PATTERSON-SULLIVAN MEASURES FOR TRANSVERSE SUBGROUPS

RICHARD CANARY, TENGREN ZHANG, AND ANDREW ZIMMER

Abstract. We study Patterson-Sullivan measures for a class of discrete subgroups of higher rank semisimple Lie groups, called transverse groups, whose limit set is well-defined and transverse in a partial flag variety. This class of groups includes both Anosov and relatively Anosov groups, as well as all discrete subgroups of rank one Lie groups. We prove an analogue of the Hopf-Tsuji-Sullivan dichotomy and then use this dichotomy to prove a variant of Burger's Manhattan curve theorem. We also use the Patterson-Sullivan measures to obtain conditions for when a subgroup has critical exponent strictly less than the original transverse group. These gap results are new even for Anosov groups.

Contents

1. Introduction

We study Patterson-Sullivan measures for a class of discrete subgroups of higher rank semisimple Lie groups, called transverse groups. This class of groups includes both Anosov and relatively Anosov groups as well as all discrete subgroups of rank one Lie groups. Transverse groups were previously studied by Kapovich, Leeb and Porti [29], who called them regular, antipodal groups. Patterson-Sullivan measures for discrete subgroups of higher rank Lie groups were first studied by Albuquerque [1] and Quint [41]. Recently Patterson-Sullivan measures for Anosov groups

Canary was partially supported by grant DMS-1906441 from the National Science Foundation. Zhang was partially supported by the NUS-MOE grant R-146-000-270-133 and A-8000458-00-00. Zimmer was partially supported by a Sloan research fellowship and grants DMS-2105580 and DMS-2104381 from the National Science Foundation.

have been extensively studied by Dey-Kapovich [22], Sambarino [45], Burger-Landesberg-Lee-Oh [12], Lee-Oh [35, 36] and others.

In this paper, we prove an analogue of the Hopf-Tsuji-Sullivan dichotomy for these measures. Using this dichotomy we prove a variant of Burger's Manhattan curve theorem [11]. We also use the Patterson-Sullivan measures to obtain conditions for when a subgroup has critical exponent strictly less than the original transverse group. These gap results are new even for Anosov groups.

In this introduction, we will restrict our discussion to the setting of transverse subgroups of $\mathsf{PSL}(d,\mathbb{K})$, where $\mathbb{K}$ is either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. In the body of the paper, we will consider transverse subgroups of connected semisimple Lie groups of non-compact type with finite center.

Let

$$\mathfrak{a} := \{\mathrm{diag}(a_1,\dots,a_d) \in \mathfrak{sl}(d,\mathbb{K}) :\ a_1+\cdots+a_d = 0\}$$

denote the standard Cartan subspace of $\mathfrak{sl}(d,\mathbb{K})$ and let $\kappa : \mathsf{PSL}(d,\mathbb{K}) \to \mathfrak{a}$ denote the *Cartan projection* which is given by

$$\kappa(g) = (\log\sigma_1(g),\cdots,\log\sigma_d(g))$$

where $\sigma_1(g) \geq \cdots \geq \sigma_d(g)$ are the singular values of any lift of g to $\mathsf{SL}(d,\mathbb{K})$. Let $\Delta := \{\alpha_1,\dots,\alpha_{d-1}\} \subset \mathfrak{a}^*$ denote the standard system of simple restricted roots, i.e.

$$\alpha_j(\mathrm{diag}(a_1,\dots,a_d)) = a_j - a_{j+1}$$

for all $\mathrm{diag}(a_1,\dots,a_d) \in \mathfrak{a}$.

When $\theta = \{\alpha_{i_1},\dots,\alpha_{i_k}\} \subset \Delta$ is symmetric (i.e. $\alpha_k \in \theta$ if and only if $\alpha_{d-k} \in \theta$), we say that a subgroup Γ of $\mathsf{PSL}(d,\mathbb{K})$ is P_θ*-divergent* if

$$+\infty = \lim_{n\to\infty} \min_{\alpha_k\in\theta} \alpha_k(\kappa(\gamma_n)) = \lim_{n\to\infty} \min_{\alpha_k\in\theta} \log\frac{\sigma_k(\gamma_n)}{\sigma_{k+1}(\gamma_n)}$$

whenever $\{\gamma_n\}$ is a sequence of distinct elements of Γ. A P_θ-divergent group is discrete and has a well-defined limit set $\Lambda_\theta(\Gamma)$ in the partial flag variety

$$\mathcal{F}_\theta := \left\{(F^{i_1},\dots,F^{i_k}) :\ \dim\left(F^j\right) = j \ \text{ for all } \alpha_j\in\theta, \text{ and } F^{i_1}\subset F^{i_2}\subset\cdots\subset F^{i_k}\right\}.$$

A P_θ-divergent subgroup $\Gamma \subset \mathsf{PSL}(d,\mathbb{K})$ is called P_θ*-transverse* if whenever $F, G \in \Lambda_\theta(\Gamma)$ are distinct, then F and G are transverse (i.e. for all $\alpha_k \in \theta$ the k-plane component F^k of F is transverse to the $(d-k)$-plane component G^{d-k} of G). We note that in the literature, divergent groups are sometimes called regular and transverse groups are sometimes called antipodal groups (e.g. [29]).

