $\{\gamma_n\}$ is an escaping sequence in Γ. If not, then

$$\inf_{n\in\mathbb{Z}^+} \frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} > 0,$$

in which case (7) implies that $\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}(\xi(x_n), \xi(y_n)) \to \infty$, which is impossible. It follows that $\mathrm{d}_\Omega(b_0, \gamma_n(b_0)) \to \infty$, and hence that $\mathrm{d}_\Omega(b_0, z_n) \to \infty$.

For all $n \in \mathbb{N}$, choose $w_n \in B_\Omega(z_n, r) \cap [b_0, y_n)_\Omega$. Since $\{\gamma_n\}$ is an escaping sequence, we may pass to a subsequence so that

$$\gamma_n^{-1}(x_n) \to \bar{x} \in \Lambda_\Omega(\Gamma), \quad \gamma_n^{-1}(y_n) \to \bar{y} \in \Lambda_\Omega(\Gamma), \quad \gamma_n^{-1}(z_n) \to \bar{z} \in \Omega,$$

$$\gamma_n^{-1}(w_n) \to \bar{w} \in \Omega \quad \text{and} \quad \gamma_n^{-1}(b_0) \to \bar{b} \in \Lambda_\Omega(\Gamma).$$

Note that $\bar{z} \in (\bar{b}, \bar{x})_\Omega$ and $\bar{w} \in (\bar{b}, \bar{y})_\Omega$, which implies that $\bar{x} \neq \bar{b}$ and $\bar{y} \neq \bar{b}$, see Figure 1.

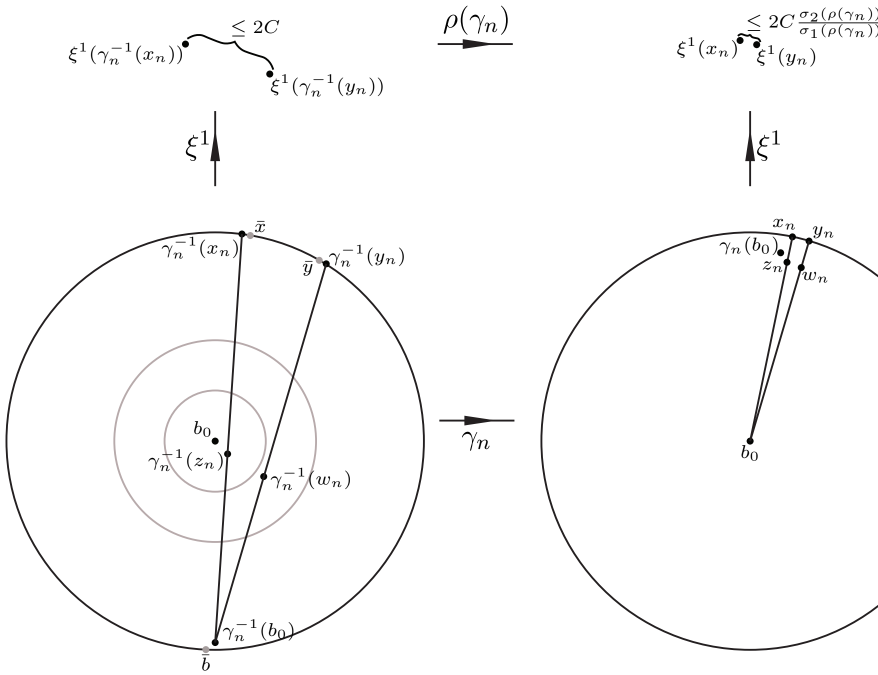

FIGURE 1. Upper bound on the size of shadow.

Let $\rho(\gamma_n) = m_n a_n \ell_n$ be the Cartan decomposition of $\rho(\gamma_n)$, where $m_n, \ell_n \in \mathsf{PU}(d, \mathbb{K})$ and $a_n \in \exp(\mathfrak{a}^+)$. For each n, let v_n and u_n be unit vectors so that

$$[v_n] = a_n^{-1} m_n^{-1}(\xi^1(x_n)) \quad \text{and} \quad [u_n] = a_n^{-1} m_n^{-1}(\xi^1(y_n)).$$

Then $\ell_n^{-1}([v_n]) = \xi^1(\gamma_n^{-1}(x_n))$ and $\ell_n^{-1}([u_n]) = \xi^1(\gamma_n^{-1}(y_n))$. Passing to a subsequence, we can suppose that $\ell_n \to \ell$, $v_n \to v$ and $u_n \to u$. Since $\gamma_n^{-1}(b_0) \to \bar{b}$ and ρ is P_θ-transverse, we may deduce from Lemma 2.1 that

$$\xi^{d-1}(\bar{b}) = \lim_{n \to \infty} U_{d-1}(\rho(\gamma_n^{-1})) = \ell^{-1}(\mathrm{Span}_\mathbb{K}(e_2, \ldots, e_d)).$$

Also, by the continuity of ξ^1,

$$\xi^1(\bar{x}) = \ell^{-1}([v]) \quad \text{and} \quad \xi^1(\bar{y}) = \ell^{-1}([u]).$$

Since $\bar{x} \neq \bar{b}$ and $\bar{y} \neq \bar{b}$, $\xi^1(\bar{x})$ and $\xi^1(\bar{y})$ are both transverse to $\xi^{d-1}(\bar{b})$, or equivalently, $[v]$ and $[u]$ are transverse to $\mathrm{Span}_\mathbb{K}(e_2, \ldots, e_d)$. This implies that there exists $C > 0$ so that

$$\tan \angle([v_n], [e_1]) \leq C \quad \text{and} \quad \tan \angle([u_n], [e_1]) \leq C$$

for all large enough n. Since $\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}$ is the angle metric,

$$\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(\xi^1(x_n), m_n([e_1])\right) = \mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(m_n^{-1}(\xi^1(x_n)), [e_1]\right) \leq \tan \angle(a_n([v_n]), [e_1]) \leq C\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}.$$

Similarly,

$$\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}(\xi^1(y_n), m_n([e_1])) \leq C\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))},$$

so

$$\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(\xi^1(x_n), \xi^1(y_n)\right) \leq 2C\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))},$$

which contradicts (7). This proves the theorem when $k = 1$.

If $k > 1$, consider the exterior power map $E^k : \mathsf{PGL}(d, \mathbb{K}) \to \mathsf{PGL}(D, \mathbb{K})$ and boundary map

$$\xi_{E^k} : \mathcal{F}_{k,d-k}(\mathbb{K}^d) \to \mathcal{F}_{1,D-1}(\mathbb{K}^D)$$

defined in Section 2.5. Recall that for all $g \in \mathsf{PGL}(d, \mathbb{K})$,

$$\alpha_k(\kappa(g)) = \alpha_1(\kappa(E^k(g))) \text{ and } U_{k,d-k}(g) = U_{1,D-1}(E^k(g)).$$

Since ρ is a P_θ-transverse representation and ξ_{E^k} is a transverse map,

$$E^k \circ \rho : \Gamma \to \mathsf{PGL}(D, \mathbb{K})$$

is a $P_{1,D-1}$-tranverse representation with limit map $\xi_{E^k} \circ \xi$. Furthermore, since ξ_{E^k} is a smooth embedding, there is some $C' > 1$ such that for any $x \in \Lambda_\Omega(\Gamma)$ and $t > 0$,

$$B_{\mathbb{P}(\mathbb{K}^D)}\left((\xi_{E^k} \circ \xi)^1(x), \frac{t}{C'}\right) \cap \xi^1_{E^k}(\mathrm{Gr}_k(\mathbb{K}^d)) \subset \xi^1_{E^k}\left(B_{\mathrm{Gr}_k(\mathbb{K}^d)}(\xi^k(x), t)\right) \subset B_{\mathbb{P}(\mathbb{K}^D)}\left((\xi_{E^k} \circ \xi)^1(x), C't\right).$$

This reduces to the case of $k = 1$.

□

As a corollary of Theorem 5.1, we get the following generalization of the work of Glorieux-Montclair-Tholozan [27, Thm. 4.1] and Pozzetti-Sambarino-Wienhard [43, Prop. 4.1].

Corollary 5.2. *If $\Gamma \subset \mathsf{PGL}(d, \mathbb{K})$ is $P_{k,d-k}$-transverse, then*

$$\dim_H\left(\Lambda_{k,d-k,c}(\Gamma)\right) \leq \delta^{\alpha_k}(\Gamma).$$

In particular, $\dim_H\left(\Lambda_{k,c}(\Gamma)\right) \leq \delta^{\alpha_k}(\Gamma)$.

Proof. By Theorem 4.2, there is a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ for some properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$, and a P_θ-transverse representation $\rho : \Gamma_0 \to \mathsf{PGL}(d, \mathbb{K})$ with limit map ξ, such that $\rho(\Gamma_0) = \Gamma$ and $\xi(\Lambda_\Omega(\Gamma_0)) = \Lambda_{k,d-k}(\Gamma)$ for all $\gamma \in \Gamma_0$. Since ξ is ρ-equivariant, injective, and continuous, we have

$$\Lambda_{k,d-k,c}(\Gamma) = \xi(\Lambda_{\Omega,c}(\Gamma_0)).$$

Fix a base point $b_0 \in \Omega$. For $r > 0$, let $L_r \subset \Lambda_{\Omega,c}(\Gamma_0)$ denote the set of conical limit points x with the property that there exists a sequence $\{\gamma_n\}$ in Γ_0 such that $\gamma_n(b_0) \to x$ and $\mathrm{d}_\Omega(\gamma_n(b_0), [b_0, x)_\Omega) < r$ for all n. Then by definition

$$\Lambda_{\Omega,c}(\Gamma_0) = \bigcup_{r=1}^{\infty} L_r.$$

Fix, for the moment, $r > 0$. For $\gamma \in \Gamma$ define

$$a(\gamma) = \min\{\alpha_k(\kappa(\rho(\gamma))), \alpha_{d-k}(\kappa(\rho(\gamma)))\}.$$

Then Theorem 5.1 guarantees that there exists $C = C(r) > 1$ so that

$$\operatorname{diam} \xi\left(\widehat{\mathcal{O}}_r(b_0, \gamma(b_0))\right) \leq Ce^{-a(\gamma)}. \tag{8}$$

Further, for any N,

$$L_r \subset \bigcup_{\{\gamma \in \Gamma_0 \ \mid \ a(\gamma) > N\}} \widehat{\mathcal{O}}_r(b_0, \gamma(b_0)).$$

By (8) the diameter of each element of this covering goes to 0 as $N \to \infty$ and

$$\sum_{\{\gamma\in\Gamma_0 \ | \ a(\gamma)>N\}} \left(\operatorname{diam} \xi\left(\widehat{\mathcal{O}}_r\big(b_0, \gamma(b_0)\big)\right)\right)^s \leq C^s \sum_{\gamma\in\Gamma} e^{-sa(\gamma)}$$
$$\leq C^s \left(\sum_{\gamma\in\Gamma} e^{-s\alpha_k(\kappa(\rho(\gamma)))} + \sum_{\gamma\in\Gamma} e^{-s\alpha_k(\kappa(\rho(\gamma)))}\right)$$

is finite when $s > \max\{\delta^{\alpha_k}(\Gamma), \delta^{\alpha_{d-k}}(\Gamma)\}$. Note that $\delta^{\alpha_k}(\Gamma) = \delta^{\alpha_{d-k}}(\Gamma)$ because $\alpha_k(\kappa(\rho(\gamma))) = \alpha_{d-k}(\kappa(\rho(\gamma^{-1})))$ for all $\gamma \in \Gamma$. Therefore,

$$\dim_H(\xi(L_r)) \leq \delta^{\alpha_k}(\Gamma).$$

Since r was arbitrary, we see that

$$\dim_H(\Lambda_{k,d-k,c}(\Gamma)) = \dim_H(\xi(\Lambda_{\Omega,c}(\Gamma_0))) = \sup_{r\in\mathbb{N}} \dim_H(\xi(L_r)) \leq \delta^{\alpha_k}(\Gamma).$$

For the second statement, observe that $\dim_H(\Lambda_{k,c}(\Gamma)) \leq \dim_H(\Lambda_{k,d-k,c}(\Gamma)) \leq \delta^{\alpha_k}(\Gamma)$. □

We use Corollary 5.2 and results from the theory of Kleinian groups to prove Theorem 1.4.

Theorem 5.3. *Let $\rho_1 : \Gamma \to \mathsf{SO}(d_1 - 1, 1)$ and $\rho_2 : \Gamma \to \mathsf{SO}(d_2 - 1, 1)$ be faithful geometrically finite representations so that $\rho_1(\alpha)$ is parabolic if and only if $\rho_2(\alpha)$ is parabolic. If we regard $\rho = \rho_1 \oplus \rho_2$ as a representation into $\mathsf{PSL}(d_1 + d_2, \mathbb{R})$, then*

$$\dim_H(\Lambda_2(\rho(\Gamma))) = \max\{\dim_H(\Lambda_1(\rho_1(\Gamma))), \dim_H(\Lambda_1(\rho_2(\Gamma)))\}.$$

Proof. Let $\mathcal{P}$ denote the collection of maximal subgroups of Γ whose images under each ρ_i consist entirely of parabolic (or trivial) elements of $\mathsf{SO}(d_i - 1, 1)$. Then $(\Gamma, \mathcal{P})$ is relatively hyperbolic and there exists a ρ_i-equivariant homeomorphism $\nu_i : \partial(\Gamma, \mathcal{P}) \to \Lambda_1(\rho_i(\Gamma))$ where $\partial(\Gamma, \mathcal{P})$ is the Bowditch boundary of $(\Gamma, \mathcal{P})$. Moreover, $\Lambda_{1,c}(\rho_i(\Gamma))$ is the complement of the images of fixed points of elements of $\mathcal{P}$ (See Bowditch [10, Sec. 9] and Tukia [55]). One may then easily check that

$$\Lambda_2(\rho(\Gamma)) = \{\langle \nu_1(x), \nu_2(x)\rangle \ : \ x \in \partial(\Gamma, \mathcal{P})\} \subset \mathrm{Gr}_2(\mathbb{R}^{d_1+d_2}),$$

that $\Lambda_{2,c}(\rho(\Gamma))$ is the complement of the images of fixed points of elements of $\mathcal{P}$, and that ρ is P_2-transverse. Then, Corollary 5.2 implies that

$$\dim_H(\Lambda_2(\rho(\Gamma))) = \dim_H(\Lambda_{2,c}(\rho(\Gamma))) \leq \delta^{\alpha_2}(\rho).$$

Since $\sigma_2(\rho(\gamma)) = \min\{\sigma_1(\rho_1(\gamma), \sigma_1(\rho_2(\gamma))\}$ for all $\gamma \in \Gamma$

$$\#\{\gamma \in \Gamma \ : \ \alpha_2(\rho(\gamma)) \leq T\} \leq \#\{\gamma \in \Gamma \ : \ \alpha_1(\rho_1(\gamma)) \leq T\} + \#\{\gamma \in \Gamma \ : \ \alpha_1(\rho_2(\gamma)) \leq T\}$$

which implies that

$$\delta^{\alpha_2}(\rho) \leq \max\{\delta^{\alpha_1}(\rho_1), \delta^{\alpha_1}(\rho_2)\}.$$

Sullivan [53] showed that $\dim_H(\Lambda_1(\rho_i(\Gamma_1))) = \delta^{\alpha_1}(\rho_i)$, so we may combine these results to see that

$$\dim_H(\Lambda_2(\rho(\Gamma))) \leq \max\{\dim_H(\Lambda_1(\rho_1(\Gamma)), \dim_H(\Lambda_1(\rho_2(\Gamma))\}.$$

Since there is a smooth projection of $\Lambda_2(\rho(\Gamma))$ onto $\Lambda_1(\rho_i(\Gamma))$, for both $i = 1, 2$, we see that equality must hold. □

Remark.

(1) It is possible to give an elementary proof of Corollary 5.2 without developing the theory of projectively visible subgroups. However, the current approach also yields additional structural informations about shadows which may be more broadly applicable.

(2) In the proof of Theorem 1.4 one may alternatively check that the assumptions of Corollary 5.2 hold by verifying that ρ is P_2-relatively dominated in the sense of Zhu [56].

6. Singular values and radial projection

In this section, we first show that singular values of products of elements in the image of a P_θ-transverse representation satisfy a coarsely multiplicative lower bound if the orbit of the elements in the domain proceed towards infinity "without backtracking." To quantify "without backtracking" we use the radial projection map introduced in Section 2.4. We will use this result to show that simple root functionals satisfy a coarsely additive lower bound in the absence of backtracking.

Lemma 6.1. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is properly convex, $\Gamma \subset \mathrm{Aut}(\Omega)$ is projectively visible and $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is a P_θ-transverse representation. For any $b_0 \in \Omega$ and $\epsilon > 0$ there exists $C > 0$ so that: if $\gamma, \eta \in \Gamma$,*

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\Big(\iota_{b_0}(\gamma^{-1}(b_0)), \iota_{b_0}(\eta(b_0))\Big) \geq \epsilon$$

and $k \in \theta$, then

$$\sigma_k(\rho(\gamma\eta)) \geq C\sigma_k(\rho(\gamma))\sigma_k(\rho(\eta)).$$

Proof. If not, there exist sequences $\{\gamma_n\}$ and $\{\eta_n\}$ in Γ and $k \in \theta$, such that

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\Big(\iota_{b_0}(\gamma_n^{-1}(b_0)), \iota_{b_0}(\eta_n(b_0))\Big) \geq \epsilon$$

for all $n \geq 1$ and

$$\lim_{n\to\infty} \frac{\sigma_k(\rho(\gamma_n\eta_n))}{\sigma_k(\rho(\gamma_n))\sigma_k(\rho(\eta_n))} = 0.$$

Since $\sigma_k(\rho(\gamma_n\eta_n)) \geq \max\{\sigma_k(\rho(\gamma_n))\sigma_d(\rho(\eta_n)), \sigma_d(\rho(\gamma_n))\sigma_k(\rho(\eta_n))\}$, this is only possible if $\{\gamma_n\}$ and $\{\eta_n\}$ are escaping sequences. So we can pass to subsequences so that $\gamma_n^{-1}(b_0) \to x \in \Lambda_\Omega(\Gamma)$ and $\eta_n(b_0) \to y \in \Lambda_\Omega(\Gamma)$, in which case $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x, y) \geq \epsilon$.

Consider the Cartan decompositions

$$\rho(\gamma_n) = m_n a_n \ell_n \quad \text{and} \quad \rho(\eta_n) = \hat{m}_n \hat{a}_n \hat{\ell}_n,$$

where $m_n, \ell_n, \hat{m}_n, \hat{\ell}_n \in \mathsf{PU}(d, \mathbb{K})$, and $a_n, \hat{a}_n \in \exp(\mathfrak{a}^+)$. Let $W = \mathrm{Span}_{\mathbb{K}}(e_1, \dots, e_k)$ and $W^\perp = \mathrm{Span}_{\mathbb{K}}(e_{k+1}, \dots, e_d)$. Then let

$$\pi_W : \mathbb{K}^d \to W \quad \text{and} \quad \pi_{W^\perp} : \mathbb{K}^d \to W^\perp$$

denote the orthogonal projections.

Let $\xi : \Lambda_\Omega(\Gamma) \to \mathcal{F}_\theta$ be the limit map of ρ. Since ρ is P_θ-transverse, Lemma 2.1 implies that

$$\ell_n^{-1}(W^\perp) = U_{d-k}(\rho(\gamma_n)^{-1}) \to \xi^{d-k}(x) \quad \text{and} \quad \hat{m}_n(W) = U_k(\rho(\eta_n)) \to \xi^k(y).$$

Since $x \neq y$, $\xi(x)$ is transverse to $\xi(y)$, so there is some $c > 0$ such that for large enough n, all $u \in W^\perp - \{0\}$, and all $v \in W - \{0\}$, we have

$$\angle(u, \ell_n\hat{m}_n(v)) = \angle(\ell_n^{-1}(u), \hat{m}_n(v)) \geq c.$$

Hence, there is some $C > 0$ such that for large enough n and all $v \in W$,

$$\|\pi_W(\ell_n\hat{m}_n(v))\| \geq C\,\|v\|\,.$$

Then for all $v \in W$ we have

$$\begin{aligned}\left\|\rho(\gamma_n\eta_n)\hat{\ell}_n^{-1}(v)\right\| &= \|a_n\ell_n\hat{m}_n\hat{a}_n(v)\| \geq \|\pi_W(a_n\ell_n\hat{m}_n\hat{a}_n(v))\| = \|a_n\pi_W(\ell_n\hat{m}_n\hat{a}_n(v))\| \\ &\geq \sigma_k(\rho(\gamma_n))\,\|\pi_W(\ell_n\hat{m}_n\hat{a}_n(v))\| \geq C\sigma_k(\rho(\gamma_n))\,\|\hat{a}_n(v)\| \geq C\sigma_k(\rho(\gamma_n))\sigma_k(\rho(\eta_n))\,\|v\|\,.\end{aligned}$$

So by the max-min characterization of singular values

$$\frac{\sigma_k(\rho(\gamma_n\eta_n))}{\sigma_k(\rho(\gamma_n))\sigma_k(\rho(\eta_n))} \geq \min_{v\in W, \|v\|=1} \frac{\left\|\rho(\gamma_n\eta_n)\hat{\ell}_n^{-1}(v)\right\|}{\sigma_k(\rho(\gamma_n))\sigma_k(\rho(\eta_n))} \geq C$$

and we have a contradiction. □

As an immediate consequence we see that the first fundamental weight is coarsely additive in the absence of backtracking.

Lemma 6.2. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is properly convex, $\Gamma \subset \mathrm{Aut}(\Omega)$ is projectively visible and $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is a P_θ-transverse representation. For any $b_0 \in \Omega$ and $\epsilon > 0$ there exists $C > 0$ so that: if $\gamma, \eta \in \Gamma$,*

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\Big(\iota_{b_0}(\gamma^{-1}(b_0)), \iota_{b_0}(\eta(b_0))\Big) \geq \epsilon$$

and $k \in \theta$, then

$$(\sigma_1 \cdots \sigma_k)(\rho(\gamma\eta)) \geq C(\sigma_1 \cdots \sigma_k)(\rho(\gamma))(\sigma_1 \cdots \sigma_k)(\rho(\eta)).$$

Proof. Let $E^k : \mathsf{PGL}(d, \mathbb{K}) \to \mathsf{PGL}(\bigwedge^k \mathbb{K}^d)$ be the exterior power representation, see Section 2.5. Since $k \in \theta$, the representation $E^k \circ \rho$ is P_{1,d_1-1}-transverse where $d_1 = \dim_{\mathbb{K}} \bigwedge^k \mathbb{K}^d$. Further, if we fix the standard inner product on $\bigwedge^k \mathbb{K}^d$, then

$$(\sigma_1 \cdots \sigma_k)(g) = \sigma_1\left(E^k(g)\right)$$

for all $g \in \mathsf{PGL}(d, \mathbb{K})$. So Lemma 6.1 immediately implies the result. □

The proofs of the next two results will use the following well known estimate.

Observation 6.3. *If $g, h \in \mathsf{PGL}(d, \mathbb{K})$, then*

$$(\sigma_1 \cdots \sigma_k)(gh) \leq (\sigma_1 \cdots \sigma_k)(g)(\sigma_1 \cdots \sigma_k)(h).$$

Proof. By definition $\sigma_1(gh) \leq \sigma_1(g)\sigma_1(h)$. For $k > 1$,

$$(\sigma_1 \cdots \sigma_k)(gh) = \sigma_1\left(E^k(gh)\right) \leq \sigma_1\left(E^k(g)\right)\sigma_1\left(E^k(h)\right) = (\sigma_1 \cdots \sigma_k)(g)(\sigma_1 \cdots \sigma_k)(h).$$ □

We next show that if $k \in \theta$, then the k^{th} simple root of the Cartan projection has a coarsely additive lower bound if there is no backtracking.

Lemma 6.4. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is properly convex, $\Gamma \subset \mathrm{Aut}(\Omega)$ is projectively visible and $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is a P_θ-transverse representation. For any $b_0 \in \Omega$ and $\epsilon > 0$ there exists $C > 0$ so that: if $\gamma, \eta \in \Gamma$,*

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\Big(\iota_{b_0}(\gamma^{-1}(b_0)), \iota_{b_0}(\eta(b_0))\Big) \geq \epsilon$$

and $k \in \theta$, then

$$\alpha_k(\kappa(\rho(\gamma\eta))) \geq \alpha_k(\kappa(\rho(\gamma))) + \alpha_k(\kappa(\rho(\eta))) - C.$$

Proof. By Lemmas 6.1 and 6.2 there exist $C_1, C_2 > 0$, which depend on ϵ, but are independent of γ and η, such that

$$\sigma_k(\rho(\gamma\eta)) \geq C_1 \sigma_k(\rho(\gamma))\sigma_k(\rho(\eta))$$

and

$$(\sigma_1 \cdots \sigma_k)(\rho(\gamma\eta)) \geq C_2(\sigma_1 \cdots \sigma_k)(\rho(\gamma))(\sigma_1 \cdots \sigma_k)(\rho(\eta)).$$

Combining these facts with Observation 6.3, we see that

$$\begin{aligned}
\alpha_k(\kappa(\rho(\gamma\eta))) &= \log\left(\frac{\sigma_k(\rho(\gamma\eta))}{\sigma_{k+1}(\rho(\gamma\eta))}\right) = \log\left(\frac{(\sigma_1 \cdots \sigma_k)(\rho(\gamma\eta))}{(\sigma_1 \cdots \sigma_{k+1})(\rho(\gamma\eta))}\sigma_k(\rho(\gamma\eta))\right) \\
&\geq \log\left(C_1 C_2 \frac{(\sigma_1 \cdots \sigma_k)(\rho(\gamma))(\sigma_1 \cdots \sigma_k)(\rho(\eta))}{(\sigma_1 \cdots \sigma_{k+1})(\rho(\gamma))(\sigma_1 \cdots \sigma_{k+1})(\rho(\eta))}\sigma_k(\rho(\gamma))\sigma_k(\rho(\eta))\right) \\
&= \log(C_1 C_2) + \alpha_k(\kappa(\rho(\gamma))) + \alpha_k(\kappa(\rho(\eta))).
\end{aligned}$$

So, Lemma 6.4 holds with $C = -\log(C_1 C_2)$. □

A very similar argument shows that simple roots of the Cartan projection sometimes admit a coarsely additive upper bound if there is no backtracking. We will apply this result to the special case of Hitchin representations

Lemma 6.5. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is properly convex, $\Gamma \subset \mathrm{Aut}(\Omega)$ is projectively visible and $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is a P_θ-transverse representation. For any $b_0 \in \Omega$ and $\epsilon > 0$ there exists $C > 0$ so that: if $\gamma, \eta \in \Gamma$,*

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\Big(\iota_{b_0}(\gamma^{-1}(b_0)), \iota_{b_0}(\eta(b_0))\Big) \geq \epsilon$$

and $k-1, k, k+1 \in \theta \cup \{0, d\}$, then

$$\alpha_k(\kappa(\rho(\gamma\eta))) \leq \alpha_k(\kappa(\rho(\gamma))) + \alpha_k(\kappa(\rho(\eta))) + C.$$

Proof. In the following argument, if $k = 1$, then we use the convention that $(\sigma_1 \cdots \sigma_{k-1})(g) = 1$ for all $g \in \mathsf{PGL}(d, \mathbb{K})$.

By Lemmas 6.1 and 6.2 there exist $C_1, C_2 > 0$, which depend only on ϵ, such that

$$\sigma_{k+1}(\rho(\gamma\eta)) \geq C_1 \sigma_{k+1}(\rho(\gamma))\sigma_{k+1}(\rho(\eta))$$

(this is obvious when $k + 1 = d$) and

$$(\sigma_1 \cdots \sigma_{k-1})(\rho(\gamma\eta)) \geq C_2(\sigma_1 \cdots \sigma_{k-1})(\rho(\gamma))(\sigma_1 \cdots \sigma_{k-1})(\rho(\eta)).$$

Combining these facts with Observation 6.3 we see that

$$\begin{aligned}
\alpha_k(\kappa(\rho(\gamma\eta))) &= \log\left(\frac{\sigma_k(\rho(\gamma\eta))}{\sigma_{k+1}(\rho(\gamma\eta))}\right) = \log\left(\frac{(\sigma_1 \cdots \sigma_k)(\rho(\gamma\eta))}{(\sigma_1 \cdots \sigma_{k-1})(\rho(\gamma\eta))\sigma_{k+1}(\rho(\gamma\eta))}\right) \\
&\leq \log\left(\frac{\sigma_k(\rho(\gamma))\sigma_k(\rho(\eta))}{C_1 C_2 \sigma_{k+1}(\rho(\gamma))\sigma_{k+1}(\rho(\eta))}\right) \\
&= -\log(C_1 C_2) + \alpha_k(\kappa(\rho(\gamma))) + \alpha_k(\kappa(\rho(\eta))),
\end{aligned}$$

which completes the proof. □

Finally, we obtain a lower bound which applies when $\gamma(b_0)$ lies within a bounded distance of the projective line segment joining b_0 to $\eta(b_0)$.

Lemma 6.6. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is properly convex, $\Gamma \subset \mathrm{Aut}(\Omega)$ is projectively visible and $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is a P_θ-transverse representation. For any $b_0 \in \Omega$ and $r > 0$ there exists $C > 0$ so that: if $\gamma, \eta \in \Gamma$,*

$$\mathrm{d}_\Omega\left(\gamma(b_0), [b_0, \eta(b_0)]_\Omega\right) \leq r$$

and $k \in \theta$, then

$$\alpha_k(\kappa(\rho(\eta))) \geq \alpha_k(\kappa(\rho(\gamma))) + \alpha_k(\kappa(\rho(\gamma^{-1}\eta))) - C.$$

Proof. If not, there exist sequences $\{\gamma_n\}$ and $\{\eta_n\}$ in Γ where

$$\mathrm{d}_\Omega\left(\gamma_n(b_0), [b_0, \eta_n(b_0)]_\Omega\right) \leq r$$

and

$$\alpha_k(\kappa(\rho(\eta_n))) \leq \alpha_k(\kappa(\rho(\gamma_n))) + \alpha_k(\kappa(\rho(\gamma_n^{-1}\eta_n))) - n.$$

We claim that $\{\gamma_n^{-1}\eta_n\}$ and $\{\gamma_n\}$ are escaping sequences. Notice that

$$\begin{aligned}\alpha_k(\kappa(\rho(\gamma_n))) &\geq n + \alpha_k(\kappa(\rho(\eta_n))) - \alpha_k(\kappa(\rho(\gamma_n^{-1}\eta_n))) \\ &\geq n - \log \frac{\sigma_1(\rho(\gamma_n)))}{\sigma_d(\rho(\gamma_n)))}.\end{aligned}$$

So $\{\gamma_n\}$ must be escaping. A similar argument shows that $\{\gamma_n^{-1}\eta_n\}$ is escaping. Then

$$\mathrm{d}_\Omega\left(b_0, [\gamma_n^{-1}(b_0), \gamma_n^{-1}\eta_n(b_0)]_\Omega\right) = \mathrm{d}_\Omega\left(\gamma_n(b_0), [b_0, \eta_n(b_0)]_\Omega\right) \leq r,$$

so

$$\liminf_{n\to\infty} \mathrm{d}_{\mathbb{P}(\mathbb{R}^d)}\left(\iota_{b_0}(\gamma_n^{-1}(b_0)), \iota_{b_0}(\gamma_n^{-1}\eta_n(b_0))\right) > 0.$$

By Lemma 6.4 there exists some $C > 0$ such that

$$\alpha_k(\kappa(\rho(\eta_n))) \geq \alpha_k(\kappa(\rho(\gamma_n))) + \alpha_k(\kappa(\rho(\gamma_n^{-1}\eta_n))) - C$$

for n sufficiently large. So we have a contradiction. □

7. Lower bounds on shadows

We will need to restrict to a special class of transverse representations to obtain lower bounds on the inradii of images of shadows. Recall, from the introduction, that a discrete subgroup $\Gamma_0 \subset \mathsf{PGL}(d, \mathbb{K})$ is $(1, 1, q)$-hypertransverse if Γ_0 is P_θ-transverse for some θ containing 1 and q, and

$$F^1 + G^1 + H^{d-q}$$

is a direct sum for all pairwise distinct $F, G, H \in \Lambda_\theta(\Gamma_0)$. Then we say that a P_θ-transverse representation $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is $(1, 1, q)$-*hypertransverse* if $1, q \in \theta$ and $\rho(\Gamma)$ is a $(1, 1, q)$-hypertransverse subgroup of $\mathsf{PGL}(d, \mathbb{K})$. Theorem 4.2 implies that every $(1, 1, q)$-hypertransverse subgroup is the image of a $(1, 1, q)$-hypertransverse representation.

We obtain a bound on the inradius of limit sets of hypertransverse representations at uniformly conical limit points, which generalizes work of Pozzetti-Sambarino-Wienhard [43] from the Anosov setting. For any $b_0 \in \Omega$ and $R > 0$, let

$$\Lambda_{b_0,R}(\Gamma) \subset \Lambda_\Omega(\Gamma)$$

denote the set of points $x \in \Lambda_\Omega(\Gamma)$ such that the geodesic ray $[b_0, x)_\Omega$ lies in a closed neighborhood of radius R of the orbit of b_0, i.e.

$$[b_0, x)_\Omega \subset \Gamma\left(\overline{B_\Omega(b_0, R)}\right).$$

We say that such limit points are R-*uniformly conical* from b_0.

Theorem 7.1. *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ *is a properly convex domain,* $\Gamma \subset \mathrm{Aut}(\Omega)$ *is a projectively visible subgroup,* $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ *is* $(1,1,q)$*-hypertransverse with limit map* $\xi : \Lambda_\Omega(\Gamma) \to \mathcal{F}_{1,q,d-q,d-1}$ *and*

$$\sigma_2(\rho(\gamma)) = \sigma_q(\rho(\gamma))$$

for all $\gamma \in \Gamma$. *For any* $b_0 \in \Omega$ *and* $r, R > 0$, *there exists* $C > 1$ *so that: if* $x \in \Lambda_{b_0,R}(\Gamma)$, $z \in [b_0, x)_\Omega$ *and* $\gamma \in \Gamma$ *satisfy* $\mathrm{d}_\Omega(z, \gamma(b_0)) < r$, *then*

$$B_{\mathbb{P}(\mathbb{K}^d)}\left(x, \frac{1}{C}\frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right) \cap \Lambda_1(\rho(\Gamma)) \subset \xi^1\left(\widehat{\mathcal{O}}_r(b_0, z)\right).$$

7.1. **Projection onto a line.** One major tool in the proof will be a projection of $\partial\Omega$ onto the projective line through a point $x \in \Lambda_\Omega(\Gamma)$ and the basepoint b_0.

If $x \in \Lambda_\Omega(\Gamma)$, let $x_{\mathrm{opp}} \in \partial\Omega \setminus \{x\}$ be the other point of intersection of the projective line through b_0 and x with $\partial\Omega$. For every $x \in \partial\Omega$, fix a supporting hyperplane H_x to Ω at x. Notice that H_x may intersect $\partial\Omega - \{x\}$, but it does not contain x_{opp} since $(x, x_{\mathrm{opp}}) \subset \Omega$. Moreover, if $x \in \Lambda_\Omega(\Gamma)$, then $H_x = T_x\partial\Omega$ and H_x cannot intersect $\Lambda_\Omega(\Gamma) - \{x\}$. Thus, for each $x \in \Lambda_\Omega(\Gamma)$, the codimension 2 projective subspace

$$W_x = H_x \cap H_{x_{\mathrm{opp}}}$$

does not intersect $\Lambda_\Omega(\Gamma)$, so we may define

$$\pi_x : \Lambda_\Omega(\Gamma) \to [x_{\mathrm{opp}}, x]_\Omega$$

by $\pi_x(y) = [x_{\mathrm{opp}}, x]_\Omega \cap (y \oplus W_x)$. Note that $\pi_x(y) = x$ if and only if $y = x$.

Remark 7.2. When Ω is the Klein-Beltrami model of real hyperbolic d-space, then $\pi_x(y)$ is the orthogonal projection of y onto the geodesic $(x_{\mathrm{opp}}, x)_\Omega$.

The construction of the map π_x involves choosing a supporting hyperplane at each boundary point. So, when $\partial\Omega$ is not C^1, there is no reason to expect that $\pi_x(y)$ or W_x varies continuously with x. However, the following weak continuity property will suffice for our purposes.

Lemma 7.3. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* $\Lambda_\Omega(\Gamma)$ *such that* $x_n \to x$ *and* $y_n \to y$. *Then* $x = y$ *if and only if* $\pi_{x_n}(y_n) \to x$.

Proof. Since $\overline{\Omega}$ is compact, it suffices to consider the case when $\lim_{n\to\infty} \pi_{x_n}(y_n)$ exists.

By taking a subsequence, we may assume that $H_{x_{n,\mathrm{opp}}} \to H$ for some supporting hyperplane H to Ω at x_{opp}. Since $x_n \in \Lambda_\Omega(\Gamma)$ for all n and Γ is a projectively visible, we have

$$H_{x_n} = T_{x_n}\partial\Omega \to T_x\partial\Omega = H_x,$$

so $W_{x_n} \to W = H_x \cap H$. (It is possible that $H \neq H_{x_{\mathrm{opp}}}$ and $W \neq W_x$.)

First, suppose that $x = y$. By definition, $\pi_{x_n}(y_n) \in y_n \oplus W_{x_n}$, which implies

$$\lim_{n\to\infty} \pi_{x_n}(y_n) \in y \oplus W = x \oplus W = H_x.$$

Since $\pi_{x_n}(y_n) \in [x_{n,\mathrm{opp}}, x_n]_\Omega$ for all n, we have

$$\lim_{n\to\infty} \pi_{x_n}(y_n) \in [x_{\mathrm{opp}}, x]_\Omega.$$

It follows that $\pi_{x_n}(y_n) \to H_x \cap [x_{\mathrm{opp}}, x]_\Omega = x$.

Conversely, suppose that $\pi_{x_n}(y_n) \to x$. Since $y_n \in W_{x_n} \oplus \pi_{x_n}(y_n)$, this implies that $y \in W \oplus x = H_x$, which is possible only if $x = y$. □

As a consequence, we prove that if y is close enough to x, then $\pi_x(y) \in (b_0, x)$.

Lemma 7.4. *There exists $\delta > 0$ so that if $x, y \in \Lambda_\Omega(\Gamma)$ and $0 < \mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x, y) \le \delta$, then $\pi_x(y) \in (b_0, x)_\Omega$.*

Proof. If not, then there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\Lambda_\Omega(\Gamma)$ such that

$$\pi_{x_n}(y_n) \in [x_{n,\mathrm{opp}}, b_0]_\Omega \quad \text{and} \quad 0 < \mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x_n, y_n) < 1/n$$

for all n. Passing to a subsequence, we can suppose that

$$x_n \to x \in \Lambda_\Omega(\Gamma), \quad y_n \to y \in \Lambda_\Omega(\Gamma) \quad \text{and} \quad \pi_{x_n}(y_n) \to p \in [x_{\mathrm{opp}}, b_0]_\Omega.$$

Since $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x_n, y_n) \to 0$, we have that $x = y$, so Lemma 7.3 implies that $p = x$, which is a contradiction. □

The following lemma shows that if $z \in [b_0, x)_\Omega$, z is near enough to the orbit $\Gamma(b_0)$ and $\pi_x(y)$ lies between z and x and far enough from z, then y lies in the shadow of z from b_0.

Lemma 7.5. *Given $r, r' > 0$, there exists $T > 0$ such that if $x, y \in \Lambda_\Omega(\Gamma)$, $z \in [b_0, x)_\Omega$, $\mathrm{d}_\Omega(z, \Gamma(b_0)) \le r'$, $\pi_x(y) \in (z, x)_\Omega$ and $d_\Omega(\pi_x(y), z) \ge T$, then $y \in \mathcal{O}_r(b_0, z)$.*

Proof. If not, there exist sequences $\{x_n\}$, $\{y_n\}$ in $\Lambda_\Omega(\Gamma)$ and $\{z_n\}$ in Ω so that

$$z_n \in [b_0, x_n)_\Omega, \quad \mathrm{d}_\Omega(z_n, \Gamma(b_0)) \le r', \quad \pi_{x_n}(y_n) \in (z_n, x_n)_\Omega,$$
$$\mathrm{d}_\Omega\left(\pi_{x_n}(y_n), z_n\right) \ge n \quad \text{and} \quad y_n \notin \mathcal{O}_r(b_0, z_n).$$

For all $n \in \mathbb{N}$, choose $\gamma_n \in \Gamma$ so that $\mathrm{d}_\Omega(\gamma_n^{-1}(z_n), b_0) \le r'$ and pass to a subsequence so that

$$\gamma_n^{-1}(z_n) \to \bar{z} \in \Omega, \quad \gamma_n^{-1}(b_0) \to \bar{b} \in \overline{\Omega}, \quad \gamma_n^{-1}(x_n) \to \bar{x} \in \Lambda_\Omega(\Gamma), \quad \gamma_n^{-1}(x_{n,\mathrm{opp}}) \to \hat{x} \in \partial\Omega,$$
$$\gamma_n^{-1}(y_n) \to \bar{y} \in \Lambda_\Omega(\Gamma) \quad \text{and} \quad \gamma_n^{-1}(H_{x_n,\mathrm{opp}}) \to \bar{H}.$$

Let $\bar{W} = \bar{H} \cap H_{\bar{x}}$, and note that $\gamma_n^{-1}(W_{x_n}) \to \bar{W}$.