Quint [41] defined a cocycle for the action of $\mathsf{PSL}(d,\mathbb{K})$ on $\mathcal{F}_\theta$ which is an analogue of the Busemann cocycle in rank one. To define this cocycle we need some preliminary definitions.

Let

$$\mathfrak{a}_\theta := \{\mathrm{diag}(a_1,\dots,a_d) \in \mathfrak{a} : a_j = a_{j+1} \text{ for all } \alpha_j \notin \theta\}.$$

denote the partial Cartan subspace and let

$$\mathfrak{a}_\theta^+ := \{\mathrm{diag}(a_1,\dots,a_d) \in \mathfrak{a}_\theta :\ a_1 \geq a_2 \geq \cdots \geq a_d\}.$$

denote the partial positive Weyl Chamber. For $\alpha \in \Delta$, let $\omega_\alpha \in \mathfrak{a}^*$ denote the fundamental weight associated to α. One can check that $\{\omega_\alpha|_{\mathfrak{a}_\theta}\}_{\alpha\in\theta}$ is a basis of $\mathfrak{a}_\theta^*$. Then there is a well-defined partial Cartan projection $\kappa_\theta : \mathsf{PSL}(d,\mathbb{K}) \to \mathfrak{a}_\theta$ with the defining property that

$$\omega_\alpha(\kappa(g)) = \omega_\alpha(\kappa_\theta(g))$$

for all $\alpha \in \theta$ and $g \in \mathsf{PSL}(d,\mathbb{K})$.

Quint [41] proved that there exists a cocycle $B_\theta : \mathsf{PSL}(d,\mathbb{K}) \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$, called the *partial Iwasawa cocycle*, with the defining property that if $g \in \mathsf{PSL}(d,\mathbb{K})$, $F \in \mathcal{F}_\theta$ and $\alpha_k \in \theta$, then

$$\omega_{\alpha_k}(B_\theta(g,F)) = \log \frac{\left\| \left(\bigwedge^k g \right)(v) \right\|}{\|v\|}$$

for any $v \in \bigwedge^k F^k - \{0\}$, where $\bigwedge^k$ is the k^{th} exterior power, and $\|\cdot\|$ denotes both the standard norm on $\mathbb{K}^d$ and the induced norm on $\bigwedge^k \mathbb{K}^d$.

Using this cocycle we can define conformal measures and Patterson-Sullivan measures.

Definition 1.1. Given $\phi \in \mathfrak{a}_\theta^*$ and a P_θ-divergent group $\Gamma \subset \mathsf{PSL}(d,\mathbb{K})$, a probability measure μ on $\mathcal{F}_\theta$ is called a *ϕ-conformal measure for Γ of dimension β* if for any $\gamma \in \Gamma$, the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$\frac{d\gamma_*\mu}{d\mu} = e^{-\beta\phi(B_\theta(\gamma^{-1},\cdot))}.$$

If, in addition, $\operatorname{supp}(\mu) \subset \Lambda_\theta(\Gamma)$, then we say that μ is a *ϕ-Patterson-Sullivan measure.*

In our setting, we do not assume that Γ has any irreducibility properties and so there can exist many non-interesting conformal densities, e.g. if Γ fixes a flag $F \in \mathcal{F}_\theta$, then a Dirac measure centered at F will be a conformal measure of dimension zero. Hence to develop an interesting theory in the setting of (non-irreducible) transverse groups, it is reasonable to restrict to the setting where the measure is supported on the limit set.

Given a discrete subgroup $\Gamma \subset \mathsf{PSL}(d,\mathbb{K})$ and $\phi \in \mathfrak{a}_\theta^*$, let $\delta^\phi(\Gamma)$ be the (possibly infinite) critical exponent of the Poincaré series

$$Q_\Gamma^\phi(s) = \sum_{\gamma\in\Gamma} e^{-s\phi(\kappa_\theta(\gamma))},$$

that is $Q_\Gamma^\phi(s)$ converges when $s > \delta^\phi(\Gamma)$ and diverges when $s < \delta^\phi(\Gamma)$.

The standard proof, originating in work of Patterson [39], implies that if $\Gamma \subset \mathsf{PSL}(d,\mathbb{K})$ is P_θ-divergent, $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$, then there exists a ϕ-Patterson-Sullivan measure for Γ of dimension $\delta^\phi(\Gamma)$, see Proposition 3.1. Dey and Kapovich [22] previously established the same result in the slightly more restrictive setting when ϕ is positive on the entire partial Weyl chamber $\mathfrak{a}_\theta^+$. It is straightforward to show that if Γ is P_θ-divergent and ϕ is positive on $\mathfrak{a}_\theta^+$, then $\delta^\phi(\Gamma) < +\infty$, see Proposition 2.7.