Notice that $\bar{z} \in [\bar{b}, \bar{x}]_\Omega \cap \Omega$, hence $\bar{b} \neq \bar{x}$, which implies that $\bar{b} \in [\hat{x}, \bar{x})_\Omega$. If $\bar{b} = \hat{x}$, then $\hat{x} \in \Lambda_\Omega(\Gamma)$, and the visibility of Γ implies that $(\hat{x}, \bar{x})_\Omega \subset \Omega$. If $\bar{b} \neq \hat{x}$, then $\bar{b} \in \Omega$, and the convexity of Ω implies that $(\hat{x}, \bar{x})_\Omega \subset \Omega$. In either case, since $\bar{H}$ is a supporting hyperplane to Ω at $\hat{x}$, it follows that $\bar{x} \notin \bar{H}$, which implies that $\bar{x} \notin \bar{W}$.

Since $\gamma_n^{-1}(\pi_{x_n}(y_n)) \in (\gamma_n^{-1}(z_n), \gamma_n^{-1}(x_n))_\Omega$, and

$$\lim_{n\to\infty} \mathrm{d}_\Omega\left(\gamma_n^{-1}(\pi_{x_n}(y_n)), \gamma_n^{-1}(z_n)\right) = \lim_{n\to\infty} \mathrm{d}_\Omega\left(\pi_{x_n}(y_n), z_n\right) = \infty,$$

we have $\gamma_n^{-1}(\pi_{x_n}(y_n)) \to \bar{x}$. Also, since $\bar{x} \notin \bar{W}$,

$$\bar{y} = \lim_{n\to\infty} \gamma_n^{-1}(y_n) \in \lim_{n\to\infty} \gamma_n^{-1}(W_{x_n} \oplus \pi_{x_n}(y_n)) = \bar{W} \oplus \bar{x} = \lim_{n\to\infty} \gamma_n^{-1}(H_{x_n}) = H_{\bar{x}}.$$

As such, $\bar{y} = \bar{x}$ because $\bar{x}, \bar{y} \in \Lambda_\Omega(\Gamma)$ and Γ is a projectively visible subgroup. Then

$$\lim_{n\to\infty} \mathrm{d}_\Omega(z_n, (b_0, y_n)_\Omega) = \lim_{n\to\infty} \mathrm{d}_\Omega(\gamma_n^{-1}(z_n), \gamma_n^{-1}(b_0, y_n)_\Omega) = \mathrm{d}_\Omega(\bar{z}, (\bar{b}, \bar{x})_\Omega) = 0$$

and so $y_n \in \mathcal{O}_r(b_0, z_n)$ for n sufficiently large. This is a contradiction. □

7.2. **Proof of Theorem 7.1.** Fix $b_0 \in \Omega$ and $r, R > 0$. Let $\delta > 0$ be the constant given by Lemma 7.4.

The following lemma is the crucial estimate in the proof. It shows that if $\gamma(b_0)$ is near $\pi_x(y)$, then one obtains a lower bound on the distance between $\xi^1(x)$ and $\xi^1(y)$ in terms of $\alpha_1(\rho(\gamma))$.

Lemma 7.6. *There exists $C_1 > 1$ so that if $x, y \in \Lambda_\Omega(\Gamma)$, $\gamma \in \Gamma$, $0 < \mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x, y) \le \delta$ and $\mathrm{d}_\Omega(\gamma(b_0), \pi_x(y)) \le R$, then*

$$\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(\xi^1(x), \xi^1(y)\right) \ge \frac{1}{C_1} \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}.$$

Proof. If not, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\Lambda_\Omega(\Gamma)$ and $\{\gamma_n\}$ in Γ such that for all n, we have $0 < \mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x_n, y_n) \leq \delta$, $\mathrm{d}_\Omega(\gamma_n(b_0), \pi_{x_n}(y_n)) \leq R$, and

$$\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(\xi^1(x_n), \xi^1(y_n)\right) \leq \frac{1}{n}\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}. \tag{9}$$

Passing to a subsequence, we may assume that

$$x_n \to x \in \Lambda_\Omega(\Gamma), \quad y_n \to y \in \Lambda_\Omega(\Gamma), \quad \gamma_n^{-1}(\pi_{x_n}(y_n)) \to \bar{z} \in \Omega,$$
$$\gamma_n^{-1}(b_0) \to \bar{b} \in \Omega \cup \Lambda_\Omega(\Gamma), \quad \gamma_n(b_0) \to b \in \Omega \cup \Lambda_\Omega(\Gamma), \quad \gamma_n^{-1}(x_n) \to \bar{x} \in \Lambda_\Omega(\Gamma),$$
$$\gamma_n^{-1}(y_n) \to \bar{y} \in \Lambda_\Omega(\Gamma), \quad \gamma_n^{-1}(x_{n,\mathrm{opp}}) \to \hat{x} \in \partial\Omega \quad \text{and} \quad \gamma_n^{-1}(H_{x_{n,\mathrm{opp}}}) \to \bar{H}.$$

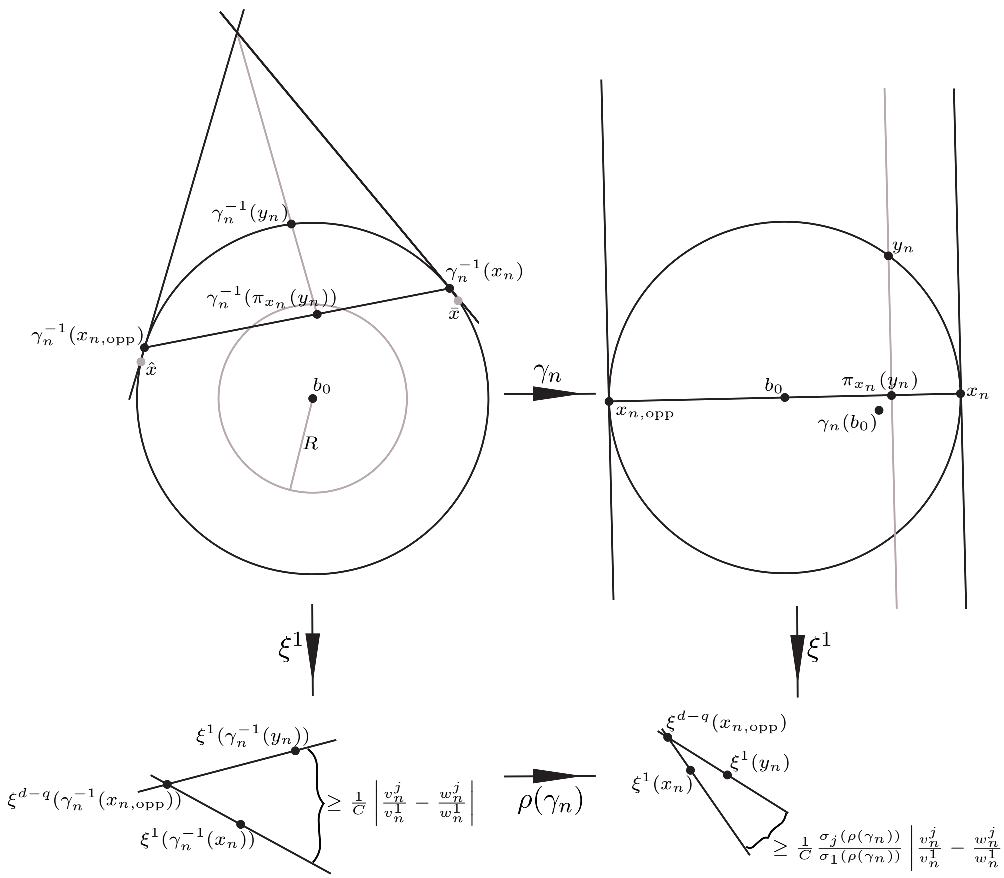

FIGURE 2. Lower bound on the size of shadow.

We first show that $\{\gamma_n\}$ is an escaping sequence, $b = x = y$, $\bar{b} = \hat{x}$ and $\hat{x} \neq \bar{x}$. Since ξ^1 is injective and continuous, and $\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}(\xi^1(x_n), \xi^1(y_n)) \to 0$, we know that $x = y$. By Lemma 7.3, we have $\pi_{x_n}(y_n) \to x \in \Lambda_\Omega(\Gamma)$. Then, since $\{\gamma_n^{-1}(\pi_{x_n}(y_n)) : n \geqslant 1\}$ is a relatively compact subset of Ω, it follows that $\{\gamma_n\}$ is an escaping sequence, so $b, \bar{b} \in \Lambda_\Omega(\Gamma)$. By Proposition 2.7(2),

$$b = \lim_{n\to\infty} \gamma_n\big(\gamma_n^{-1}(\pi_{x_n}(y_n))\big) = \lim_{n\to\infty} \pi_{x_n}(y_n) = x.$$

Since Γ acts as a convergence group on $\Lambda(\Gamma)$ (see Proposition 3.5(3)), γ_n^{-1} converges to $\bar{b}$ uniformly on compacta in $\Lambda(\Gamma) - \{x\}$. Since $\{x_{n,\mathrm{opp}} : n \geqslant 1\}$ is relatively compact in $\Lambda(\Gamma) - \{x\}$, we have $\hat{x} = \bar{b}$. Since $d\left(b_0, \gamma_n^{-1}\left([x_n, x_{n,opp}]_\Omega\right)\right) \leq R$, we see that $\hat{x} \neq \bar{x}$.

Note that $\bar{H}$ is a supporting hyperplane to Ω at $\hat{x}$. Since $\hat{x} = \bar{b} \in \Lambda_\Omega(\Gamma)$, the supporting hyperplane to Ω at $\hat{x}$ is unique, so $\bar{H} = H_{\hat{x}}$. Thus, if we denote $\bar{W} = H_{\bar{x}} \cap H_{\hat{x}}$, then

$$\bar{z} = \left(\bar{W} \oplus \bar{y}\right) \cap [\hat{x}, \bar{x}]_\Omega.$$

In particular, $\hat{x}$, $\bar{x}$ and $\bar{y}$ are pairwise distinct points in $\Lambda_\Omega(\Gamma)$ (see Figure 2), so the transversality of ρ gives

$$\xi^1(\bar{x}) + \xi^{d-1}(\hat{x}) = \mathbb{K}^d = \xi^1(\bar{y}) + \xi^{d-1}(\hat{x}), \tag{10}$$

and the assumption that ρ is $(1,1,q)$-hypertransverse implies

$$\xi^1(\bar{x}) + \xi^1(\bar{y}) + \xi^{d-q}(\hat{x}) \tag{11}$$

is a direct sum.

For each n, let $\rho(\gamma_n) = m_n a_n \ell_n$ be the Cartan decomposition of $\rho(\gamma_n)$, where $m_n, \ell_n \in \mathsf{PU}(d,\mathbb{K})$ and $a_n \in \exp(\mathfrak{a}^+)$. Also let $v_n = (v_n^1, \dots, v_n^d)$ and $w_n = (w_n^1, \dots, w_n^d)$ be unit vectors so that

$$[v_n] = a_n^{-1} m_n^{-1}(\xi^1(x_n)) \quad \text{and} \quad [w_n] = a_n^{-1} m_n^{-1}(\xi^1(y_n)).$$

Then $\ell_n^{-1}([v_n]) = \xi^1(\gamma_n^{-1} x_n)$ and $\ell_n^{-1}([w_n]) = \xi^1(\gamma_n^{-1} y_n)$. Passing to a subsequence, we can suppose that

$$\ell_n \to \ell, \quad v_n \to v = (v^1, \dots, v^d) \quad \text{and} \quad w_n \to w = (w^1, \dots, w^d).$$

Since ρ is a $P_{1,q,d-q,d-1}$-transverse representation, and $\{\gamma_n\}$ is an escaping sequence such $\gamma_n^{-1}(b_0) \to \hat{x}$ and $\gamma_n(b_0) \to x$, Lemma 2.1 implies that

$$\xi^{d-i}(\hat{x}) = \lim_{n\to\infty} U_{d-i}(\rho(\gamma_n^{-1})) = \ell^{-1}(\mathrm{Span}_\mathbb{K}(e_{i+1}, \dots, e_d)) \tag{12}$$

for $i = 1$ and $i = q$. Also, since ξ^1 is continuous,

$$\xi^1(\bar{x}) = \ell^{-1}([v]) \quad \text{and} \quad \xi^1(\bar{y}) = \ell^{-1}([w]).$$

Hence, Equations (10) and (12) (with $i = 1$) imply that $v^1 \neq 0 \neq w^1$, so $v_n^1 \neq 0 \neq w_n^1$ for sufficiently large n. Equations (11) and (12) (with $i = q$) imply that the collection of vectors $\{v, w, e_{q+1}, \dots, e_d\}$ are linearly independent over $\mathbb{K}$. In particular, there is some $j \in \{2, \dots, q\}$ such that

$$\frac{v^j}{v^1} - \frac{w^j}{w^1} \neq 0$$

Consider the affine chart

$$\mathbb{A}^{d-1} = \left\{\left[(u^1, \dots, u^d)\right] \in \mathbb{P}(\mathbb{K}^d) : u^1 \neq 0\right\}$$

of $\mathbb{P}(\mathbb{K}^d)$. Let $\mathrm{d}_{\mathbb{A}^{d-1}}$ denote the pullback to $\mathbb{A}^{d-1}$ of the standard metric on $\mathbb{K}^{d-1}$ via the identification $\mathbb{A}^{d-1} \simeq \mathbb{K}^{d-1}$ given by

$$\left[(u^1, \dots, u^d)\right] \mapsto \left(\frac{u^2}{u^1}, \dots, \frac{u^d}{u^1}\right).$$

Since $v_n^1 \neq 0 \neq w_n^1$ for sufficiently large n and $\alpha_1(\kappa(\rho(\gamma_n))) \to \infty$, we have

$$\lim_{n\to\infty} a_n([v_n]) = [e_1] = \lim_{n\to\infty} a_n([w_n]),$$

so there is a compact $\mathcal{K} \subset \mathbb{A}^{d-1}$ that contains $a_n([v_n])$ and $a_n([w_n])$ for sufficiently large n. Choose $C > 1$ so that $\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}$ and $\mathrm{d}_{\mathbb{A}^{d-1}}$ are C-bilipschitz on $\mathcal{K}$. Then

$$\begin{aligned}\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(\xi^1(x_n),\xi^1(y_n)\right) &= \mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(\rho(\gamma_n)\xi^1(\gamma_n^{-1}(x_n)),\rho(\gamma_n)\xi^1(\gamma_n^{-1}(y_n))\right) = \mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(a_n([v_n]),a_n([w_n])\right)\\ &\geq \frac{1}{C}\mathrm{d}_{\mathbb{A}^{d-1}}\left(a_n([v_n]),a_n([w_n])\right) = \frac{1}{C}\sqrt{\sum_{i=2}^{d}\frac{\sigma_i(\rho(\gamma_n))^2}{\sigma_1(\rho(\gamma_n))^2}\left|\frac{v_n^i}{v_n^1}-\frac{w_n^i}{w_n^1}\right|^2}\\ &\geq \frac{1}{C}\frac{\sigma_j(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}\left|\frac{v_n^j}{v_n^1}-\frac{w_n^j}{w_n^1}\right|.\end{aligned}$$

Thus, by (9) and the hypothesis that $\sigma_q(\rho(\gamma_n)) = \sigma_2(\rho(\gamma_n))$,

$$\begin{aligned}0 &= \lim_{n\to\infty}\frac{\sigma_1(\rho(\gamma_n))}{\sigma_2(\rho(\gamma_n))}\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(\xi^1(x_n),\xi^1(y_n)\right) \geq \frac{1}{C}\lim_{n\to\infty}\frac{\sigma_1(\rho(\gamma_n))}{\sigma_2(\rho(\gamma_n))}\left(\left|\frac{v_n^j}{v_n^1}-\frac{w_n^j}{w_n^1}\right|\frac{\sigma_j(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}\right)\\ &= \frac{1}{C}\left|\frac{v^j}{v^1}-\frac{w^j}{w^1}\right| \neq 0\end{aligned}$$

and we have achieved a contradiction. □

If Theorem 7.1 does not hold, then for every n there exist $x_n \in \Lambda_{b_0,R}(\Gamma)$, $y_n \in \Lambda_\Omega(\Gamma)$, $z_n \in [b_0, x_n)_\Omega$ and $\gamma_n \in \Gamma$ such that

$$\mathrm{d}_\Omega(z_n, \gamma_n(b_0)) \leq r, \quad y_n \notin \mathcal{O}_r(b_0, z_n) \quad \text{and} \quad \mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}(\xi^1(x_n), \xi^1(y_n)) \leq \frac{1}{n}\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}.$$

In particular, $x_n \neq y_n$ for all n. Also, by taking subsequences, we may assume that

$$x_n \to x \in \Lambda_\Omega(\Gamma), \quad y_n \to y \in \Lambda_\Omega(\Gamma) \quad \text{and} \quad \gamma_n(b_0) \to b \in \Omega \cup \Lambda_\Omega(\Gamma).$$

Since ξ is a continuous embedding, $x = y$. So by passing to a tail of our sequences, we can assume that $0 < \mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x_n, y_n) \leq \delta$ for all n. Also, since $x_n \in \Lambda_{b_0,R}(\Gamma)$, there exists $\beta_n \in \Gamma$ with $\mathrm{d}_\Omega(\pi_{x_n}(y_n), \beta_n(b_0)) \leq R$. Then, by Lemma 7.6 (for the first inequality) and assumption (for the second), there exists $C_1 > 1$ that does not depend on n, so that

$$\frac{\sigma_2(\rho(\beta_n))}{\sigma_1(\rho(\beta_n))} \leq C_1\mathrm{d}_{\mathbb{P}(\mathbb{K}^d)}\left(\xi^1(x_n),\xi^1(y_n)\right) \leq \frac{C_1}{n}\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}. \tag{13}$$

Notice that $\sigma_j(\rho(\beta_n)) = \sigma_j(\rho(\gamma_n)\rho(\gamma_n^{-1}\beta_n))$ and

$$\mathrm{d}_\Omega(\gamma_n^{-1}\beta_n(b_0), b_0) = \mathrm{d}_\Omega(\beta_n(b_0), \gamma_n(b_0)) \leq \mathrm{d}_\Omega(\pi_{x_n}(y_n), z_n) + R + r.$$

So if we set

$$S_n = \max\left\{\sigma_1(\rho(\eta)) : \mathrm{d}_\Omega(\eta(b_0), b_0) \leq d_\Omega(\pi_{x_n}(y_n), z_n) + R + r\right\},$$

then for all $j \in \{1, \dots, d\}$, we have

$$\frac{1}{S_n}\sigma_j(\rho(\gamma_n)) \leq \sigma_j(\rho(\beta_n)) \leq S_n\sigma_j(\rho(\gamma_n)).$$

Thus, (13) implies that $S_n \geq \sqrt{\frac{n}{C_1}} \to \infty$, so

$$\lim_{n\to\infty}\mathrm{d}_\Omega(\pi_{x_n}(y_n), z_n) = \infty. \tag{14}$$

Since $y_n \notin \mathcal{O}_r(b_0, z_n)$ and $\mathrm{d}_\Omega(\pi_{x_n}(y_n), z_n) \to \infty$, Lemma 7.5 (with $r' = r$) implies that by taking the tail end of the sequence, we may assume that $\pi_{x_n}(y_n) \notin (z_n, x_n)_\Omega$ for all n. At the same time, since $0 < \mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x_n, y_n) \le \delta$ for all n, Lemma 7.4 implies that $\pi_{x_n}(y_n) \in (b_0, x_n)_\Omega$. Thus $\pi_{x_n}(y_n) \in [b_0, z_n]_\Omega$ for all n.

Equation (1) implies that

$$\mathrm{d}_\Omega^{\mathrm{Haus}}\Big([b_0, z_n]_\Omega, [b_0, \gamma_n(b_0)]_\Omega\Big) \le \mathrm{d}_\Omega(z_n, \gamma_n(b_0)) \le r.$$

Then, since $\pi_{x_n}(y_n) \in [b_0, z_n]_\Omega$ and $d(\beta_n(b_0), \pi_{x_n}(y_n)) \le R$, we see that

$$\mathrm{d}_\Omega\left(\beta_n(b_0), [b_0, \gamma_n(b_0)]_\Omega\right) \le R + r.$$

So by Lemma 6.6 there exists $C > 0$, depending only on $R + r$, such that

$$\alpha_1(\kappa(\rho(\gamma_n))) \ge \alpha_1(\kappa(\rho(\beta_n))) + \alpha_1(\kappa(\rho(\beta_n^{-1}\gamma_n))) - C \ge \alpha_1(\kappa(\rho(\beta_n))) - C$$

for all n. But this contradicts Equation (13). □

8. Lower bounds on Hausdorff dimension

In this section we complete the proof of Theorem 1.5 which we restate here.

Theorem 8.1. *Suppose that $\Gamma \subset \mathsf{PGL}(d, \mathbb{K})$ is $(1,1,q)$-hypertransverse and $\sigma_2(\gamma) = \sigma_q(\gamma)$ for all $\gamma \in \Gamma$. Then*

$$\dim_H(\Lambda_{1,c}(\Gamma)) = \delta^{\alpha_1}(\Gamma).$$

As mentioned in the introduction, Theorem 8.1 generalizes earlier results of Pozzetti-Sambarino-Wienhard [43] for $(1,1,2)$-hyperconvex Anosov representations and Bishop-Jones [6] for Kleinian groups.

8.1. **Proof of Theorem 8.1.** Let $\theta = \{1, q, d-q, d-1\}$. By Theorem 4.2, there exist a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$, a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ and a P_θ-transverse representation $\rho : \Gamma_0 \to \mathsf{PGL}(d, \mathbb{K})$ with limit map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$, such that $\rho(\Gamma_0) = \Gamma$ and $\xi(\Lambda_\Omega(\Gamma_0)) = \Lambda_\theta(\Gamma)$. Then, by definition, ρ is a $(1,1,q)$-hypertransverse representation.

Choose $b_0 \in \Omega$ so that $\mathrm{Stab}_{\Gamma_0}(b_0) = \mathrm{id}$. Then the orbit map $\gamma \mapsto \gamma(b_0) \in \Omega$ is injective and $\Gamma_0(b_0)$ is a closed discrete subset of Ω. Further, the function $c : \Gamma_0(b_0) \to (0,1]$ given by

$$c(\gamma(b_0)) := \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}$$

is well defined.

The following technical result places us in the situation where we may apply the argument of Bishop and Jones.

Proposition 8.2. *For any $0 < \delta < \delta^{\alpha_1}(\Gamma)$, there exist $r_0, D_0 > 0$ such that if $z \in \Gamma_0(b_0)$, then there exists a finite subset $\mathcal{C}(z)$ of $\Gamma_0(b_0) - \{z\}$ with the following properties*

1. *if $w \in \mathcal{C}(z)$, then $\mathcal{O}_{2r_0}(b_0, w) \subset \mathcal{O}_{r_0}(b_0, z)$,*
2. *the sets $\{\mathcal{O}_{2r_0}(b_0, w)\}_{w \in \mathcal{C}(z)}$ are pairwise disjoint,*
3. *if $w \in \mathcal{C}(z)$, then $\mathrm{d}_\Omega(z, w) \le D_0$,*
4. *$\sum_{w \in \mathcal{C}(z)} c(w)^\delta \ge c(z)^\delta$.*

Assuming Proposition 8.2 for the moment we prove Theorem 8.1.

Outline of proof: We will construct, for each $0 < \delta < \delta^{\alpha_1}(\Gamma)$, a set $E_\delta \subset \Lambda_{\Omega,c}(\Gamma_0)$, a measure μ_δ on $\partial\Omega$ supported on E_δ and constants $C, t_0 > 0$ such that:

$$\xi^1_* \mu_\delta\left(B_{\mathbb{P}(\mathbb{K}^d)}(p, t)\right) \le C t^\delta$$

for all $p \in \mathbb{P}(\mathbb{K}^d)$ and $0 < t < t_0$. Once we do so, then we may apply simple covering arguments in the spirit of Frostman's Lemma, see for instance [7, Theorem 1.2.8 and Lemma 3.1.1], to show that

$$\dim_H(\Lambda_{1,c}(\Gamma)) \geq \dim_H(\xi^1(E_\delta)) \geq \delta.$$

Taking the limits as $\delta \to \delta^{\alpha_1}(\Gamma)$ will yield that $\dim_H(\Lambda_{1,c}(\Gamma)) \geq \delta^{\alpha_1}(\Gamma)$. Since Corollary 5.2 implies that $\dim_H(\Lambda_{1,c}(\Gamma)) \leq \delta^{\alpha_1}(\Gamma)$, our theorem will follow.

Fix $0 < \delta < \delta^{\alpha_1}(\Gamma)$, and let $r_0, D_0 > 0$ be as given by Proposition 8.2.

We inductively construct a tree $\mathcal{T} = \mathcal{T}_\delta \subset \Omega$ with root b_0, whose vertices are a subset of $\Gamma_0(b_0)$, and with the property that for every vertex $z \in \mathcal{T}$, its children $\mathcal{C}(z)$ satisfy the conditions in Proposition 8.2. (Notice that, by definition, $\mathcal{O}_{r_0}(b_0, z) = \partial\Omega$ if $d_\Omega(b_0, z) < r_0$.) Properties (1) and (2) in Proposition 8.2 guarantee that our inductive construction does indeed produce a tree.

Let $E = E_\delta \subset \Lambda_\Omega(\Gamma_0)$ be the set of accumulation points of the vertices of $\mathcal{T}$. Since $\Gamma_0(b_0)$ is discrete and $\mathcal{T}$ is infinite, E is non-empty. We now observe that E is uniformly conical.

Lemma 8.3.

(1) For any $x \in E$ there exists a (discrete) geodesic ray $\{x_n\}$ in $\mathcal{T}$ such that $x_0 = b_0$, $x_n \to x$ and

$$x \in \bigcap_{n \in \mathbb{N}} \mathcal{O}_{r_0}(b_0, x_n).$$

(2) $E \subset \Lambda_{b_0,R_0}(\Gamma_0) \subset \Lambda_{\Omega,c}(\Gamma_0)$ where $R_0 = \frac{1}{2}D_0 + 2r_0$.

Proof. (1): Fix $x \in E$. Then there exist a sequence of vertices $\{w_m\}$ in $\mathcal{T}$ with $w_m \to x$. We may pass to a subsequence so that $w_m \to w \in \partial_\infty \mathcal{T}$, the abstract visual boundary of $\mathcal{T}$. Let $\sigma : \mathbb{Z}_{\geq 0} \to \mathcal{T}$ be a geodesic ray that starts at b_0 and limits to w. Let $x_n = \sigma(n)$ and let

$$k_m = \max\{n \ : \ w_m \ \text{ is a descendent of } \ x_n\}.$$

Observe that $k_m \to \infty$, so we may pass to a subsequence so that the sequence $\{k_m\}$ is strictly increasing.

Since x_{n+1} is a child of x_n for all n and w_m is a descendent of x_{k_m} for all m, Proposition 8.2 part (1) gives

$$\mathcal{O}_{2r_0}(b_0, x_{n+1}) \subset \mathcal{O}_{r_0}(b_0, x_n) \ \text{ and } \ \mathcal{O}_{2r_0}(b_0, w_m) \subset \mathcal{O}_{r_0}(b_0, x_{k_m})$$

for all n and m. Hence,

$$\iota_{b_0}(w_m) \in \mathcal{O}_{r_0}(b_0, x_{k_m}) = \bigcap_{n \leq k_m} \mathcal{O}_{r_0}(b_0, x_n),$$

where ι_{b_0} is the radial projection. Since $\{k_m\}$ is strictly increasing,

$$x = \lim_{m\to\infty} w_m = \lim_{m\to\infty} \iota_{b_0}(w_m) \in \bigcap_{n\in\mathbb{N}} \mathcal{O}_{r_0}(b_0, x_n). \tag{15}$$

Finally, we prove that $x_n \to x$. By (15), there exists a sequence $\{y_n\}$ along the geodesic ray $[b_0, x)_\Omega$ in Ω such that $\mathrm{d}_\Omega(y_n, x_n) \leq r_0$ for all n. Suppose for contradiction that the sequence $\{x_n\}$ does not converge to x. Then by taking a subsequence, we may assume that $x_n \to y \in E$ for some $y \neq x$. Let $\{\gamma_n\}$ be the sequence in Γ_0 so that $\gamma_n(b_0) = x_n$ for all n. Then Proposition 2.7 part (2) implies that $\gamma_n(b) \to y$ for all $b \in \Omega$ and the convergence is locally uniform. Since $\{\gamma_n^{-1}(y_n)\}$ is relatively compact in Ω, we have $y_n = \gamma_n(\gamma_n^{-1}(y_n)) \to y$. However, by construction $y_n \subset [b_0, x)$ for all n, so this is not possible.

(2): Fix $x \in E$. By (1), there is a geodesic ray $\{x_n\}$ in $\mathcal{T}$ such that $x_n \to x$, and there exists a sequence $\{y_n\}$ along the geodesic ray $[b_0, x)_\Omega$ in Ω such that $\mathrm{d}_\Omega(y_n, x_n) \leq r_0$ for all n. Then by Proposition 8.2 part (3),

$$\mathrm{d}_\Omega(y_n, y_{n+1}) \leq \mathrm{d}_\Omega(x_n, x_{n+1}) + 2r_0 \leq D_0 + 2r_0.$$

It then follows that

$$[b_0, x)_\Omega \subset \bigcup_{n=1}^{\infty} \overline{B_\Omega\left(y_n, \frac{1}{2}D_0 + r_0\right)} \subset \bigcup_{n=1}^{\infty} \overline{B_\Omega\left(x_n, \frac{1}{2}D_0 + 2r_0\right)}.$$

Since $x_n \in \Gamma_0(b_0)$ for all n, we have $x \in \Lambda_{b_0,R_0}(\Gamma_0)$, where $R_0 = \frac{1}{2}D_0 + 2r_0$. Therefore, $E \subset \Lambda_{b_0,R_0}(\Gamma_0)$. Lemma 3.6 implies that $\Lambda_{b_0,R_0}(\Gamma_0) \subset \Lambda_{\Omega,c}(\Gamma_0)$, which completes the proof. □

We now construct a well-behaved probability measure μ on E.

Lemma 8.4. *There exists a Borel probability measure $\mu = \mu_\delta$ on $\partial\Omega$ such that $\mu(E) = 1$ and*

(1) $\mu\left(\mathcal{O}_{2r_0}(b_0, z)\right) \leq c(z)^\delta$ for every vertex z of $\mathcal{T}$ and

(2) there exists $C > 0$ and $t_0 > 0$ so that if $0 < t < t_0$ and $p \in \mathbb{P}(\mathbb{K}^d)$, then

$$\xi^1_*\mu(B(p,t)) \leq Ct^\delta,$$

where $B(p,t)$ is the ball of radius t in $\mathbb{P}(\mathbb{K}^d)$ centered at p.

Proof. Let $\mathcal{V}_n$ denote the set of vertices of $\mathcal{T}$ at distance n from b_0 (with respect to the integer-valued metric on the vertices of $\mathcal{T}$). We inductively define a sequence of Borel probability measures supported on $\mathcal{V}_n$. First let μ_0 be the Dirac measure δ_{b_0} at b_0. Then, inductively define

$$\mu_n = \sum_{z \in \mathcal{V}_{n-1}} \left(\frac{\mu_{n-1}(z)}{\sum_{w \in \mathcal{C}(z)} c(w)^\delta} \sum_{w \in \mathcal{C}(z)} c(w)^\delta \delta_w \right).$$

We may view $\{\mu_n\}$ as a sequence of probability measures on the compact space $\mathcal{T} \cup E$, so it has a weak-$*$ subsequential limit μ, which is a probability measure on $\mathcal{T} \cup E$. Further, since $\mathrm{d}_\Omega(b_0, \mathcal{V}_n) \to \infty$, the support of μ lies in E.

(1): Let z be a vertex of $\mathcal{T}$, and let n be the integer such that $z \in \mathcal{V}_n$. If z' is a descendent of z, then Proposition 8.2 part (1) implies that $\mathcal{O}_{2r_0}(b_0, z') \subset \mathcal{O}_{2r_0}(b_0, z)$. Also, by Proposition 8.2 part (2) and induction, if $w \in \mathcal{V}_n - \{z\}$, then

$$\mathcal{O}_{2r_0}(b_0, z) \cap \mathcal{O}_{2r_0}(b_0, w) = \emptyset.$$

As such, if we denote the cone over $\mathcal{O}_{2r_0}(b_0, z) \subset \partial\Omega$ based at b_0 by

$$\mathcal{C}(\mathcal{O}_{2r_0}(b_0, z)) \subset \Omega,$$

then the set of vertices in $\cup_{j \geq n} \mathcal{V}_j$ that lie in $\mathcal{C}(\mathcal{O}_{2r_0}(b_0, z))$ is precisely the set of descendants of z. This, together with the definition of μ_n, implies that

$$\mu_m(\mathcal{C}(\mathcal{O}_{2r_0}(b_0, z))) = \mu_n(\mathcal{C}(\mathcal{O}_{2r_0}(b_0, z)))$$

for all $m \geq n$. Furthermore, if $\{b_0 = x_0, x_1, \dots, x_{n-1}, x_n = z\}$ is the geodesic in $\mathcal{T}$ between b_0 and z, then by induction and Proposition 8.2 part (4), we have

$$\mu_n(z) = c(z)^\delta \prod_{i=0}^{n-1} \frac{c(x_i)^\delta}{\sum_{w \in \mathcal{C}(x_i)} c(w)^\delta} \leq c(z)^\delta.$$

Since $\mathcal{C}(\mathcal{O}_{2r_0}(b_0, z)) \cap \mathcal{V}_n = \{z\}$, it now follows that for all $m \geq n$

$$\mu_m(\mathcal{C}(\mathcal{O}_{2r_0}(b_0, z))) \leq c(z)^\delta.$$

Thus, $\mu(\mathcal{O}_{2r_0}(b_0,z)) = \lim_{m\to\infty}\mu_m(\mathcal{C}(\mathcal{O}_{2r_0}(b_0,z))) \le c(z)^\delta$, so (1) holds.

(2): Let $z, w \in \mathcal{T}$ be any two adjacent vertices. By Proposition 8.2 part (3), $\mathrm{d}_\Omega(z,w) \le D_0$. So if we set

$$S = \max\left\{\frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))} : \mathrm{d}_\Omega(b_0,\gamma(b_0)) \le D_0\right\},$$

then

$$\frac{1}{S}c(z) \le c(w) \le Sc(z). \tag{16}$$

By Lemma 8.3 part (1), for any $x \in E$ there exists a geodesic ray $\{x_n\}$ in $\mathcal{T}$ starting at $x_0 = b_0$ such that $x_n \to x$ in $\overline{\Omega}$ and

$$x \in \bigcap_{n\in\mathbb{N}} \mathcal{O}_{r_0}(b_0,x_n).$$

So there exists a sequence $\{y_n\}$ along the geodesic ray $[b_0,x)$ in Ω with $\mathrm{d}_\Omega(y_n,x_n) \le r_0$ for all n. By Theorem 7.1 and Lemma 8.3 part (2), there exists $C_1 > 1$ (that depends on R_0 but not x) such that

$$B\left(\xi^1(x), \frac{c(x_n)}{C_1}\right) \cap \Lambda_1(\Gamma) \subset \xi^1\left(\widehat{\mathcal{O}}_{r_0}(b_0,y_n)\right) \subset \xi^1\left(\widehat{\mathcal{O}}_{2r_0}(b_0,x_n)\right) \tag{17}$$

for all n. Then by (1),

$$\xi^1_*\mu\left(B\left(\xi^1(x), \frac{c(x_n)}{C_1}\right)\right) \le c(x_n)^\delta. \tag{18}$$

Since Γ_0 is P_1-divergent, $c(x_n) \to 0$. Let

$$t_1 = \frac{1}{C_1}\min\{c(z) \ : \ z \in \mathcal{V}_1\}.$$

If $0 < t < t_1$, there is some positive integer n such that $c(x_{n+1}) \le tC_1 \le c(x_n)$. By (16),

$$\frac{1}{S}\frac{c(x_n)}{C_1} \le t \le \frac{c(x_n)}{C_1},$$

so, Equation (18) implies that

$$\xi^1_*\mu\left(B\left(\xi^1(x),t\right)\right) \le \xi^1_*\mu\left(B\left(\xi^1(x), \frac{c(x_n)}{C_1}\right)\right) \le c(x_n)^\delta \le (C_1S)^\delta t^\delta.$$

Let $t_0 = t_1/2$. Suppose that $p \in \mathbb{P}(\mathbb{K}^d)$ and $0 < t < t_0$. Since μ is supported on E, either $B(p,t) \cap \xi^1(E)$ is empty, in which case $(\xi^1)^*\mu(B(p,t)) = 0$ or there exists $x \in E$ so that $\xi^1(x) \in B(p,t) \cap \xi^1(E)$, in which case $B(p,t) \subset B(\xi^1(x),2t)$, so

$$\xi^1_*\mu\left(B(p,t)\right) \le \xi^1_*\mu\left(B(\xi^1(x),2t)\right) \le (C_1S)^\delta(2t)^\delta \le (2C_1S)^\delta t^\delta.$$

Therefore, (2) holds with $C = (2C_1S)^\delta$. □

Let $\{B(p_i,r_i)\}$ be a countable covering of $\xi^1(E)$ by open balls. Lemma 8.4 implies that

$$\sum_i r_i^\delta \ge \frac{1}{C}\sum_i \xi^1_*\mu(B(p_i,r_i)) \ge \frac{1}{C},$$

so $\dim_H(\xi^1(E)) \ge \delta$. Since $\xi^1 : \Lambda_\Omega(\Gamma_0) \to \Lambda_1(\Gamma)$ is a ρ-equivariant homeomorphism and $E \subset \Lambda_{\Omega,c}(\Gamma)$, we see that $\xi^1(E) \subset \Lambda_{1,c}(\Gamma)$. Therefore,

$$\dim_H(\Lambda_{1,c}(\Gamma)) \ge \dim_H(\xi^1(E)) \ge \delta.$$

Since δ can be chosen to be any positive number less than $\delta^{\alpha_1}(\Gamma)$, it follows that

$$\dim_H(\Lambda_{1,c}(\Gamma)) \geq \delta^{\alpha_1}(\Gamma).$$

By Corollary 5.2, $\dim_H(\Lambda_{1,c}(\Gamma)) \leq \delta^{\alpha_1}(\Gamma)$, and this completes the proof of Theorem 8.1.

8.2. **Proof of Proposition 8.2.** It remains to prove Proposition 8.2. We start with some observations about shadows. For any integer $n \geq 0$, let

$$\mathcal{A}_n = \left\{ z \in \Gamma(b_0) : e^{-(n+1)} < c(z) \leq e^{-n} \right\}.$$

Notice that, since Γ is P_1-divergent, $\mathcal{A}_n$ is finite. We first show that there is a uniform lower bound on distance which implies that two points in $\mathcal{A}_n$ have disjoint shadows.

Lemma 8.5. *For any $r > 0$ there exists $C_0 = C_0(r) > 0$ such that if $n \geq 0$, $z, w \in \mathcal{A}_n$ and $\mathrm{d}_\Omega(z,w) > C_0$, then*

$$\mathcal{O}_r(b_0, z) \cap \mathcal{O}_r(b_0, w) = \emptyset.$$

Proof. We prove the contrapositive. Suppose

$$x \in \mathcal{O}_r(b_0, \gamma(b_0)) \cap \mathcal{O}_r(b_0, \eta(b_0))$$

and $\gamma(b_0), \eta(b_0) \in \mathcal{A}_n$. Then there exist $z', w' \in [b_0, x)_\Omega$ such that $\mathrm{d}_\Omega(\gamma(b_0), z') \leq r$ and $\mathrm{d}_\Omega(\eta(b_0), w') \leq r$. Without loss of generality we can assume that $z' \in [b_0, w']_\Omega$. Then Equation (1) implies that

$$\mathrm{d}_\Omega(\gamma(b_0), [b_0, \eta(b_0)]_\Omega) \leq r + \mathrm{d}_\Omega(z', [b_0, \eta(b_0)]_\Omega) \leq 2r.$$

So, by Lemma 6.6, there exists $C > 0$, which only depends on $2r$, such that

$$\alpha_1(\rho(\eta)) \geq \alpha_1(\rho(\gamma)) + \alpha_1(\rho(\gamma^{-1}\eta)) - C.$$

Since $\gamma(b_0), \eta(b_0) \in \mathcal{A}_n$,

$$\alpha_1(\rho(\gamma^{-1}\eta)) \leq C + 1.$$

So if

$$C_0 = \max\left\{ \mathrm{d}_\Omega(b_0, \beta(b_0)) : \ \beta \in \Gamma \text{ and } \alpha_1(\rho(\beta)) \leq C + 1 \right\},$$

then

$$\mathrm{d}_\Omega(\gamma(b_0), \eta(b_0)) = \mathrm{d}_\Omega(b_0, \gamma^{-1}\eta(b_0)) \leq C_0.$$

Since C_0 only depends on r this completes the proof. $\square$

The next lemma is needed to define the constants r_0 and D_0 in Proposition 8.2.