One immediate consequence of the existence of Patterson-Sullivan measures is a criterion for when there is strict inequality between the critical exponent associated to a transverse group and a subgroup. The study of this "entropy gap" was initiated by Brooks [10] in the setting of convex cocompact Kleinian groups. Coulon, Dal'bo and Sambusetti [18] showed that if Γ admits a cocompact, properly discontinuous action on a CAT(-1)-space, then a subgroup of Γ has strictly smaller critical exponent if and only if is is co-amenable. The most general current results are due to Coulon, Dougall, Schapira and Tapie [19] who work in the setting of strongly positively recurrent actions on Gromov hyperbolic spaces. Our criterion is obtained using techniques due to Dal'bo, Otal and Peigné [21] .

Theorem 1.2 (see Theorem 4.1)**.** *Suppose $\Gamma \subset \mathsf{PSL}(d,\mathbb{K})$ is a non-elementary P_θ-transverse subgroup, $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$. If G is a subgroup of Γ such that $Q_G^\phi(\delta^\phi(G)) = +\infty$ and $\Lambda_\theta(G)$ is a proper subset of $\Lambda_\theta(\Gamma)$, then*

$$\delta^\phi(\Gamma) > \delta^\phi(G).$$

In the setting of Anosov groups, we see that there is always an entropy gap for infinite index, quasiconvex subgroups.

Corollary 1.3 (see Corollary 4.2)**.** *Suppose $\Gamma \subset \mathsf{PSL}(d,\mathbb{K})$ is a non-elementary P_θ-Anosov subgroup and G is an infinite index quasiconvex subgroup of Γ. If $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$, then $\delta^\phi(\Gamma) > \delta^\phi(G)$.*

For Fuchsian and Kleinian groups, there is a stark contrast in the dynamics of the action of the group which depends on whether or not the Poincaré series diverges at its critical exponent. The analysis of this contrast is known as the Hopf-Tsuji-Sullivan dichotomy and has many aspects. We obtain a version of this dichotomy for transverse groups.

To state the dichotomy precisely we need a few more definitions. A P_θ-transverse subgroup $\Gamma \subset \mathsf{PSL}(d,\mathbb{R})$ acts on its limit set $\Lambda_\theta(\Gamma)$ as a convergence group [30, Proposition 5.38], and hence one can define the set of conical limit points $\Lambda_\theta^{\mathrm{con}}(\Gamma) \subset \Lambda_\theta(\Gamma)$. In the case when Γ is P_θ-Anosov, $\Lambda_\theta^{\mathrm{con}}(\Gamma) = \Lambda_\theta(\Gamma)$.

Let $\iota : \mathfrak{a} \to \mathfrak{a}$ be the involution given by

$$\iota\big(\mathrm{diag}(a_1, a_2 \dots, a_d)\big) = \mathrm{diag}(-a_d, -a_{d-1}, \dots, -a_1).$$

Then given $\phi \in \mathfrak{a}_\theta^*$, let $\bar\phi := \phi \circ \iota \in \mathfrak{a}_\theta^*$. More explicitly, if $\phi = \sum_{k\in\theta} a_k \omega_{\alpha_k}$, then $\bar\phi = \sum_{k\in\theta} a_k \omega_{\alpha_{d-k}}$. The following theorem is our version of the Hopf-Sullivan-Tsuji dichotomy for transverse groups.

Theorem 1.4 (see Proposition 8.1, Proposition 9.1 and Corollary 11.1)**.** *Suppose $\Gamma \subset \mathsf{PSL}(d,\mathbb{K})$ is a non-elementary P_θ-transverse subgroup, $\phi \in \mathfrak{a}_\theta^*$ and $\delta := \delta^\phi(\Gamma) < +\infty$.*

- *If $Q_\Gamma^\phi(\delta) = +\infty$, then there exists a unique ϕ-Patterson-Sullivan measure μ_ϕ for Γ of dimension δ and there exists a unique $\bar\phi$-Patterson-Sullivan measure $\mu_{\bar\phi}$ for Γ of dimension δ. Moreover:*
  (1) $\mu_\phi(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = \mu_{\bar\phi}(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 1$.
  (2) μ_ϕ and $\mu_{\bar\phi}$ have no atoms.
  (3) The action of Γ on $(\Lambda_\theta(\Gamma), \mu_\phi)$ is ergodic.
  (4) The action of Γ on $(\Lambda_\theta(\Gamma)^2, \mu_\phi \otimes \mu_{\bar\phi})$ is ergodic.
- *If $Q_\Gamma^\phi(\delta) < +\infty$, then $\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 0$ for any ϕ-Patterson-Sullivan measure μ for Γ.*

We note that the "divergent" case contains several important classes of groups. Sambarino [45, Cor. 5.7.2] proved that for an Anosov group, the Poincaré series diverges whenever the critical exponent is finite (this was previously established by Lee-Oh [35, Lem. 7.11] and Dey-Kapovich [22, Thm. A] in certain cases). In the sequel to this paper we will prove the same result for relatively Anosov groups.