Lemma 8.6. *Given $t > 0$, there exists $r_0 = r_0(t) > 0$ and $N_0 = N_0(t) > 0$ such that if $z, w \in \Gamma_0(b_0) - \{b_0\}$, $\mathrm{d}_\Omega(b_0, w) \geq N_0$, and $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\big(\iota_{b_0}(z), \iota_{b_0}(w)\big) \geq t$, then*

$$\mathcal{O}_{2r_0}(z, w) \subset \mathcal{O}_{r_0}(z, b_0) \cap \mathcal{O}_{2r_0+1}(b_0, w).$$

Proof. Since $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ is projectively visible, there exists $r_0 > 0$ such that if $p, q \in \Lambda_\Omega(\Gamma_0)$ and $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(p,q) \geq t$, then

$$\mathrm{d}_\Omega(b_0, (p,q)_\Omega) \leq r_0.$$

Suppose, for contradiction, that the lemma fails for this value of r_0. Then for every positive integer n, there exist $z_n, w_n \in \Gamma_0(b_0) - \{b_0\}$ and

$$x_n \in \mathcal{O}_{2r_0}(z_n, w_n) - (\mathcal{O}_{r_0}(z_n, b_0) \cap \mathcal{O}_{2r_0+1}(b_0, w_n))$$

such that $\mathrm{d}_\Omega(b_0, w_n) \geq n$ and $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(\iota_{b_0}(z_n), \iota_{b_0}(w_n)) \geq t$. For each n, let $\gamma_n \in \Gamma_0$ be the element with $w_n = \gamma_n(b_0)$, and choose $u_n \in [z_n, x_n)_\Omega$ so that $d(w_n, u_n) \leq R$. By passing to a subsequence, we can suppose that

$$w_n \to w \in \Lambda_\Omega(\Gamma_0), \quad z_n \to z \in \overline{\Omega}, \quad x_n \to x \in \partial\Omega, \quad \gamma_n^{-1}(b_0) \to \bar{b} \in \Lambda_\Omega(\Gamma_0),$$

$$\gamma_n^{-1}(u_n) \to \bar{u} \in \Omega \quad \text{and} \quad \gamma_n^{-1}(x_n) \to \bar{x} \in \partial\Omega.$$

Then $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(\iota_{b_0}(z), w) \geq t$ and so (see Figure 3)

$$\mathrm{d}_\Omega(b_0, (z, w)_\Omega) \leq \mathrm{d}_\Omega(b_0, (\iota_{b_0}(z), w)_\Omega) \leq r_0. \tag{19}$$

We will obtain a contradiction by showing that $x \in \mathcal{O}_{r_0}(z, b_0) \cap \mathcal{O}_{2r_0+1}(b_0, w)$.

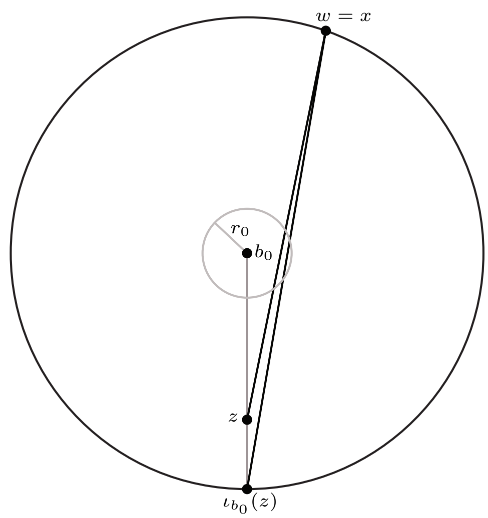

FIGURE 3. $x \in \mathcal{O}_{r_0}(z, b_0)$.

Proposition 2.7 part (2) implies that

$$\gamma_n(v) \to w \tag{20}$$

locally uniformly over all $v \in \mathbb{P}(\mathbb{R}^{d_0}) - T_{\bar{b}}\partial\Omega$ and

$$\gamma_n^{-1}(v) \to \bar{b} \tag{21}$$

locally uniformly over all $v \in \mathbb{P}(\mathbb{R}^{d_0}) - T_w\partial\Omega$. Since $\{\gamma_n^{-1}(u_n)\}$ is relatively compact in Ω, (20) implies that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \gamma_n(\gamma_n^{-1}(u_n)) = w.$$

Since $u_n \in [z_n, x_n)_\Omega$ for all n, it follows that $w \in [z, x]_\Omega$. Since $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(\iota_{b_0}(z), w) \geq t$ and $w \in \partial\Omega$, we must have $x = w$. Then (19) implies that $\mathrm{d}_\Omega(b_0, (z, x)_\Omega) \leq r_0$, or equivalently, $x \in \mathcal{O}_{r_0}(z, b_0)$, see Figure 3.

Since $\mathrm{d}_\Omega(b_0, (z, w)_\Omega) \leq r_0$, we see that $z \notin T_w\partial\Omega$. Hence, $\{z_n\}$ is relatively compact in $\mathbb{P}(\mathbb{R}^{d_0}) - T_w\partial\Omega$, so (21) implies that $\gamma_n^{-1}(z_n) \to \bar{b}$. Then $\bar{u} \in (\bar{b}, \bar{x})_\Omega$, which gives

$$\lim_{n\to\infty} \mathrm{d}_\Omega(u_n, (b_0, x_n)_\Omega) = \lim_{n\to\infty} \mathrm{d}_\Omega\left(\gamma_n^{-1}(u_n), \gamma_n^{-1}(b_0, x_n)_\Omega\right) = \mathrm{d}_\Omega(\bar{u}, (\bar{b}, \bar{x})_\Omega) = 0.$$

So for sufficiently large n,

$$\emptyset \neq \overline{B_\Omega(u_n, 1)} \cap (b_0, x_n)_\Omega \subset \overline{B_\Omega(w_n, 2r_0 + 1)} \cap (b_0, x_n)_\Omega,$$

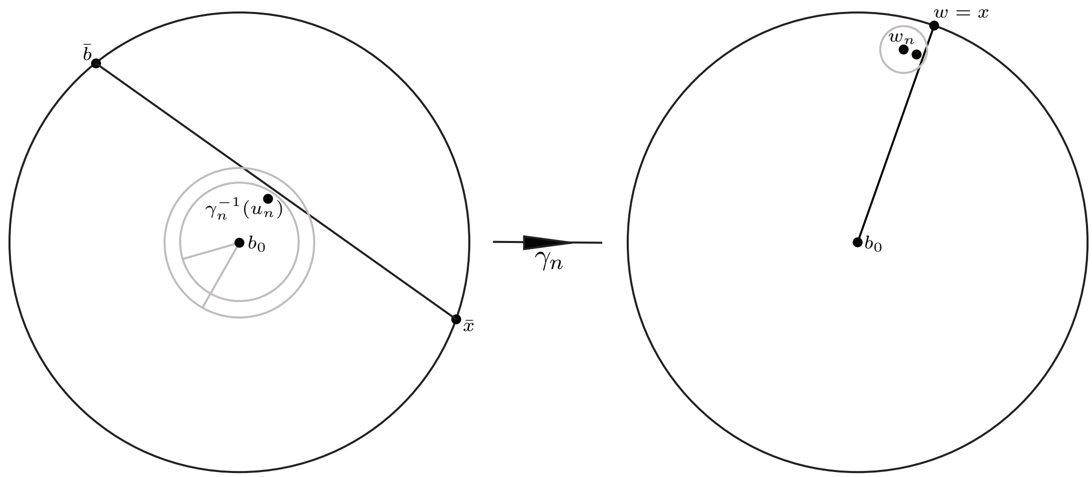

FIGURE 4. $x \in \mathcal{O}_{2r_0+1}(b_0, w)$.

or equivalently, $x_n \in \mathcal{O}_{2r_0+1}(b_0, w_n)$, see Figure 4. Thus, $x \in \mathcal{O}_{2r_0+1}(b_0, w)$, which completes the proof. □

For any $y \in \partial\Omega$ and any $t > 0$, set

$$\mathcal{B}(y,t) = \{w \in \Gamma_0(b_0) - \{b_0\} : \mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(\iota_{b_0}(w), y) < t\}.$$

Fix $\epsilon > 0$ such that $0 < \delta < \delta + \epsilon < \delta^{\alpha_1}(\Gamma)$. Then

$$\sum_{w \in \Gamma_0(b_0)} c(w)^{\delta+\epsilon} = \infty.$$

Thus, for each $n > 0$, there is some $x_n \in \Lambda_\Omega(\Gamma_0)$ such that

$$\sum_{w \in \mathcal{B}(x_n, 1/n)} c(w)^{\delta+\epsilon} = \infty.$$

By taking a subsequence, we may assume that $x_n \to x \in \Lambda_\Omega(\Gamma_0)$. Then for every $t > 0$,

$$\sum_{w \in \mathcal{B}(x,t)} c(w)^{\delta+\epsilon} = \infty \tag{22}$$

We prove the following refinement of Equation (22).

Lemma 8.7. *If $y \in \Gamma_0(x)$ and $t > 0$, then*

$$\limsup_{n\to\infty} \sum_{w \in \mathcal{A}_n \cap \mathcal{B}(y,t)} c(w)^{\delta} = \infty.$$

Proof. Choose $\gamma \in \Gamma_0$ so that $\gamma(x) = y$. Since γ is a diffeomorphism and $\mathbb{P}(\mathbb{R}^{d_0})$ is compact, there exists $D = D(\gamma) > 1$ so that

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(\gamma(a), \gamma(b)) \leq D \mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(a, b)$$

for all $a, b \in \mathbb{P}(\mathbb{R}^{d_0})$. Also, since

$$\sigma_d(\rho(\gamma))\sigma_j(\rho(\eta)) \leq \sigma_j(\rho(\gamma\eta)) \leq \sigma_1(\rho(\gamma))\sigma_j(\rho(\eta))$$

for all $j \in \{1, \dots, d\}$, we may enlarge D if necessary to ensure that

$$c(\gamma w) \geq \frac{1}{D} c(w)$$

for all $w \in \Gamma_0(b_0)$. Then

$$\sum_{w \in \mathcal{B}(y,t)} c(w)^{\delta+\epsilon} = \sum_{w \in \mathcal{B}(\gamma(x),t)} c(w)^{\delta+\epsilon} \geq \frac{1}{D^{\delta+\epsilon}} \sum_{w \in \mathcal{B}(x,t/D)} c(w)^{\delta+\epsilon} = \infty.$$

If $\limsup_{n\to\infty} \sum_{w \in \mathcal{A}_n \cap \mathcal{B}(y,t)} c(w)^\delta < \infty$, then there exists $C > 0$ such that

$$\sum_{w \in \mathcal{A}_n \cap \mathcal{B}(y,t)} c(w)^\delta \leq C.$$

for all $n \in \mathbb{N}$. Then

$$\sum_{w \in \mathcal{B}(y,t)} c(w)^{\delta+\epsilon} \leq \sum_{n=0}^{\infty} \left(e^{-\epsilon n} \sum_{w \in \mathcal{A}_n \cap \mathcal{B}(y,t)} c(w)^\delta \right) \leq C \sum_{n=0}^{\infty} e^{-\epsilon n} < \infty,$$

which is a contradiction. □

We are now ready to finish the proof of Proposition 8.2. Choose $x' \in \Gamma(x) - \{x\}$ and fix

$$0 < t_0 < \frac{1}{4} \mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x, x').$$

Note that $\sigma_1(\rho(\gamma\eta)) \leq \sigma_1(\rho(\gamma))\sigma_1(\rho(\eta))$ for all $\gamma, \eta \in \Gamma$. Thus, by Lemma 6.1, there exists $C_1 = C_1(t_0)$ so that if $\gamma, \eta \in \Gamma_0$ satisfy $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(\iota_{b_0}(\eta(b_0)), \iota_{b_0}(\gamma^{-1}(b_0))) \geq t_0$, then

$$\frac{\sigma_2(\rho(\gamma\eta))}{\sigma_1(\rho(\gamma\eta))} \geq C_1 \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \frac{\sigma_2(\rho(\eta))}{\sigma_1(\rho(\eta))}. \tag{23}$$

Also, let $r_0 = r_0(t_0) > 0$ and $N_0 = N_0(t_0) > 0$ be the constants given by Lemma 8.6. Then let $C_0 = C_0(2r_0 + 1) > 0$ be the constant from Lemma 8.5. Since Γ_0 is a discrete group, there exist a finite partition

$$\Gamma_0(b_0) = P_1 \cup \cdots \cup P_L$$

such that each P_i is C_0-separated. By definition

$$\lim_{n\to\infty} \min_{z \in \mathcal{A}_n} \mathrm{d}_\Omega(b_0, z) = \infty.$$

So by Lemma 8.7, there exist $n, n' \geq 1$ such that

$$\min_{z \in \mathcal{A}_n \cup \mathcal{A}_{n'}} \mathrm{d}_\Omega(z, b_0) \geq N_0, \tag{24}$$

$$\sum_{w \in \mathcal{A}_n \cap \mathcal{B}(x,t_0)} c(w)^\delta \geq \frac{L}{C_1^\delta}, \text{ and}$$

$$\sum_{w \in \mathcal{A}_{n'} \cap \mathcal{B}(x',t_0)} c(w)^\delta \geq \frac{L}{C_1^\delta}.$$

Thus, there exist $i_x, i_{x'} \in \{1, \dots, L\}$ such that

$$\sum_{w \in S_x} c(w)^\delta \geq \frac{1}{C_1^\delta} \quad \text{and} \quad \sum_{w \in S_{x'}} c(w)^\delta \geq \frac{1}{C_1^\delta}, \tag{25}$$

where $S_x = P_{i_x} \cap \mathcal{A}_n \cap \mathcal{B}(x, t_0)$ and $S_{x'} = P_{i_{x'}} \cap \mathcal{A}_{n'} \cap \mathcal{B}(x', t_0)$. Set

$$D_0 = \max_{z \in \mathcal{A}_n \cup \mathcal{A}_{n'}} \mathrm{d}_\Omega(b_0, z).$$

Fix $z = \gamma(b_0) \in \Gamma(b_0)$. Since $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(x, x') > 4t_0$, there exists $y \in \{x, x'\}$ such that

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\left(y, \iota_{b_0}(\gamma^{-1}(b_0))\right) > 2t_0. \tag{26}$$

Let $\mathcal{C}(z) = \gamma(S_y) \subset \Gamma_0(b_0) - \{z\}$. We check that $\mathcal{C}(z)$ satisfies parts (1)–(4) of Proposition 8.2.

Since $S_y \subset \mathcal{B}(y, t_0)$, (26) implies that

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\left(\iota_{b_0}(\gamma^{-1}(w)), \iota_{b_0}(\gamma^{-1}(b_0))\right) > t_0 \tag{27}$$

for all $w \in \mathcal{C}(z)$. Since $S_y \subset \mathcal{A}_n \cup \mathcal{A}_{n'}$, (24) implies that

$$\mathrm{d}_\Omega(b_0, \gamma^{-1}(w)) \geq N_0 \tag{28}$$

for all $w \in \mathcal{C}(z)$. Since $S_y \subset P_{i_y}$, S_y is C_0-separated, so by Lemma 8.5,

$$\mathcal{O}_{2r_0+1}(b_0, \gamma^{-1}(w)) \cap \mathcal{O}_{2r_0+1}(b_0, \gamma^{-1}(w')) = \emptyset \tag{29}$$

for all distinct $w, w' \in \mathcal{C}(z)$.

Lemma 8.6, (27), and (28) imply that

$$\mathcal{O}_{2r_0}(b_0, w) = \gamma\Big(\mathcal{O}_{2r_0}(\gamma^{-1}(b_0), \gamma^{-1}(w))\Big) \subset \gamma\Big(\mathcal{O}_{r_0}(\gamma^{-1}(b_0), b_0)\Big) = \mathcal{O}_{r_0}(b_0, z).$$

for all $w \in \mathcal{C}(z)$, so part (1) holds. Lemma 8.6 and (29) imply that

$$\begin{aligned}\gamma^{-1}\Big(\mathcal{O}_{2r_0}(b_0, w) \cap \mathcal{O}_{2r_0}(b_0, w')\Big) &= \mathcal{O}_{2r_0}(\gamma^{-1}(b_0), \gamma^{-1}(w)) \cap \mathcal{O}_{2r_0}(\gamma^{-1}(b_0), \gamma^{-1}(w')) \\ &\subset \mathcal{O}_{2r_0+1}(b_0, \gamma^{-1}(w)) \cap \mathcal{O}_{2r_0+1}(b_0, \gamma^{-1}(w')) = \emptyset.\end{aligned}$$

for all distinct $w, w' \in \mathcal{C}(z)$, so part (2) holds. Since $S_y \subset \mathcal{A}_n \cup \mathcal{A}_{n'}$, we know that

$$\mathrm{d}_\Omega(z, w) = \mathrm{d}_\Omega(b_0, \gamma^{-1}(w)) \leq D_0,$$

for all $w \in \mathcal{C}(z)$, so part (3) holds. Finally, for each $w \in \mathcal{C}(z)$, choose $\eta_w \in \Gamma_0$ so that

$$\eta_w(b_0) = \gamma^{-1}(w) \in S_y.$$

By (27), we have $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}(\iota_{b_0}(\eta_w(b_0)), \iota_{b_0}(\gamma^{-1}(b_0)) > t_0$, so by (23),

$$c(w) = \frac{\sigma_2(\rho(\gamma\eta_w))}{\sigma_1(\rho(\gamma\eta_w))} \geq C_1 \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \frac{\sigma_2(\rho(\eta_w))}{\sigma_1(\rho(\eta_w))} = C_1 c(z) c(\gamma^{-1}(w)).$$

Then, by (25),

$$\sum_{w \in \mathcal{C}(z)} c(w)^\delta \geq C_1^\delta c(z)^\delta \sum_{w \in S_y} c(w)^\delta \geq c(z)^\delta.$$

This completes the proof of Proposition 8.2 and hence the proof of Theorem 8.1.

9. Critical exponents and entropies

In this section, we show that critical exponents and entropies agree for cusped P_θ-Anosov representations of geometrically finite Fuchsian groups. This generalizes results of Glorieux-Montclair-Tholozan [27, Thm. 3.1] and Pozzetti-Sambarino-Weinhard [43, Prop. 4.1] from the Anosov setting. Notice that Proposition 1.8 from the introduction is a special case of this result.

Proposition 9.1. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is geometrically finite, $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{K})$ is P_θ-Anosov, and $\phi \in \mathcal{B}_\theta(\rho)^+$, then $h^\phi(\rho) = \delta^\phi(\rho)$.*

T: Decide position of +.

Proof. We will make crucial use of the fact that if there is a definite angle between the attracting k-plane and the repelling $(d-k)$-plane of a P_k-proximal element, then the k^{th} fundamental weights of the Cartan and Jordan projections are uniformly close.

Lemma 9.2. [27, Lemma 2.36] *Given $\eta > 0$, there exists $C > 0$ so that if $g \in \mathsf{PGL}(d,\mathbb{K})$ is P_k-proximal, V is its attracting k-plane and W is its repelling $(d-k)$-plane, and $\angle(V,W) \geq \eta$, then*

$$\left|\omega_k(\nu(g)) - \omega_k(\kappa(g))\right| \leq C.$$

We use this to control the difference between the Jordan and Cartan projections of images of elements whose axes in $\mathbb{H}^2$ pass through a compact subset of $\mathbb{H}^2$.

Lemma 9.3. *Given a compact subset $K \subset \mathbb{H}^2$, there exists $C > 0$ so that if $\gamma \in \Gamma_{hyp}$ and its axis passes through K, then*

$$\left|\omega_k(\kappa(\rho(\gamma))) - \omega_k(\nu(\rho(\gamma)))\right| \leq C$$

for all $k \in \theta$.

Proof. By compactness and the transversality of ξ, there exists $\eta > 0$ so that if the geodesic with endpoints $z, w \in \Lambda(\Gamma)$ intersect K, then $\angle(\xi^k(z), \xi^{d-k}(w)) \geq \eta$ for all $k \in \theta$. We may now apply Lemma 9.2 to deduce the lemma. ☐

We first prove that $h^\phi(\rho) \geq \delta^\phi(\rho)$. Denote $a(\phi) = \sum_{k\in\theta} |a_k|$, where

$$\phi = \sum_{k\in\theta} a_k \omega_k.$$

Since Γ is geometrically finite, there is a compact subset $K \subset \mathbb{H}^2$ so that if $\gamma \in \Gamma$ is hyperbolic, then there exists a conjugate of γ whose axis intersects K. Moreover, by Lemma 9.3, there exists $C > 0$ such that

$$\left|\phi(\kappa(\rho(\gamma))) - \phi(\nu(\rho(\gamma)))\right| \leq a(\phi)C$$

for all $\gamma \in \Gamma_{hyp}$ whose axis intersects K. It follows that

$$\begin{aligned}\#\left\{[\gamma] \in [\Gamma_{hyp}] \;\middle|\; \ell^\phi(\gamma) \leq T\right\} &\leq \#\left\{\gamma \in \Gamma_{hyp} \;\middle|\; \text{the axis of } \gamma \text{ intersects } K \text{ and } \phi(\nu(\rho(\gamma))) \leq T\right\}\\ &\leq \#\left\{\gamma \in \Gamma \;\middle|\; \phi(\kappa(\rho(\gamma))) \leq T + a(\phi)C\right\}\end{aligned}$$

for all $T > 0$. Since

$$\delta^\phi(\rho) = \limsup_{T\to\infty} \frac{1}{T} \log \#\left\{\gamma \in \Gamma \;\middle|\; \phi(\kappa(\rho(\gamma))) \leq T\right\},$$

this implies that $h^\phi(\rho) \leq \delta^\phi(\rho)$.

We now prove the opposite inequality. A special case of a result of Abels-Margulis-Soifer [1, Theorem 4.1] implies that there exist $\mu > 0$ and a finite subset $\mathcal{A}$ of Γ such that if $\gamma \in \Gamma$, then there exists $\alpha \in \mathcal{A}$ so that $\gamma\alpha \in \Gamma_{hyp}$ and $\mathrm{d}_{\partial\mathbb{H}^2}((\gamma\alpha)^+, (\gamma\alpha)^-) \geq \mu$.

Let $K' \subset \mathbb{H}^2$ be a compact set so that any bi-infinite geodesic whose endpoints are at least μ apart pass through K'. By Lemma 9.3, there is some $C' > 0$ such that

$$\left|\phi(\kappa(\rho(\gamma))) - \phi(\nu(\rho(\gamma)))\right| \leq a(\phi)C'$$

for all $\gamma \in \Gamma_{hyp}$ whose axis intersects K'. For all $\gamma \in \Gamma$, set

$$\mathcal{A}_\gamma = \{\eta \in (\gamma\mathcal{A}) \cap \Gamma_{hyp} : \text{ the axis of } \eta \text{ intersects } K'\}.$$

Since $\mathcal{A}$ is finite, there exists $E > 0$ so that if $\gamma \in \Gamma$ and $\eta \in \mathcal{A}_\gamma$, then

$$\phi(\kappa(\rho(\eta))) \leq E + \phi(\kappa(\rho(\gamma))).$$

For any $\eta \in \Gamma_{hyp}$, let $\ell(\eta)$ denote the minimal translation distance of η in $\mathbb{H}^2$. Since K' is compact, there exists a positive integer D such that for any $\eta \in \Gamma_{hyp}$, at most $D\ell(\eta)$ of the Γ-translates of the axis of η intersect K', or equivalently, at most $D\ell(\eta)$ conjugates of η have axes that intersect K'. Also, since ρ is P_θ-Anosov, there are constants $C > 0$ and $c > 1$ such that

$$\ell(\eta) \leq \mathrm{d}_{\mathbb{H}^2}(\eta(b_0), b_0) \leq c\phi(\kappa(\rho(\eta))) + C$$

for all $\eta \in \Gamma_{hyp}$. Furthermore, note that any element $\eta \in \Gamma$ can lie in $\mathcal{A}_\gamma$ for at most $|\mathcal{A}|$ different $\gamma \in \Gamma$. Thus, for all $T > 0$,

$$\begin{aligned}
\#\left\{\gamma \in \Gamma \;\middle|\; \phi(\kappa(\rho(\gamma))) \leq T\right\} &\leq |\mathcal{A}|\#\left\{\eta \in \Gamma \;\middle|\; \eta \in \mathcal{A}_\gamma \text{ for some } \gamma \in \Gamma \text{ such that } \phi(\kappa(\rho(\gamma))) \leq T\right\} \\
&\leq |\mathcal{A}|\#\left\{\eta \in \Gamma_{hyp} \;\middle|\; \text{the axis of } \eta \text{ intersects } K' \text{ and } \phi(\kappa(\rho(\eta))) \leq E + T\right\} \\
&\leq |\mathcal{A}|D(c(E+T)+C)\#\left\{[\eta] \in [\Gamma_{hyp}] \;\middle|\; \phi(\nu(\rho(\gamma))) \leq E + T + a(\phi)C'\right\}.
\end{aligned}$$

This implies that $\delta^{\alpha_1}(\rho) \leq h^{\alpha_1}(\rho)$. □

We use a similar argument to show that the ϕ-Poincaré series $Q^\phi_{\rho(\Gamma)}$ diverges at its critical exponent.

Proposition 9.4. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is geometrically finite, $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{K})$ is P_θ-Anosov, and $\phi \in \mathcal{B}_\theta(\rho)^+$, then $Q^\phi_{\rho(\Gamma)}(\delta^\phi(\rho)) = +\infty$.*

Proof. It follows immediately from Theorem 2.3 that there exists $N \in \mathbb{N}$ and $A > 0$ so that if $n \geq N$ and

$$S^\phi_n(\rho) = \{[\gamma] \in [\Gamma_{hyp}] \;:\; n \leq \phi(\nu(\rho(\gamma))) < n+1\}, \qquad \text{then} \qquad \#S^\phi_n(\rho) \geq \frac{Ae^{n\delta^\phi(\rho)}}{n}.$$

The argument above shows that if $[\gamma] \in S^\phi_n(\rho)$, then there exists $\hat\gamma \in [\gamma]$ so that

$$\phi(\kappa(\rho(\hat\gamma)) \leq n + 1 + a(\phi)C.$$

Therefore, if $n \geq N$,

$$\sum_{\{\gamma\ :[\gamma]\in S^\phi_n(\rho)\}} e^{-\delta^\phi(\rho)\phi(\kappa(\rho(\gamma)))} \geq \frac{Ae^{n\delta^\phi(\rho)}}{n} e^{-\delta^\phi(\rho)(n+1+a(\phi)C)} \geq \frac{Ae^{-\delta^\phi(\rho)(1+a(\phi)C)}}{n}$$

which implies that $Q^\phi_{\rho(\Gamma)}(\delta^\phi(\rho)) = +\infty$. □

Remark 9.5. Propositions 9.1 and 9.4 do not hold for all P_θ-transverse representations of geometrically finite Fuchsian groups. If $\rho : \pi_1(S) \to \mathsf{PO}(3,1)$ is a geometrically infinite discrete faithful representation such that $\Lambda(\rho(\pi_1(S)))$ is not all of $\partial\mathbb{H}^3$, then ρ is a P_1-transverse representation into $\mathsf{PSL}(4,\mathbb{R})$. However, $R^{\omega_1}_T(\rho(\Gamma))$ is infinite for all sufficiently large T (see Bonahon [9]), so

$h^{\omega_1}(\rho) = +\infty$ with our definition. Moreover, $\delta^{\omega_1}(\rho) = 2$ (see Canary [17, Cor. 4.2]) and the α_1-Poincaré series converges at $s = 2$ (see Sullivan [52, Thm. II]).

10. Quint's indicator set

In this section, we develop analyticity and convexity properties for the set of linear functionals with entropy 1.

Suppose that Γ is a geometrically finite Fuchsian group and $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ is P_θ-Anosov. Following Sambarino [46], we define the *Quint indicator set*

$$\mathcal{Q}_\theta(\rho) = \{\phi \in \mathcal{B}^+_\theta(\rho(\Gamma)) \mid h^\phi(\rho) = 1\}$$

which arises as the boundary of

$$\mathcal{R}_\theta(\rho) = \{\phi \in \mathcal{B}^+_\theta(\rho(\Gamma)) \mid h^\phi(\rho) \leq 1\}.$$

The following result generalizes a result of Sambarino [46, Prop. 4.7] for Hitchin representations, see also [42, Prop. 4.11] and [48]. (Sambarino [46] was inspired by work of Quint [44] who studied the behavior of the entropy on the space of linear functionals in the case that the representation is Zariski dense.) Theorem 1.11 from the introduction is a special case of the result below.

Theorem 10.1. *Suppose that $\Gamma \subset \mathsf{PSL}(2, \mathbb{R})$ is geometrically finite and $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ is a P_θ-Anosov representation. Then*

(1) $\mathcal{Q}_\theta(\rho)$ intersects each ray (based at $\mathbf{0}$) in $\mathcal{B}^+_\theta(\rho(\Gamma))$ exactly once.
(2) $\mathcal{Q}_\theta(\rho)$ is an analytic submanifold of $\mathfrak{a}^*_\theta$.
(3) $\mathcal{R}_\theta(\rho)$ is a convex subset of $\mathfrak{a}^*_\theta$.
(4) If $\phi_1, \phi_2 \in \mathcal{Q}_\theta(\rho)$, then the line segment in $\mathfrak{a}^*_\theta$ between ϕ_1 and ϕ_2 lies in $\mathcal{Q}_\theta(\rho)$ if and only if

$$\phi_1(\nu(\rho(\gamma))) = \phi_2(\nu(\rho(\gamma)))$$

for all $\gamma \in \Gamma$.

In this proof, we will use the technology developed in Bray-Canary-Kao-Martone [11]. We note that the results of that paper were stated for representations into $\mathsf{SL}(d, \mathbb{R})$, but a careful reading verifies that the same arguments taken verbatim work for representations into $\mathsf{PGL}(d, \mathbb{R})$.

One first associates to any torsion-free, geometrically finite Fuchsian group Γ, a topologically mixing, countable, one-sided Markov shift (Σ^+, σ) with the big images and pre-images property and countable alphabet $\mathcal{A}$. If Γ is convex cocompact, we use the Bowen-Series coding [14], if Γ is a non-cocompact lattice, we use the Stadlbauer-Ledrappier-Sarig coding [38, 51], and in the remaining cases, we use the coding of Dal'bo-Peigné [24]. In all cases, such a coding gives a pair of maps

$$G : \mathcal{A} \to \Gamma \text{ and } \omega : \Sigma^+ \to \Lambda(\Gamma)$$

with the property that if $x = \overline{x_1 \cdots x_n} \in \mathrm{Fix}^n(\Sigma^+)$, then $\omega(x)$ is the attracting fixed point of $G(x_1) \ldots G(x_n)$. If $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ is P_θ-Anosov, then one can use Quint's Iwasawa cocycle [44] to construct a locally Hölder continuous, vector-valued function $\tau_\rho : \Sigma^+ \to \mathfrak{a}_\theta$ so that if $x = \overline{x_1 \cdots x_n} \in \mathrm{Fix}^n(\Sigma^+)$, then

$$S_n\tau_\rho(x) = \sum_{i=0}^{n-1} \tau_\rho(\sigma^i(x)) = p_\theta(\nu(G(x_1) \cdots G(x_n)))$$

see [11, Thm. D*]. See [11] for careful definitions and more detailed statements.

Given a locally Hölder continuous function $g : \Sigma^+ \to \mathbb{R}$ one may define its *Gurevich pressure*

$$P(g) = \sup_{\mu \in \mathcal{M}} h_\sigma(\mu) + \int_{\Sigma^+} g d\mu$$

where $\mathcal{M}$ is the space of σ-invariant probability measures on Σ^+ and $h_\sigma(\mu)$ is the measure-theoretic entopy of σ with respect to μ. Notice that this pressure need not be finite. However, it is analytic and convex on the space of finite pressure, locally α-Hölder continuous functions for any $\alpha > 0$ (see Mauldin-Urbanski [41, Thm. 2.6.12] or Sarig [49, Cor. 4]-[50, Prop. 4.4]). A measure $\mu_g \in \mathcal{M}$ is said to be an *equilibrium measure* for a locally Hölder continuous function $g : \Sigma^+ \to \mathbb{R}$ if

$$P(g) = h_\sigma(\mu_g) + \int_{\Sigma^+} g d\mu_g.$$

Proof of Theorem 10.1. First, notice that we may assume throughout that Γ is torsion-free, since any geometrically finite Fuchsian group Γ has a finite index torsion-free subgroup Γ_0 and $h^\phi(\rho(\Gamma)) = h^\phi(\rho(\Gamma_0))$ for all $\phi \in \mathcal{B}^+_\theta(\rho(\Gamma)) = \mathcal{B}^+_\theta(\rho(\Gamma_0))$.

If α is a parabolic element of Γ, then, by [19, Cor. 4.2], the quantity

$$c_k(\rho, \alpha) = \lim_{s \to \infty} \frac{\omega_k(\rho(\alpha^s))}{\log s}$$

is well-defined positive integer. If $\phi = \sum_{k \in \theta} a_k \omega_k \in \mathcal{B}^+_\theta(\rho(\Gamma))$, let

$$c(\rho, \phi) = \inf \left\{ \sum_{k \in \theta} a_k c_k(\rho, \alpha) : \alpha \in \Gamma \text{ parabolic} \right\}.$$

We recall the following results from Bray-Canary-Kao-Martone [11].

Proposition 10.2. *Let $\phi \in \mathcal{B}^+_\theta(\rho(\Gamma))$.*

(1) [11, Cor. 1.2, Lem. 3.3, Thm. D*] $0 < h^\phi(\rho) < +\infty$ *and* $P(-t\phi \circ \tau_\rho) = 0$ *if and only if* $t = h^\phi(\rho)$.

(2) [11, Lem. 3.3, Thm. D*] $P(-\phi \circ \tau_\rho) < +\infty$ *if and only if* $c(\rho, \phi) > 1$. *In particular,*

$$\mathcal{E}(\rho) = \{\phi \in \mathcal{B}^+_\theta(\rho(\Gamma)) \mid P(-\phi \circ \tau_\rho) < +\infty\}$$

is an open subset of $\mathfrak{a}^*_\theta$.

(3) [11, Lem. 3.4, Thm. D*] *If* $P(-\phi \circ \tau_\rho) < +\infty$, *then there is a unique equilibrium measure* $dm_{-\phi \circ \tau_\rho}$ *for* $-\phi \circ \tau_\rho$.

Notice that (1) follows immediately from Proposition 10.2 part (1) and the observation that $h^{k\phi} = \frac{1}{k} h^\phi$ for all $k > 0$.

The function $\hat{P} : \mathcal{E}(\rho) \to \mathbb{R}$ given by

$$\hat{P}(\phi) = P(-\phi \circ \tau_\rho)$$

is convex and analytic since P is convex and analytic. By Proposition 10.2 part (1), $\hat{P}(\phi) = 0$ if and only if $h^\phi(\rho) = 1$. Also, note that $\hat{P}(k\phi) < \hat{P}(\phi)$ for all $1 < k < \infty$. Thus,

$$\mathcal{R}_\theta(\rho) = \hat{P}^{-1}((-\infty, 0]) \text{ and } \mathcal{Q}_\theta(\rho) = \hat{P}^{-1}(0).$$

This proves (2) and (3).

Let $\phi_1, \phi_2 \in \mathcal{Q}_\theta(\rho)$ be distinct. Suppose that the line segment in $\mathfrak{a}^*_\theta$ between ϕ_1 and ϕ_2 also lies in $\mathcal{Q}_\theta(\rho)$, or equivalently, that

$$f(t) = P((t-1)\phi_1 \circ \tau_\rho - t\phi_2 \circ \tau_\rho) = 0$$

for all $t \in [0,1]$. For any $t \in [0,1]$, Proposition 10.2 part (3) states that there is a unique equilibrium measure m_t for the function $(t-1)\phi_1 \circ \tau_\rho - t\phi_2 \circ \tau_\rho$, so Mauldin and Urbanski's formula for the derivative of the pressure function [41, Prop. 2.6.13] (see also the discussion in [11, Section 2]) implies that

$$f'(t) = \int_{\Sigma^+} \left(\phi_1 \circ \tau_\rho - \phi_2 \circ \tau_\rho\right) \, dm_t = 0.$$

(Notice that by Proposition 10.2 part (2), $\mathcal{E}(\rho)$ is open, so f can be defined on some open interval containing $[0,1]$.) In particular, we can write the equality $f'(1) = 0$ as

$$\int_{\Sigma^+} \phi_1 \circ \tau_\rho \, dm_1 = \int_{\Sigma^+} \phi_2 \circ \tau_\rho \, dm_1.$$

Since, $P(-\phi_2 \circ \tau_\rho) = 0$ and m_1 is an equilibrium measure for $-\phi_2 \circ \tau_\rho$,

$$h_\sigma(m_1) = \int_{\Sigma^+} \phi_2 \circ \tau_\rho \, dm_1$$

so we see that

$$h_\sigma(m_1) - \int_{\Sigma^+} \phi_1 \circ \tau_\rho \, dm_1 = \int_{\Sigma^+} \phi_2 \circ \tau_\rho \, dm_1 - \int_{\Sigma^+} \phi_1 \circ \tau_\rho \, dm_1 = 0.$$

So, m_1 is also an equilibrium measure for $-\phi_1 \circ \tau_\rho$. Since Proposition 10.2 part (3) implies that $-\phi_1 \circ \tau_\rho$ has a unique equilibrium measure, $m_0 = m_1$. Sarig [50, Thm. 4.8] showed that this only happens when $\phi_1 \circ \tau_\rho$ and $\phi_2 \circ \tau_\rho$ are cohomologous, so the Livsic Theorem [50, Thm. 1.1] (see also Mauldin-Urbanski [41, Thm. 2.2.7]) implies that $\phi_1(\nu(\rho(\gamma))) = \phi_2(\nu(\rho(\gamma)))$ for all $\gamma \in \Gamma$. We have completed the proof. □

Since domain groups of (traditional) Anosov representations admit topologically mixing, finite Markov codings (see Bridgeman-Canary-Labourie-Sambarino [13] or Constantine-LaFont-Thompson [20]), one may apply the exact same argument to obtain the analogous result in this setting, see also Sambarino [48].

Theorem 10.3. *Suppose that Γ is a hyperbolic group and $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{R})$ is a P_θ-Anosov representation. Then*

(1) $\mathcal{Q}_\theta(\rho)$ intersects each ray (based at $\mathbf{0}$) in $\mathcal{B}_\theta^+(\rho(\Gamma))$ exactly once.
(2) $\mathcal{Q}_\theta(\rho)$ is a analytic submanifold of $\mathfrak{a}_\theta^*$.
(3) $\mathcal{R}_\theta(\rho)$ is a convex subset of $\mathfrak{a}_\theta^*$.
(4) If $\phi_1, \phi_2 \in \mathcal{Q}_\theta(\rho)$, then the line segment in $\mathfrak{a}_\theta^*$ between ϕ_1 and ϕ_2 lies in $\mathcal{Q}_\theta(\rho)$ if and only if

$$\phi_1(\nu(\rho(\gamma))) = \phi_2(\nu(\rho(\gamma)))$$

for all $\gamma \in \Gamma$.

11. Entropy rigidity

In this section we establish our main result. It remains to prove Proposition 1.6, which bounds the Hausdorff dimension of the limit set of a Hitchin representation, check that all exterior powers of Hitchin representations are $(1,1,2)$-hypertransverse and carefully apply Sambarino's classification of Zariski closures of Hitchin representations.

11.1. **Hausdorff dimension of positive curves.** We prove that the image of a continuous, positive map has Hausdorff dimension at most 1. The proof below is a mild generalization of the standard proof that the graph of a monotonic function $f : [a, b] \to \mathbb{R}$ has Hausdorff dimension 1. Notice that Proposition 1.6 is an immediate consequence.

Proposition 11.1. *Let $X \subset \mathbb{S}^1$ be a closed subset. If $\xi : X \to \mathcal{F}$ is a continuous, positive map, then $\dim_H(\xi(X)) \leq 1$.*

Proof. If X is a finite set, then $\dim_H(\xi(X)) = 0$. Hence, we may assume that X is infinite.

Let $x, y, z \in X$ be mutually distinct points and let

$$I := \{w \in X : x \leq w \leq y < z < x\}.$$

Since any positive n-tuple of flags consists of mutually transverse flags, see Lemma 2.4, $\xi(I)$ is a compact subset of the affine chart

$$\mathbb{A}_{\xi(z)} := \{F \in \mathcal{F} : F \text{ is transverse to } \xi(z)\}.$$

Furthermore, if we let $U_{\xi(z)} \subset \mathsf{SL}(d, \mathbb{R})$ denote the subgroup of unipotent elements that fix $\xi(z)$, then we have a real analytic diffeomorphism

$$L : U_{\xi(z)} \to \mathbb{A}_{\xi(z)}$$

given by $L(w) = w(\xi(x))$. Let

$$u := L^{-1} \circ \xi : I \to U_{\xi(z)}.$$

To prove the lemma, it is sufficient to show that $\dim_H(u(I)) \leq 1$ (with respect to any Riemannian metric on $U_{\xi(z)}$).