As an application of Theorem 1.4, we show that if Γ is P_θ-transverse, then the critical exponent is a concave function on the space of linear functionals which diverge at their finite critical exponent. Moreover, we characterize exactly when it fails to be strictly concave in terms of the associated length functions. More precisely, given $\phi \in \mathfrak{a}_\theta^*$, the *$\phi$-length* of $g \in \mathsf{PSL}(d,\mathbb{K})$ is

$$\ell^\phi(g) := \lim_{n\to\infty} \frac{1}{n}\phi(\kappa_\theta(g^n)).$$

Theorem 1.5 (see Theorem 12.1)**.** *Suppose $\Gamma \subset \mathsf{PSL}(d,\mathbb{K})$ is a non-elementary P_θ-transverse subgroup, $\phi_1, \phi_2 \in \mathfrak{a}_\theta^*$ and $\delta^{\phi_1}(\Gamma) = \delta^{\phi_2}(\Gamma) = 1$. If $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$ where $\lambda \in (0,1)$, then*

$$\delta^\phi(\Gamma) \le 1.$$

Moreover, if Q_Γ^ϕ diverges at its critical exponent, then equality occurs if and only if $\ell^{\phi_1}(\gamma) = \ell^{\phi_2}(\gamma)$ for all $\gamma \in \Gamma$.

We will explain in Section 12 why one might regard this as a variant of Burger's Manhattan Curve Theorem. By applying a result of Benoist [2], we can conclude that strict concavity holds whenever Γ is Zariski dense.

Corollary 1.6 (see Corollary 12.2)**.** *Suppose $\Gamma \subset \mathsf{PSL}(d, \mathbb{K})$ is Zariski dense and P_θ-transverse, $\phi_1, \phi_2 \in \mathfrak{a}_\theta^*$, $\phi_1 \neq \phi_2$ and $\delta^{\phi_1}(\Gamma) = \delta^{\phi_2}(\Gamma) = 1$. If $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$ where $\lambda \in (0,1)$ and Q_Γ^ϕ diverges at its critical exponent, then $\delta^\phi(\Gamma) < 1$.*

1.1. **The geometric framework for the proofs.** The key idea in our proofs is to associate to any P_θ-transverse group Γ a metric space that Γ acts on by isometries, where the boundary action of Γ on $\Lambda_\theta(\Gamma)$ embeds into the action of Γ on a compactification of that metric space. The metric space we construct has enough hyperbolic-like behavior that some of the classical arguments in hyperbolic geometry can be adapted to work in our setting. This approach to studying transverse groups builds upon on our earlier work in [15].

The metric spaces we consider in this construction are properly convex domains $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ endowed with their Hilbert metrics. A discrete subgroup $\Gamma_0 \subset \mathsf{PSL}(d_0, \mathbb{R})$ which preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is called *projectively visible* when the limit set $\Lambda_\Omega(\Gamma_0) \subset \partial\Omega$ is C^1-smooth and strictly convex (precise definitions are given in Sections 5 and 6).

The class of projectively visible groups contains the class of Kleinian groups, i.e. discrete subgroups of the isometry group $\mathsf{Isom}(\mathbb{H}^d_{\mathbb{R}})$ of real hyperbolic d-space. This follows from the identification of $\mathsf{PO}(m,1) = \mathsf{Isom}(\mathbb{H}^d_{\mathbb{R}})$ using the Klein-Beltrami model and the fact that $\mathsf{PO}(m,1)$ preserves the unit ball in an affine chart.

Given a projectively visible group $\Gamma_0 \subset \mathsf{PSL}(d_0, \mathbb{R})$, a representation $\rho : \Gamma_0 \to \mathsf{PSL}(d, \mathbb{K})$ is called P_θ-*transverse* if its image $\Gamma := \rho(\Gamma_0)$ is a P_θ-transverse subgroup and there exists a ρ-equivariant boundary map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$ which is a homeomorphism onto $\Lambda_\theta(\Gamma)$ (again, precise definitions are given in Sections 5 and 6).

To continue our analogy with hyperbolic geometry, we note that if $\Gamma \subset \mathsf{Isom}(\mathbb{H}^d_{\mathbb{R}}) = \mathsf{PO}(m,1)$ is convex co-compact, then the class of P_θ-transverse representations of Γ coincides with the class of P_θ-Anosov representations of Γ.

In [15], we proved that any P_θ-transverse subgroup of $\mathsf{PSL}(d, \mathbb{K})$ can be realized as the image of a P_θ-transverse representation. In this paper we extend this result to the general semisimple Lie group case, see Theorem 6.2. Using this perspective we will prove a version of the shadow lemma, which is one of the foundational tools in our arguments.

Shadows in Hilbert geometries can be defined exactly as in hyperbolic geometry: Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$, points $b, p \in \Omega$, and $r > 0$, let $\mathcal{O}_r(b,p)$ denote the set of points $x \in \partial\Omega$ for which the projective line segment in $\overline{\Omega}$ with endpoints b and x intersects that open ball of radius r (with respect to the Hilbert metric on Ω) centered at p.

Proposition 1.7 (see Proposition 7.1)**.** *Suppose $\theta \subset \Delta$ is symmetric, $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ is a properly convex domain, $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ is a non-elementary projectively visible subgroup, $\rho : \Gamma_0 \to \mathsf{PSL}(d, \mathbb{K})$ a P_θ-transverse representation with limit map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$, $\Gamma := \rho(\Gamma_0)$, $\phi \in \mathfrak{a}_\theta^*$ and μ is a ϕ-Patterson-Sullivan measure for Γ of dimension β. For any $b_0 \in \Omega$, there exists R_0 such that: if $r > R_0$, then there exists $C = C(b_0, r) > 1$ so that*

$$C^{-1} e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))} \leq \mu\Big(\xi\left(\mathcal{O}_r(b_0, \gamma(b_0)) \cap \Lambda_\Omega(\Gamma_0)\right)\Big) \leq C e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))}$$

for all $\gamma \in \Gamma_0$.

The transverse representations perspective also allow us to construct a dynamical system associated to a transverse group. In particular, given a transverse representation $\rho : \Gamma_0 \to \mathsf{PSL}(d, \mathbb{K})$ of a projectively visible group $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ we can consider the unit tangent bundle $T^1\Omega$ of Ω (relative to the Hilbert metric) and the subspace $\mathsf{U}(\Gamma_0) \subset T^1\Omega$ of directions where the associated projective geodesic lines has forward and backward endpoints in $\Lambda_\Omega(\Gamma_0)$, the limit set of Γ_0. The subspace $\mathsf{U}(\Gamma_0)$ is invariant under the geodesic flow and, by the projectively visible assumption, homeomorphic to $\Lambda_\Omega^{(2)}(\Gamma_0) \times \mathbb{R}$. We then use our Patterson-Sullivan measures to construct a Bowen-Margulis-Sullivan measure on the quotient $\Gamma_0 \backslash \mathsf{U}(\Gamma_0)$.

This dynamical system is critical in our work. For instance to prove that the boundary actions are ergodic in Theorem 1.4, we use a general version of the Hopf Lemma, due to Coudène [17], to show that the geodesic flow is ergodic with respect to the Bowen-Margulis-Sullivan measure.

Historical remarks: In this section we briefly discuss some important prior works concerning Patterson-Sullivan measures for discrete subgroups in higher rank semisimple Lie groups.

(1) Both Albuquerque [1] and Quint [41] study Patterson-Sullivan measures in the setting of Zariski dense, discrete subgroups of a semisimple group with finite center. Quint's measures live on flag varieties, as ours do, while Albuquerque's lie on the visual boundary of the associated symmetric space. Link [37] showed if the ray limit set has positive measure, then the action of the group on the ray limit set is ergodic with respect to the measures constructed by Albuquerque.

(2) Dey and Kapovich [22] study Patterson-Sullivan measures in the setting of P_θ-Anosov subgroups. They proved that when Γ is a P_θ-divergent subgroup and $\phi \in \mathfrak{a}_\theta^*$ is positive on $\mathfrak{a}_\theta^+$, that there is a ϕ-Patterson-Sullivan measure. In addition, when Γ is P_θ-Anosov, they also prove that the Patterson-Sullivan measure is unique, the conical limit set has full measure and the action of Γ on $\Lambda_\theta(\Gamma)$ is ergodic. Their approach is based heavily on studying the action of Γ on the associated symmetric space.

(3) Sambarino [44, 45] used the thermodynamical formalism to provide an alternative proof of Dey and Kapovich's results for all $\phi \in \mathfrak{a}_\theta^*$ such that $\delta^\phi(\Gamma) < \infty$. Further, he shows that the action of Γ on $\Lambda_\theta(\Gamma)^2$ is ergodic and characterizes linear functionals with critical exponent as exactly those which are strictly positive on the Benoist limit cone. The thermodynamical formalism requires the existence of an associated dynamical system with a Markov coding and this is currently only known to exist for Anosov subgroups and a few other specific groups.

(4) In the case when Γ is a P_θ-Anosov group which is isomorphic to the fundamental group of a closed negatively curved manifold one can use the perspective in [34] to obtain nicely behaved Patterson-Sullivan measures, for details of this approach see [43].