Observe that any ordered basis compatible with $(\xi(z), \xi(x))$ defines an identification of $U_{\xi(z)}$ with the group of unipotent, upper triangular matrices in $\mathsf{SL}(d, \mathbb{R})$. Since ξ is a positive map, there exists an ordered basis $\mathcal{B}$ compatible with $(\xi(z), \xi(x))$ so that if $s, t \in I$ such that $x \leq s < t \leq y$, then

$$u(s)^{-1} u(t) \in U_{>0}(\mathcal{B}) \subset U_{\xi(z)}.$$

In particular, every upper triangular entry of $u(t)$ is strictly larger than the corresponding upper triangular entry of $u(s)$. Thus, it is now sufficient to prove that if $I \subset [0, 1]$ is a closed subset and

$$f = (f_1, \dots, f_D) : I \to \mathbb{R}^D$$

is a continuous map such that $f_i : I \to \mathbb{R}$ is an increasing function for all i, then $\dim_H(f(I)) \leq 1$ (in the Euclidean metric) in $\mathbb{R}^\ell$. We will verify the stronger condition that the 1-dimensional Hausdorff measure of $f(I)$, denoted $\mathcal{H}^1(f(I))$, is finite.

For every $n \in \mathbb{Z}^+$ and $k \in \{0, \dots, n-1\}$, let

$$I_{n,k} = \left[\frac{k}{n}, \frac{k+1}{n}\right] \cap I,$$

and let

$$R_{n,k} = [f_1(\min I_{n,k}), f_1(\max I_{n,k})] \times \cdots \times [f_D(\min I_{n,k}), f_D(\max I_{n,k})] \subset \mathbb{R}^D.$$

Observe that for all $n \in \mathbb{Z}^+$, $\{R_{n,k} : k = 0, \dots, n-1\}$ covers $f(I)$ because f_i is increasing for all i. Furthermore,

$$\sum_{k=0}^{n-1} \operatorname{diam}(R_{n,k}) \leq \sum_{k=0}^{n-1} \left(\sum_{i=1}^{D} f_i\left(\max I_{n,k}\right) - f_i\left(\min I_{n,k}\right) \right) \leq \sum_{i=1}^{D} f_i\left(\max I\right) - f_i\left(\min I\right).$$

Since f is continuous,

$$\lim_{n\to\infty} \sup\{\operatorname{diam}(R_{n,k}) : k = 0, \dots, n-1\} = 0.$$

It follows that $\mathcal{H}^1(f(I)) \le \sum_{i=1}^D f_i(\max I) - f_i(\min I) < \infty$. □

11.2. **Simple root critical exponents of Hitchin representations.** We first observe that all exterior powers of Hitchin representations are $(1,1,2)$-hypertransverse. Let $E^k : \mathsf{PGL}(d,\mathbb{R}) \to \mathsf{PGL}(\bigwedge^k \mathbb{R}^d)$ be the representation defined in Section 2.5.1.

Proposition 11.2. *If $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation, then it is P_Δ-transverse. Furthermore, $E^k \circ \rho(\Gamma)$ is $(1,1,2)$-hypertransverse for all $k \in \{1,\dots,d-1\}$.*

In the case where Γ is a cocompact lattice, the hypertransversality follows from the work of Labourie [36], see [43, Proposition 9.6]. We give a more direct proof using positivity, which holds for any discrete $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$.

Proof. In [19, Thm. 7.1] we proved that if Γ is finitely generated, then ρ is P_Δ-Anosov, hence P_Δ-transverse. If Γ is infinitely generated, the proof given there shows that ρ is P_Δ-transverse. Let $\xi : \Lambda(\Gamma) \to \mathcal{F}$ be the boundary map (which is positive).

Fix $k \in \{1,\dots,d-1\}$. We begin by defining the boundary map of $E^k \circ \rho$. For all $F \in \mathcal{F}$, choose a basis $(v_1,\dots,v_d)$ of $\mathbb{R}^d$ that is compatible with F, i.e. $F^i = \operatorname{Span}_{\mathbb{R}}(v_1,\dots,v_i)$ for all $i = 1,\dots,d$. This induces a basis

$$(w_1,\dots,w_\ell) = (v_{i_1} \wedge \dots \wedge v_{i_k})_{1 \le i_1 < \dots < i_k \le d} \quad \text{of} \quad \bigwedge^k \mathbb{R}^d,$$

enumerated according to the dictionary ordering in the subscripts. Let $D = \dim \bigwedge^k \mathbb{R}^d$ and $\hat{\mathcal{F}} = \mathcal{F}_{1,2,D-2,D-1}(\bigwedge^k \mathbb{R}^d)$. One can verify that we have a well-defined map $W_k : \mathcal{F} \to \hat{\mathcal{F}}$ given by

$$W_k(F) = \left(\operatorname{Span}_{\mathbb{R}}(w_1), \operatorname{Span}_{\mathbb{R}}(w_1,w_2), \operatorname{Span}_{\mathbb{R}}(w_1,\dots,w_{D-2}), \operatorname{Span}_{\mathbb{R}}(w_1,\dots,w_{D-1})\right).$$

i.e. $W_k(F)$ does not depend on the choice of the basis compatible with F. Then define

$$\zeta := W_k \circ \xi : \Lambda(\Gamma) \to \hat{\mathcal{F}}.$$

It is clear that ζ is continuous and $E^k \circ \rho$-equivariant.

To show that ζ is transverse, fix $x, y \in \Lambda(\Gamma)$ distinct. Then fix a basis $\mathcal{B} = (v_1,\dots,v_d)$ of $\mathbb{R}^d$ such that $v_i \in \xi(x)^i \cap \xi(y)^{d-i+1}$ for all $i = 1,\dots,d$. Then writing $\zeta(x)$ and $\zeta(y)$ in terms of the basis $(v_{i_1} \wedge \dots \wedge v_{i_k})_{1 \le i_1 < \dots < i_k \le d}$ shows that $\zeta(x)$ and $\zeta(y)$ are transverse.

Next we show that ζ is dynamics preserving. Fix a sequence $\{\gamma_n\}$ in Γ such that $\gamma_n \to x$ and $\gamma_n^{-1} \to y$. Since ρ is P_Δ-transverse, Lemma 2.1 implies that $\alpha_k(\rho(\gamma_n)) \to \infty$ for all $k \in \Delta$,

$$U_\Delta(\rho(\gamma_n)) \to \xi(x) \quad \text{and} \quad U_\Delta(\rho(\gamma_n^{-1})) \to \xi(y).$$

If we identify $\bigwedge^k \mathbb{R}^d$ with $\mathbb{R}^D$ using the basis

$$(e_{i_1} \wedge \dots \wedge e_{i_k})_{1 \le i_1 < \dots < i_k \le d}$$

then one can check that $\alpha_i(E^k \circ \rho(\gamma_n)) \to \infty$ for $i \in \{1, 2, D-1, D-2\}$,

$$U_{1,2,D-2,D-1}(E^k \circ \rho(\gamma_n)) = W_k\left(U_\Delta(\rho(\gamma_n))\right) \to \zeta(x) \quad \text{and} \quad U_{1,2,D-2,D-1}(E^k \circ \rho(\gamma_n^{-1})) \to \zeta(y).$$

Hence by Lemma 2.1,

$$E^k \circ \rho(\gamma_n)(F) \to \zeta(x)$$

for all $F \in \hat{\mathcal{F}}$ transverse to $\zeta(y)$.

Finally we show that $E^k\circ\rho$ is $(1,1,2)$-hypertransverse. Fix $x_1,x_2,x_3\in\Lambda(\Gamma)$ pairwise distinct. Then there is a basis $\mathcal{B}=(b_1,\dots,b_d)$ of $\mathbb{R}^d$ such that $b_i\in\xi(x_1)^i\cap\xi(x_3)^{d-i+1}$ for all $i=1,\dots,d$, and $\xi(x_2)=u(\xi(x_3))$ for some $u\in U^+_{>0}(\mathcal{B})$. Observe that

- $\zeta^1(x_1)=\mathrm{Span}_{\mathbb{R}}(b_1\wedge\dots\wedge b_k)$,
- $\zeta^{\ell-2}(x_3)=\mathrm{Span}_{\mathbb{R}}(b_{i_1}\wedge\dots\wedge b_{i_k}:\{i_1,\dots,i_k\}\neq\{1,\dots,k\},\{1,\dots,k-1,k+1\})$
- $\zeta^1(x_2)=\mathrm{Span}_{\mathbb{R}}\left(\sum_{1\le i_1<\dots<i_k\le d}u^{i_1,\dots,i_k}_{d-k+1,\dots,d}\,b_{i_1}\wedge\dots\wedge b_{i_k}\right)$, where $u^{i_1,\dots,i_k}_{j_1,\dots,j_k}$ denotes the minor given by the rows $1\le i_1<\dots<i_k\le d$ and the columns $1\le j_1<\dots<j_k\le d$ of the $d\times d$ matrix representing u in the basis $\mathcal{B}$.

In particular, $u^{1,\dots,k-1,k+1}_{d-k+1,\dots,d}\neq 0$, so

$$\zeta^1(x_1)+\zeta^1(x_2)+\zeta^{D-2}(x_3)$$

is a direct sum. □

We may now assemble the proof of Corollary 1.7 which we restate here.

Corollary 11.3. *If $\Gamma\subset\mathsf{PSL}(2,\mathbb{R})$ is discrete and $\rho:\Gamma\to\mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation, then*

$$\delta^{\alpha_k}(\rho)\le 1$$

for all $k\in\{1,\dots,d-1\}$. Moreover, if Γ is a lattice, then $\delta^{\alpha_k}(\rho)=1$ for all $k\in\{1,\dots,d-1\}$.

Proof. By Proposition 11.2, $E^k\circ\rho$ is $(1,1,2)$-hypertransverse. Theorem 1.5 implies that

$$\dim_H\left(\Lambda_{1,c}(E^k\circ\rho(\Gamma))\right)=\delta^{\alpha_1}\left(E^k\circ\rho\right)=\delta^{\alpha_k}(\rho).$$

Let $W_k:\mathcal{F}\to\hat{\mathcal{F}}$ be as in the proof of Proposition 11.2. Since W_k is smooth and $\Lambda_{1,c}(E^k\circ\rho(\Gamma))=W_k^1(\Lambda_{k,c}(\rho(\Gamma)))$, Proposition 1.6 implies that

$$\dim_H\left(\Lambda_{1,c}(E^k\circ\rho(\Gamma))\right)\le 1.$$

Moreover, if Γ is a lattice, then $\Lambda_1(E^k\circ\rho(\Gamma))$ is a curve and $\Lambda_1(E^k(\rho(\Gamma)))\setminus\Lambda_{1,c}(E^k(\rho(\Gamma))$ is countable, so $\dim_H\left(\Lambda_{1,c}(E^k\circ\rho(\Gamma))\right)=1$. □

We next show that $\dim_H\left(\Lambda_1(E^k\circ\rho(\Gamma))\right)<1$ if Γ is geometrically finite but not a lattice. We make use of the following generalization of a result of Labourie-McShane [37] which will be established in the appendix.

Proposition 11.4. *Suppose that $\Gamma\subset\mathsf{PSL}(2,\mathbb{R})$ is geometrically finite and torsion-free. If $\rho:\Gamma\to\mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation, then there exists a torsion-free lattice $\Gamma^D\subset\mathsf{PSL}(2,\mathbb{R})$ so that $\Gamma\subset\Gamma^D$ and a Hitchin representation $\rho^D:\Gamma^D\to\mathsf{PSL}(d,\mathbb{R})$ so that $\rho=\rho^D|_\Gamma$.*

We then use an argument due to Furusawa [26] to show that the critical exponent drops when one passes to an infinite index geometrically finite subgroup. Corollary 1.10 then follows from Propositions 11.4 and 11.5 and Corollary 1.7.

Proposition 11.5. *Suppose that $\Gamma\subset\mathsf{PSL}(2,\mathbb{R})$ is discrete and $G\subset\Gamma$ is a geometrically finite, infinite index subgroup. If $\rho:\Gamma\to\mathsf{PGL}(d,\mathbb{R})$ is a P_k-transverse representation, then $\delta^{\alpha_k}(\rho|_G)<\delta^{\alpha_k}(\rho)$.*

Proof. Since $G\subset\Gamma$ is geometrically finite and infinite index, $\Lambda(G)$ is a proper closed subset of $\Lambda(\Gamma)$. (If $C(G)$ is the convex hull of $\Lambda(G)$ in $\mathbb{H}^2$, then $G\backslash C(G)$ has finite area. But if $\Lambda(G)=\Lambda(\Gamma)$, then Γ preserves $C(G)$ which contradicts the fact that Γ is discrete and G has infinite index.)

We may assume, without loss of generality, that G is torsion-free. Let D be a finite-sided convex fundamental polygon for the action of G on $\mathbb{H}^2$ and let I be a component of the intersection of the closure of D with $\partial\mathbb{H}^2$ whose interior has a non-empty intersection with $\Lambda(\Gamma)$. Since fixed point pairs of hyperbolic elements are dense in $\Lambda(\Gamma)\times\Lambda(\Gamma)$, we may find a hyperbolic element $\gamma\in\Gamma$, both of whose fixed points lie in the interior of I. By passing to a power of γ, if necessary, we may assume that there exists disjoint closed geodesic half-planes H^+ and H^- centered at γ^+ and γ^- and contained in the interior of D so that $\gamma(H^-)=\mathbb{H}^2-\mathrm{int}(H^+)$, see Figure 5. Klein's combination theorem then implies that $\hat{D}=D-\mathrm{int}(H^+\cup H^-)$ is a fundamental domain for $\hat{\Gamma}=\langle G,\gamma\rangle = G*\langle\gamma\rangle$.

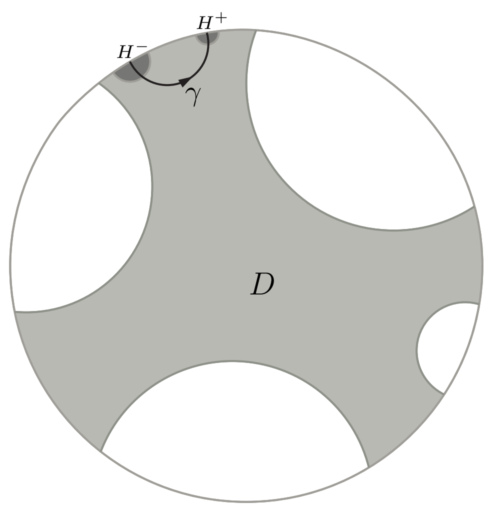

FIGURE 5. D, H^-, and H^+.

Fix $b_0\in\hat{D}$. Let $\iota_{b_0}:\mathbb{H}^2-\{b_0\}\to\partial\,\mathbb{H}^2$ be the map so that z is contained in the geodesic ray starting at b_0 and limiting to $\iota_{b_0}(z)$. Choose $\epsilon>0$ small enough so that

$$\mathrm{d}_{\partial\,\mathbb{H}^2}(\iota_{b_0}(H^+\cup H^-),\Lambda(G))\geq 2\epsilon. \tag{30}$$

There is a finite subset $F\subset G$ such that

$$\mathrm{d}_{\partial\,\mathbb{H}^2}(\iota_{b_0}(\eta(p)),\Lambda(G))<\epsilon \tag{31}$$

for all $p\in D$ and all $\eta\in G-F$. Hence, by the residual finiteness of G, we may replace G with a finite index subgroup to assume that (31) holds for all $\eta\in G-\mathrm{id}$ and all $p\in D$.

Let $\Gamma(n)\subset\Gamma$ denote the set of elements of the form $\eta_1\gamma_1\dots\eta_n\gamma_n$, where each η_i is a non-trivial element of G, and each γ_i is a non-trivial element of $\langle\gamma\rangle$. (We adopt the convention that $\Gamma(0)=\{\mathrm{id}\}$.)

Lemma 11.6. *For all* $\zeta\in\bigcup_{n>0}\Gamma(n)$, $\eta\in G-\mathrm{id}$, *and* $m\neq 0$, *we have*

$$\mathrm{d}_{\partial\,\mathbb{H}^2}(\iota_{b_0}(\zeta^{-1}(b_0),\iota_{b_0}(\eta(b_0))\geq\epsilon \ \ \textit{and}\ \ \mathrm{d}_{\partial\,\mathbb{H}^2}(\iota_{b_0}(\eta^{-1}\zeta^{-1}(b_0),\iota_{b_0}(\gamma^m(b_0))\geq\epsilon.$$

Proof. Since $b_0\in\hat{D}$,$\gamma^m(b_0)\in H^+\cup H^-$ if $m\neq 0$, so (30) gives

$$\mathrm{d}_{\partial\,\mathbb{H}^2}(\iota_{b_0}(\gamma^m(b_0)),\Lambda(G))\geq 2\epsilon.$$

Moreover, by (31),

$$\mathrm{d}_{\partial\,\mathbb{H}^2}(\iota_{b_0}(\eta(b_0)),\Lambda(G))<\epsilon$$

for all $\eta\in G-\mathrm{id}$.

We may apply a ping-pong argument to observe that $\zeta^{-1}(b_0) \in H^+ \cup H^-$ for all $\zeta \in \bigcup_{n>0} \Gamma(n)$, so

$$d_{\partial \mathbb{H}^2}(\iota_{b_0}(\zeta^{-1}(b_0)), \Lambda(G)) \geq 2\epsilon.$$

Then since $\zeta^{-1}(b_0) \in D$, (31) implies that

$$d_{\partial \mathbb{H}^2}(\iota_{b_0}(\eta^{-1}\zeta^{-1}(b_0)), \Lambda(G)) < \epsilon$$

for all $\eta \in G - \mathrm{id}$. The lemma follows easily from these four inequalities. □

For any subset $S \subset \Gamma$, denote

$$Q_S^{\alpha_k}(s) = \sum_{\gamma \in S} e^{-s\alpha_k(\kappa(\rho(\gamma)))}.$$

Lemma 11.7. *There is some $C > 0$ such that for all integers $n \geq 0$,*

$$Q_{\Gamma(n)}^{\alpha_k}(s) \geq \left(e^{-2Cs} Q_{G-\mathrm{id}}^{\alpha_k}(s) Q_{\langle\gamma\rangle-\mathrm{id}}^{\alpha_k}(s)\right)^n$$

Proof. We prove this by induction. For the base case $n = 0$ is trivial. For the inductive step, suppose that $n > 1$ and note that

$$Q_{\Gamma(n)}^{\alpha_k}(s) = \sum_{(\zeta,\eta,\gamma^m) \in \Gamma(n-1)\times(G-\mathrm{id})\times(\langle\gamma\rangle - id)} e^{-s\alpha_k(\kappa(\rho(\zeta\eta\gamma^m)))}$$

By Lemma 11.6 and Lemma 6.5, there is some $C > 0$ such that

$$\begin{aligned} \alpha_k(\kappa(\rho(\zeta\eta\gamma^m))) &\leq \alpha_k(\kappa(\rho(\zeta\eta))) + \alpha_k(\kappa(\rho(\gamma^m))) + C \\ &\leq \alpha_k(\kappa(\rho(\zeta))) + \alpha_k(\kappa(\rho(\eta))) + \alpha_k(\kappa(\rho(\gamma^m))) + 2C \end{aligned}$$

for all $\zeta \in \bigcup_{n>0} \Gamma(n)$, $\eta \in G - \mathrm{id}$, and $m \neq 0$. Thus,

$$\begin{aligned} Q_{\Gamma(n)}^{\alpha_k}(s) &\geq \sum_{\zeta \in \Gamma(n-1)} \sum_{\eta \in G-\mathrm{id}} \sum_{m \neq 0} e^{-2Cs} e^{-s\alpha_k(\kappa(\rho(\zeta)))} e^{-s\alpha_k(\kappa(\rho(\eta)))} e^{-s\alpha_k(\kappa(\rho(\gamma^m)))} \\ &= e^{-2Cs} Q_{\Gamma(n-1)}^{\alpha_k}(s) Q_{G-\mathrm{id}}^{\alpha_k}(s) Q_{\langle\gamma\rangle-\mathrm{id}}^{\alpha_k}(s) \\ &\geq \left(e^{-2Cs} Q_{G-\mathrm{id}}^{\alpha_k}(s) Q_{\langle\gamma\rangle-\mathrm{id}}^{\alpha_k}(s)\right)^n, \end{aligned}$$

where the last inequality is a consequence of the inductive hypothesis. □

Proposition 9.4 implies that $Q_{G-\mathrm{id}}^{\alpha_k}(\delta^{\alpha_k}(\rho|_G)) = \infty$. Thus, there is some $s_0 > \delta^{\alpha_k}(\rho|_G)$ such that

$$e^{-2Cs_0} Q_{G-\mathrm{id}}^{\alpha_k}(s_0) Q_{\langle\gamma\rangle-\mathrm{id}}^{\alpha_k}(s_0) \geq 1.$$

By Lemma 11.7,

$$\sum_{n=0}^{\infty} \left(e^{-2Cs} Q_{G-\mathrm{id}}^{\alpha_k}(s) Q_{\langle\gamma\rangle-\mathrm{id}}^{\alpha_k}(s)\right)^n \leq \sum_{n=0}^{\infty} Q_{\Gamma(n)}^{\alpha_k}(s) \leq Q_\rho^{\alpha_k}(s)$$

for all $s \geq 0$. Hence, $Q_\rho^{\alpha_k}(s_0) = \infty$, which implies that

$$\delta^{\alpha_k}(\rho) \geq s_0 > \delta^{\alpha_k}(\rho|_G).$$

□

11.3. **Proof of entropy rigidity.** We are now ready to prove our main result, Theorem 1.1, which we restate here for reference.