(5) Lee-Oh [36] prove that if Γ is Zariski dense and Anosov with respect to a minimal parabolic subgroup, then any ϕ-conformal measure of dimension $\delta^\phi(\Gamma)$ is supported on the limit set and hence a Patterson-Sullivan measure. They also show that the ϕ-Patterson-Sullivan measure is unique. They derive their result as a consequence of a Hopf-Tsuji-Sullivan dichotomy for the maximal diagonal actions.

(6) Burger-Landesberg-Lee-Oh [12] establish a Hopf-Tsuji-Sullivan dichotomy for the actions of discrete Zariski dense subgroups on directional limit sets with respect to a directional Poincaré series. This version of the dichotomy is different than the one we consider, for instance in Burger-Landesberg-Lee-Oh's dichotomy Anosov groups always fall into the convergent case when the rank of the semisimple Lie group is at least four. Using different techniques, Sambarino [45] gave an extension of this dichotomy to more general subsets of simple roots.

(7) Quint [41] proves the analogue of our shadow lemma for Zariski dense groups. His proof makes crucial use of Zariski density in place of our transversality assumption. Albuquerque [1] and Link [37] also establish shadow lemmas in their setting.
(8) Bray [8], Blayac [5], Zhu [49] and Blayac-Zhu [7] study Patterson-Sullivan measures for discrete subgroups $\Gamma \subset \mathsf{PGL}(d, \mathbb{R})$ which preserve a properly convex domain Ω. In their work, the measures have Radon-Nikodym derivatives which involve the Busemann functions obtained from the Hilbert metric, instead of partial Iwasawa cocycles used in other works (including this one).

Acknowledgements. We thank Hee Oh and Andrés Sambarino for helpful comments on an earlier version of this paper.

2. Background and notation

In this section, we recall some required background from the theory of semisimple Lie groups, as well as certain properties of discrete subgroups of semisimple Lie groups.

2.1. **Semisimple Lie groups.** First, we recall some basic terminology and facts from the theory of semisimple Lie groups. **For the rest of the paper**, let G be a connected semisimple Lie group without compact factors and with finite center, let $\mathfrak{g}$ denote the Lie algebra of G, and let b be the Killing form on $\mathfrak{g}$.

Fix a *Cartan involution* τ of $\mathfrak{g}$, i.e. an involution for which the bilinear pairing $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ given by $\langle X, Y \rangle := -b(X, \tau(Y))$ is an inner product. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

denote the associated *Cartan decomposition*, i.e. $\mathfrak{k}$ and $\mathfrak{p}$ are respectively the 1 and -1 eigenspaces of τ. Note that the Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, so $\mathfrak{k}$ is a maximal compact Lie subalgebra of $\mathfrak{g}$. Let $\mathsf{K} \subset \mathsf{G}$ denote the maximal compact Lie subgroup whose Lie algebra is $\mathfrak{k}$.

Next, fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, also called a *Cartan subspace*. Then let

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

be the *restricted root space decomposition* associated to $\mathfrak{a}$, i.e. for any $\alpha \in \mathfrak{a}^*$

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\},$$

and

$$\Sigma := \{\alpha \in \mathfrak{a}^* - \{0\} : \mathfrak{g}_\alpha \neq 0\}$$

is the set of *restricted roots*. One can verify that $\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, [31, Chap. VI, Prop. 6.52], so $\Sigma = -\Sigma$.

Next fix an element $H_0 \in \mathfrak{a} - \bigcup_{\alpha \in \Sigma} \ker \alpha$, and let

$$\Sigma^+ := \{\alpha \in \Sigma : \alpha(H_0) > 0\} \quad \text{and} \quad \Sigma^- := -\Sigma^+.$$

Note that $\Sigma = \Sigma^+ \cup \Sigma^-$. Let $\Delta \subset \Sigma^+$ be the associated system of *simple restricted roots*, i.e. Δ consists of all the elements in Σ^+ that cannot be written as a non-trivial linear combination of elements in Σ^+. Since Σ is an abstract root system on $\mathfrak{a}^*$, see [31, Chap. VI, Cor. 6.53], it follows that Δ is a basis of $\mathfrak{a}^*$ and every $\alpha \in \Sigma^+$ is a non-negative (integral) linear combination of elements in Δ, see [26, Chap. III, Thm. 10.1].

2.1.1. *The Weyl group and the opposition involution.* The *Weyl group* of $\mathfrak{a}$ is

$$W := \mathsf{N}_{\mathsf{K}}(\mathfrak{a})/\mathsf{Z}_{\mathsf{K}}(\mathfrak{a}),$$

where $\mathsf{N}_{\mathsf{K}}(\mathfrak{a}) \subset \mathsf{K}$ is the normalizer of $\mathfrak{a}$ in K and $\mathsf{Z}_{\mathsf{K}}(\mathfrak{a}) \subset \mathsf{K}$ is the centralizer of $\mathfrak{a}$ in K. Then W is a finite group that is generated by the reflections of $\mathfrak{a}$ (equipped with $\langle \cdot, \cdot \rangle$) about the kernels of the restricted roots in Δ, see [31, Chap. VI, Thm. 6.57]. As such, W acts transitively on the set of *Weyl chambers*, that is the closure of the components of

$$\mathfrak{a} - \bigcup_{\alpha \in \Sigma} \ker \alpha.$$

Of these, we refer to

$$\mathfrak{a}^+ := \{X \in \mathfrak{a} : \alpha(X) \geq 0 \text{ for all } \alpha \in \Delta\}$$

as the *positive Weyl chamber*.