Theorem 11.8. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is geometrically finite, $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is Hitchin and $\phi = \sum c_j \alpha_j \in (\mathfrak{a}^*)^+$, then*

$$h^\phi(\rho) \le \frac{1}{c_1 + \cdots + c_{d-1}}.$$

Moreover, equality occurs if and only if Γ is a lattice and either

(1) $\phi = c_k \alpha_k$ for some k.
(2) $\rho(\Gamma)$ lies in an irreducible image of $\mathsf{PSL}(2,\mathbb{R})$.
(3) $d = 2n-1$, the Zariski closure of $\rho(\Gamma)$ is conjugate to $\mathsf{PSO}(n, n-1)$ and $\phi = c_k\alpha_k + c_{d-k}\alpha_{d-k}$ for some k.
(4) $d = 2n$, the Zariski closure of $\rho(\Gamma)$ is conjugate to $\mathsf{PSp}(2n,\mathbb{R})$ and $\phi = c_k\alpha_k + c_{d-k}\alpha_{d-k}$ for some k.
(5) $d = 7$, the Zariski closure of $\rho(\Gamma)$ is conjugate to G_2 and $\phi = c_1\alpha_1 + c_3\alpha_3 + c_4\alpha_4 + c_6\alpha_6$, or $\phi = c_2\alpha_2 + c_5\alpha_5$.

Proof. Corollary 1.7 and 1.10, and Proposition 9.1 imply that for all $k \in \Delta$,

$$h^{\alpha_k}(\rho) = \delta^{\alpha_k}(\rho) \le 1$$

with equality if and only if Γ is a lattice.

Since $h^{a\phi}(\rho) = \frac{1}{a} h^\phi(\rho)$ for all $\phi \in \mathcal{B}^+(\rho)$ and $a > 0$, it follows from Theorem 10.1 part (3) that, if $\phi = \sum c_j\alpha_j \neq 0$ and $c_j \ge 0$ for all j, then

$$h^\phi(\rho) \le \frac{1}{\frac{c_1}{h^{\alpha_1}(\rho)} + \cdots + \frac{c_{d-1}}{h^{\alpha_{d-1}}(\rho)}} \le \frac{1}{c_1 + \cdots + c_{d-1}}.$$

In particular, if Γ is not a lattice then,

$$h^\phi(\rho) < \frac{1}{c_1 + \cdots + c_{d-1}}.$$

Now, suppose that Γ is a lattice. Then

$$h^\phi(\rho) \le \frac{1}{c_1 + \cdots + c_{d-1}}.$$

Let T_ϕ denote the simplex in $\mathfrak{a}^*$ whose vertices are the set $\{\alpha_j : c_j \neq 0\}$. Theorem 10.1 part (3) implies that if equality holds then $T_\phi \subset \mathcal{Q}_\Delta(\rho)$, and so by Theorem 10.1 part (4), $\alpha_i(\nu(\rho(\gamma))) = \alpha_j(\nu(\rho(\gamma)))$ if $c_i \neq 0$ and $c_j \neq 0$.

We recall that Sambarino [47] classified the possible Zariski closures of images of Hitchin representations.

Theorem 11.9. [47, Theorem A] *Suppose that $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is a lattice, and $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation. Then the Zariski closure of $\rho(\Gamma)$ either lies in an irreducible image of $\mathsf{PSL}(2,\mathbb{R})$ or is conjugate to either $\mathsf{PSL}(d,\mathbb{R})$, $\mathsf{PSp}(2n,\mathbb{R})$ when $d = 2n$, $\mathsf{PSO}(n,n-1)$ when $d = 2n-1$, or G_2 when $n = 7$.*

Since the Zariski closure of $\rho(\Gamma)$ is semi-simple, a result of Benoist [3] implies that $\mathcal{B}(\rho)$ is a convex open subset of the limit cone of its Zariski closure with non-empty interior. Therefore, if g lies in the Zariski closure of $\rho(\Gamma)$, then $\alpha_i(\nu(g)) = \alpha_j(\nu(g))$, if $c_i \neq 0$ and $c_j \neq 0$.

Our result then follows from the following observation about subgroups of $\mathsf{PSL}(d,\mathbb{R})$ (see, for example the discussion in [57, Appendix B]).

Proposition 11.10. *Suppose* $\mathsf{G} \subset \mathsf{PSL}(d,\mathbb{R})$ *is a group and*

$$S(\mathsf{G}) = \{(i,j) : i < j \text{ and } \alpha_i(\nu(g)) = \alpha_j(\nu(g)) \text{ for all } g \in G\}.$$

1. *If* $G = \mathsf{PSL}(d,\mathbb{R})$, *then* $S(\mathsf{G}) = \emptyset$.
2. *If* G *lies in an irreducible image of* $\mathsf{PSL}(2,\mathbb{R})$, *then* $S(\mathsf{G}) = \{(i,j) : i < j\}$.
3. *If* $d = 2n$ *and* $G = \mathsf{PSp}(2n,\mathbb{R})$, *then* $S(\mathsf{G}) = \{(k, d-k) : 1 \le k \le n-1\}$.
4. *If* $d = 2n-1$ *and* $G = \mathsf{PSO}(n, n-1)$, *then* $S(\mathsf{G}) = \{(k, d-k) : 1 \le k \le n-1\}$.
5. *If* $d = 7$ *and* $G = \mathsf{G}_2$, *then* $S(\mathsf{G}) = \{(i,j) : i < j \text{ and } i,j \in \{1,3,4,6\}\} \cup \{(2,5)\}$.

□

12. Proof of Corollary 1.12

In this section we prove Corollary 1.12 which we restate here.

Corollary 12.1. *If* $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ *is geometrically finite and* $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ *is Hitchin, then*

$$\delta_X(\rho) \le 1.$$

Moreover, $\delta_X(\rho) = 1$ *if and only if* Γ *is a lattice and* $\rho(\Gamma)$ *lies in the image of an irreducible representation* $\mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$.

Proof. Let $x_0 = [\mathsf{PSO}(d)] \in X$ and

$$C = (d-1)^2 + (d-3)^2 + \cdots + (3-d)^2 + (1-d)^2.$$

Then

$$\mathrm{d}_X(g(x_0), x_0) = \frac{2}{\sqrt{C}} \sqrt{\sum_{j=1}^{d} (\log \sigma_j(g))^2} = \frac{2}{\sqrt{C}} \left\| \kappa(g) \right\|_2$$

for all $g \in \mathsf{PSL}(d,\mathbb{R})$.

Let $e_1^*, \ldots, e_d^* \in \mathfrak{a}^*$ be the elements with

$$e_j^*(\mathrm{diag}(A_1, \ldots, A_d)) = A_j.$$

Then define

$$\begin{aligned}
\phi &= \frac{2}{C}\Big((d-1)e_1^* + (d-3)e_2^* + \cdots + (3-d)e_{d-1}^* + (1-d)e_d^*\Big) \\
&= \frac{2}{C}\left(\sum_{j=1}^{\lfloor \frac{d-1}{2} \rfloor} (d+1-2j)(\alpha_j + \cdots + \alpha_{d-j}) + \left(d - 2\left\lfloor \frac{d-1}{2} \right\rfloor - 1\right) \alpha_{\lfloor d/2 \rfloor} \right).
\end{aligned}$$

Observe that the sum of all the coefficients of

$$\sum_{j=1}^{\lfloor \frac{d-1}{2} \rfloor} (d+1-2j)(\alpha_j + \cdots + \alpha_{d-j}) + \left(d - 2\left\lfloor \frac{d-1}{2} \right\rfloor - 1\right) \alpha_{\lfloor d/2 \rfloor} \quad \text{is} \quad \frac{C}{2}.$$

Thus, by Theorem 1.1 and Proposition 1.8, we have

$$\delta^{\phi}(\rho) \le 1.$$

Further,

$$\phi(\kappa(g)) \le \left\|\phi\right\|_2 \left\|\kappa(g)\right\|_2 = \frac{2}{\sqrt{C}} \left\|\kappa(g)\right\|_2 = \mathrm{d}_X(g(x_0), x_0)$$

for all $g \in \mathsf{PSL}(d,\mathbb{R})$. Hence

$$\delta_X(\rho) \le \delta^{\phi}(\rho) \le 1.$$

If $\delta_X(\rho) = 1$, then $\delta^{\phi}(\rho) = 1$ and hence Theorem 1.1 implies that Γ is a lattice and $\rho(\Gamma)$ lies in the image of an irreducible representation $\mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$.

For the other direction, suppose that Γ is a lattice and $\rho(\Gamma)$ lies in the image of an irreducible representation $\tau : \mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$. Let $\Gamma_1 := \tau^{-1}(\rho(\Gamma))$. Since τ induces an isometry $\mathbb{H}^2 \hookrightarrow X$,

$$\delta_X(\rho) = \delta_{\mathbb{H}^2}(\Gamma_1).$$

Thus it suffices to prove that Γ_1 is a lattice in $\mathsf{PSL}(2,\mathbb{R})$.

Since $\Lambda(\Gamma) = \partial\,\mathbb{H}^2$, each element of Γ acts non-trivially on $\Lambda(\Gamma)$. Then by the equivariance of the Anosov boundary map, ρ must be faithful. So

$$\gamma \in \Gamma \mapsto \tau^{-1}(\rho(\gamma)) \in \Gamma_1$$

is an isomorphism. Further, by [19, Theorem 1.4], $\alpha \in \Gamma$ is parabolic if and only if $\tau^{-1}(\rho(\alpha))$ is parabolic. So [55, Theorem 3.3] implies that $\Lambda(\Gamma_1)$ is homeomorphic to $\Lambda(\Gamma) = \partial\,\mathbb{H}^2$. So $\Lambda(\Gamma_1) = \partial\,\mathbb{H}^2$. Hence Γ_1 is a lattice and

$$\delta_X(\rho) = \delta_{\mathbb{H}^2}(\Gamma_1) = 1. \qquad \square$$

Appendix A. Doubling Hitchin representations

In this appendix, we show that if Γ is a geometrically finite Fuchsian group uniformizing a finite area hyperbolic 2-orbifold with non-empty totally geodesic boundary Q and $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation, then we may extend ρ to a Hitchin representation of Γ_D where Γ_D uniformizes the double of Q along its boundary and $\Gamma \subset \Gamma_D$. Proposition 11.4 is an immediate consequence of this result.

When Γ is geometrically finite but not a lattice, $\Lambda(\Gamma)$ is a proper subset of $\partial\,\mathbb{H}^2$. Let $\mathcal{C}(\Gamma) \subset \mathbb{H}^2$ denote the convex hull of $\Lambda(\Gamma)$, and let $\mathcal{S}(\Gamma)$ denote the set of boundary components in $\mathbb{H}^2$ of $\mathcal{C}(\Gamma)$. Note that every $b \in \mathcal{S}(\Gamma)$ is the axis of some hyperbolic $\beta_b \in \Gamma$. For each $b \in \mathcal{S}(\Gamma)$, let $r_b \in \mathsf{PGL}(2,\mathbb{R})$ be the reflection about b. Let $\Gamma^D \subset \mathsf{PSL}(2,\mathbb{R})$ denote the index two subgroup of the group $\hat{\Gamma} = \left\langle \Gamma, \{r_b\}_{b\in\mathcal{S}(\Gamma)} \right\rangle$ consisting of orientation-preserving isometries. Then $\hat{\Gamma}\backslash\mathbb{H}^2$ is the orbifold obtained from $\mathcal{C}(\Gamma)$ by regarding the boundary components as mirrors and $\Gamma^D\backslash\mathbb{H}^2$ is its orientable double cover. In particular, Γ^D is a lattice. We refer to Γ^D as the *double* of Γ.

Labourie and McShane [37, Cor. 9.2.6] showed that any Hitchin representation of a torsion-free convex cocompact Fuchsian group extends to a Hitchin representation of its double. We observe that their result generalizes to arbitrary geometrically finite Fuchsian groups. This is probably known to experts, but we provide a proof for completeness.

Proposition A.1. *Let $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ be a geometrically finite Fuchsian group that is not a lattice, and let $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ be a Hitchin representation. There exists a Hitchin representation $\rho^D : \Gamma^D \to \mathsf{PSL}(d,\mathbb{R})$ such that $\rho^D|_\Gamma = \rho$.*

Choose a convex, finite sided fundamental polygon D for the action of Γ on $\mathcal{C}(\Gamma)$ so that D intersects each Γ-orbit in $\mathcal{S}(\Gamma)$ exactly once. Let $\mathcal{E}(\Gamma)$ denote the finite collection of geodesics in $\mathcal{S}(\Gamma)$ which intersect D. Notice that if Γ has finite presentation $\langle X \ : \ R\rangle$, then

$$\hat{\Gamma} = \left\langle X, \{r_b\}_{b\in\mathcal{E}(\Gamma)} \ : \ R, \ \ \{r_b^2\}_{b\in\mathcal{E}(\Gamma)} \right\rangle$$

is a finite presentation for $\hat{\Gamma}$.

Since ρ is a Hitchin representation, there is a continuous, ρ-equivariant positive map $\xi : \Lambda(\Gamma) \to \mathcal{F}$. For each $b \in \mathcal{S}(\Gamma)$, let R_b be the projectivization of the linear map that fixes both $\rho(\beta_b^+)$ and $\rho(\beta_b^-)$, and acts on the line $\xi^j(\beta_b^+) \cap \xi^{d-j+1}(\beta_b^-)$ by scaling by $(-1)^{j-1}$ for all

$j \in \{1, \ldots, d\}$. We define $\hat\rho : \hat\Gamma \to \mathsf{PGL}(d,\mathbb{R})$ by setting $\hat\rho(r_b) = R_b$ for all $b \in \mathcal{E}(\Gamma)$. Notice that since $R_b^2 = I$, $\hat\rho$ is a representation. We then let $\rho^D = \hat\rho|_{\Gamma^D}$.

Let $G \subset \mathsf{PGL}(2,\mathbb{R})$ be the subgroup generated by $\{r_b : b \in \mathcal{S}(\Gamma)\}$. By a ping-pong type argument, we see that every element in G can be written uniquely as a reduced word in the alphabet $\{r_b : b \in \mathcal{S}(\Gamma)\}$, and we deduce the following lemma.

Lemma A.2. *If $r_1, r_2 \in G$, then one of the following holds:*

- $r_1 = r_2$,
- $r_1 = r_2 r_b$ *for some* $b \in \mathcal{S}(\Gamma)$, *in which case* $r_1(\mathcal{C}(\Gamma)) \cap r_2(\mathcal{C}(\Gamma)) = r_1(b) = r_2(b)$,
- $r_1(\mathcal{C}(\Gamma)) \cap r_2(\mathcal{C}(\Gamma))$ *is empty.*

If $r \in G$, we may write r uniquely as a reduced word $r_{b_1} \ldots r_{b_l}$ for some $b_1, \ldots, b_l \in \mathcal{S}(\Gamma)$, so

$$\hat\rho(r) = R_{b_1} \ldots R_{b_l} \in \mathsf{PGL}(d,\mathbb{R}).$$

Then by Lemma A.2, if $r_1, r_2 \in G$ are reflections such that the intersection

$$r_1(\Lambda(\Gamma)) \cap r_2(\Lambda(\Gamma))$$

is non-empty, then either $r_1 = r_2$, or $r_1 = r_2 r_b$ for some $b \in \mathcal{S}(\Gamma)$, in which case this intersection is $r_1(b) = r_2(b)$. Hence, we have a well-defined map

$$\xi^D : \Lambda(\Gamma^D) = \bigcup_{r \in G} r(\Lambda(\Gamma)) \to \mathcal{F}_d$$

given by $\xi^D(r(x)) = \hat\rho(r)(\xi(x))$ for all $r \in G$ and $x \in \Lambda(\Gamma)$. It is straightforward to verify that ξ^D is ρ^D-equivariant. It now suffices to show that ξ^D is positive, since by [19, Thm. 9.2], a representation of a geometrically finite Fuchsian group with a positive equivariant limit map is Hitchin.

Lemma A.3. *ξ^D is a positive map.*

Proof. It is sufficient to show that ξ^D restricted to $\bigcup_{r \in S'} r(\Lambda(\Gamma))$ is a positive map for any finite subset $S' \subset G$ such that $\bigcup_{r \in S'} r(\mathcal{C}(\Gamma))$ is connected. We prove this by induction on the size of S'. The base case where $\#S' = 1$ is trivial because positivity is invariant under projective transformations.

For the inductive step, let $S'' \subset S'$ be a subset such that $\#S'' = \#S' - 1$ and $\bigcup_{r \in S''} r(\mathcal{C}(\Gamma))$ is connected. Let $r_1 \in S' - S''$. Since $\bigcup_{r \in S'} r(\mathcal{C}(\Gamma))$ is connected, there is some $r_2 \in S''$ such that $r_1(\mathcal{C}(\Gamma))$ and $r_2(\mathcal{C}(\Gamma))$ share a common boundary component d, which lies in the Γ-orbit of a geodesic in $\mathcal{E}(\Gamma)$. Let $y_1, y_2 \in \partial \mathbb{H}^2$ be the endpoints of d, with notation chosen so that $y_1 < z_1 < y_2 < z_2$ for some (any) $z_1 \in r_1(\Lambda(\Gamma))$ and $z_2 \in r_2(\Lambda(\Gamma))$.

Pick any ℓ-tuple of points $x_1 < \cdots < x_\ell < x_1$ in $\bigcup_{r \in S'} r(\Lambda(\Gamma))$ with $\ell \geq 3$. By adding points and cyclically permuting, we may assume that there is some $t \in \{2, \ldots, \ell\}$ such that $y_1 = x_1$, $y_2 = x_t$, and

$$\{x_i : 1 \leq i \leq t\} \subset r_1(\Lambda(\Gamma)), \quad \{x_i : t \leq i \leq \ell\} \in \bigcup_{r \in S''} r(\Lambda(\Gamma)) \quad \text{and} \quad x_\ell \in r_2(\Lambda(\Gamma)).$$

By the inductive hypothesis, the restrictions of ξ_D to $\bigcup_{r \in S''} r(\Lambda(\Gamma))$ and $r_1(\Lambda(\Gamma))$ are positive maps. Thus,

$$\big(\xi^D(x_1), \ldots, \xi^D(x_t)\big) \text{ and } \big(\xi^D(x_t), \ldots, \xi^D(x_\ell), \xi^D(x_1)\big) \tag{32}$$

are positive tuples of flags. In the cases when $t = 2$ or $t = \ell$, it is immediate that $(\xi(x_1), \ldots, \xi(x_\ell))$ is positive.

Suppose now that $t \in \{3, \ldots, \ell - 1\}$. Let r_d denote the reflection in d and notice that $r_d \in \hat{\Gamma}$ and $r_d(x_\ell) \in r_1(\Lambda(\Gamma))$. Since ξ^D is positive on $r_1(\Lambda(\Gamma))$, there is an ordered basis $\mathcal{B}$ associated to $(\xi^D(x_1), \xi^D(x_t))$ and $u, w \in U_{>0}(\mathcal{B})$ so that $\xi^D(x_2) = u\xi^D(x_t)$ and $R_d(\xi^D(x_\ell)) = \xi^D(r_d(x_\ell)) = w\xi^D(x_t) = wR_d\xi^D(x_t)$. By the cofactor formula for computing minors, observe that $(R_d w R_d)^{-1} = R_d w^{-1} R_d \in U_{>0}(\mathcal{B})$. Thus,

$$\big(\xi^D(x_1), \xi^D(x_2), \xi^D(x_t), \xi^D(x_\ell)\big) = \big(\xi^D(x_1), u\xi^D(x_t), \xi^D(x_t), R_d w R_d \xi^D(x_t)\big)$$

is a positive quadruple of flags (see Lemma 7.5 of [19]).

Combining this with (32), Lemma 2.4(5) implies that

$$(\xi^D(x_1), \xi^D(x_2), \ldots, \xi^D(x_\ell))$$

is positive. This completes the proof of Lemma A.3 and thus the proof of Proposition A.1. □

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