In W, there exists a unique element w_0, called the *longest element*, such that

$$w_0(\mathfrak{a}^+) = -\mathfrak{a}^+.$$

We can then define an involution $\iota : \mathfrak{a} \to \mathfrak{a}$ by $\iota(H) = -w_0 \cdot H$. This is known as the *opposition involution*, and has the following properties.

Observation 2.1.

(1) If $k_0 \in \mathsf{N}_{\mathsf{K}}(\mathfrak{a})$ is a representative of the longest element $w_0 \in W$, then

$$\mathrm{Ad}(k_0)\mathfrak{g}_\alpha = \mathfrak{g}_{-\iota^*(\alpha)} \tag{1}$$

for all $\alpha \in \Sigma$.

(2) $\iota^*(\Delta) = \Delta$.

2.1.2. *Parabolic subgroups and flag manifolds.* Given a subset $\theta \subset \Delta$, the *parabolic subgroup associated to* θ, denoted by $\mathsf{P}_\theta = \mathsf{P}_\theta^+ \subset \mathsf{G}$, is the normalizer of

$$\mathfrak{u}_\theta = \mathfrak{u}_\theta^+ := \bigoplus_{\alpha \in \Sigma_\theta^+} \mathfrak{g}_\alpha$$

where $\Sigma_\theta^+ := \Sigma^+ \setminus \mathrm{Span}(\Delta \setminus \theta)$. The *flag manifold associated to* θ is

$$\mathcal{F}_\theta = \mathcal{F}_\theta^+ := \mathsf{G}/\mathsf{P}_\theta.$$

Similarly, the *standard parabolic subgroup opposite to* P_θ, denoted by P_θ^-, is the normalizer of

$$\mathfrak{u}_\theta^- := \bigoplus_{\alpha \in \Sigma_\theta^+} \mathfrak{g}_{-\iota^*(\alpha)},$$

and the *standard flag manifold opposite to* $\mathcal{F}_\theta$ is

$$\mathcal{F}_\theta^- := \mathsf{G}/\,\mathsf{P}_\theta^-\,.$$

Notice that if $k_0 \in \mathsf{N}_{\mathsf{K}}(\mathfrak{a})$ is a representative of the longest element $w_0 \in W$, then Equation (1) implies that

$$k_0\, \mathsf{P}_\theta^\pm\, k_0^{-1} = k_0^{-1}\, \mathsf{P}_\theta^\pm\, k_0 = \mathsf{P}_{\iota^*(\theta)}^\mp\,. \tag{2}$$

We say that two flags $F_1 \in \mathcal{F}_\theta^+$ and $F_2 \in \mathcal{F}_\theta^-$ are *transverse* if (F_1, F_2) is contained in the G-orbit of $(\mathsf{P}_\theta^+, \mathsf{P}_\theta^-)$ in $\mathcal{F}_\theta^+ \times \mathcal{F}_\theta^-$. Then for any flag $F \in \mathcal{F}_\theta^\pm$, let $\mathcal{Z}_F \subset \mathcal{F}_\theta^\mp$ denote the set of flags that are not transverse to F. One can verify that the set of transverse pairs in $\mathcal{F}_\theta^+ \times \mathcal{F}_\theta^-$ is an open and dense subset, so $\mathcal{Z}_F$ is a closed subset with empty interior. Furthermore, $\mathcal{Z}_F = \mathcal{Z}_{F'}$ if and only if $F = F'$.

2.1.3. *Cartan projection.* Let $\kappa : \mathsf{G} \to \mathfrak{a}^+$ denote the *Cartan projection*, that is $\kappa(g) \in \mathfrak{a}^+$ is the unique element such that

$$g = me^{\kappa(g)}\ell$$

for some $m, \ell \in \mathsf{K}$ (in general m and ℓ are not uniquely determined by g). Such a decomposition $g = me^{\kappa(g)}\ell$ is called a KAK-*decomposition* of g, see [27, Chap. IX, Thm. 1.1]. Since $\iota(-\mathfrak{a}^+) = \mathfrak{a}^+$, we have the following observation.

Observation 2.2. $\iota(\kappa(g)) = \kappa(g^{-1})$ for all $g \in \mathsf{G}$.

In terms of the KAK-decomposition, the actions of G on $\mathcal{F}_\theta^+$ and $\mathcal{F}_\theta^-$ have the following behavior. See Appendix A for a proof.

Proposition 2.3. *Suppose $F^\pm \in \mathcal{F}_\theta^\pm$, $\{g_n\}$ is a sequence in G and $g_n = m_n e^{\kappa(g_n)}\ell_n$ is a KAK-decomposition for each $n \geq 1$. Then the following are equivalent:*

1. *$m_n \mathsf{P}_\theta \to F^+$, $\ell_g^{-1} \mathsf{P}_\theta^- \to F^-$ and $\lim_{n\to\infty} \alpha(\kappa(g_n)) = \infty$ for every $\alpha \in \theta$,*
2. *$g_n(F) \to F^+$ for all $F \in \mathcal{F}_\theta^+ \setminus \mathcal{Z}_{F^-}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$.*
3. *$g_n^{-1}(F) = F^-$ for all $F \in \mathcal{F}_\theta^- \setminus \mathcal{Z}_{F^+}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta^- \setminus \mathcal{Z}_{F^+}$.*
4. *There are open sets $\mathcal{U}^\pm \subset \mathcal{F}_\theta^\pm$ such that $g_n(F) \to F^+$ for all $F \in \mathcal{U}^+$ and $g_n^{-1}(F) \to F^-$ for all $F \in \mathcal{U}^-$.*

2.1.4. *Weights and partial Cartan projections.* For any $\alpha \in \Sigma$, let $H_\alpha \in \mathfrak{a}$ satisfy the defining property

$$\langle H_\alpha, X \rangle = \alpha(X)$$

for all $X \in \mathfrak{a}$. Then for any non-zero $E \in \mathfrak{g}_\alpha$, $\mathrm{Span}_\mathbb{R}(E, \tau(E), H_\alpha) \subset \mathfrak{g}$ is a Lie sub-algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, and this isomorphism identifies

$$H'_\alpha := \frac{2H_\alpha}{\langle H_\alpha, H_\alpha \rangle} \in \mathrm{Span}_\mathbb{R}(E, \tau(E), H_\alpha) \quad \text{with} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}),$$

see [31, Chap. VI, Prop. 6.52]. The element H'_α is called the *coroot* associated to α. If $\alpha \in \Delta$, the *fundamental weight* associated to α is then the element $\omega_\alpha \in \mathfrak{a}^*$ such that

$$\omega_\alpha(H'_\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

for all $\beta \in \Delta$.

Given a subset $\theta \subset \Delta$, the *partial Cartan subspace associated to θ* is

$$\mathfrak{a}_\theta := \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \Delta \setminus \theta\}.$$

Since $(\Delta \setminus \theta) \cup \{\omega_\alpha : \alpha \in \theta\}$ is a basis of $\mathfrak{a}^*$, there is a unique projection

$$p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$$

such that $\omega_\alpha(X) = \omega_\alpha(p_\theta(X))$ for all $\alpha \in \theta$ and $X \in \mathfrak{a}$. Then the *partial Cartan projection associated to θ* is

$$\kappa_\theta := p_\theta \circ \kappa : \mathsf{G} \to \mathfrak{a}_\theta.$$

One can show that $\{\omega_\alpha|_{\mathfrak{a}_\theta} : \alpha \in \theta\}$ is a basis of $\mathfrak{a}_\theta^*$ and hence we will identify

$$\mathfrak{a}_\theta^* = \mathrm{Span}\{\omega_\alpha : \alpha \in \theta\} \subset \mathfrak{a}^*.$$

Note that $\omega_\alpha(\kappa_\theta(g)) = \omega_\alpha(\kappa(g))$ for all $\alpha \in \theta$ and $g \in \mathsf{G}$. So

$$\phi(\kappa_\theta(g)) = \phi(\kappa(g)) \tag{3}$$

for all $\phi \in \mathfrak{a}_\theta^*$ and $g \in \mathsf{G}$.

Given $\phi \in \mathfrak{a}_\theta^*$ we define the *ϕ-length* of an element $g \in \mathsf{G}$ as

$$\ell^\phi(g) = \lim_{n\to\infty} \frac{1}{n}\phi(\kappa_\theta(g^n))$$

(notice that this limit exists by Fekete's Subadditive Lemma). Equivalently, one can define the length using the Jordan projection.

2.1.5. *The partial Iwasawa cocycle.* Let $\mathsf{U} := \exp(\mathfrak{u}_\Delta)$. The *Iwasawa decomposition* states that the map

$$(k, a, u) \in \mathsf{K} \times \exp(\mathfrak{a}) \times \mathsf{U} \mapsto kau \in \mathsf{G}$$

is a diffeomorphism, see [31, Chap. VI, Prop. 6.46]. Using this, Quint [41] defined the Iwasawa cocycle

$$B : \mathsf{G} \times \mathcal{F}_\Delta \to \mathfrak{a}$$

with the defining property that $gk \in \mathsf{K} \cdot \exp(B(g, F)) \cdot \mathsf{U}$ for all $(g, F) \in \mathsf{G} \times \mathcal{F}_\Delta$, where $k \in \mathsf{K}$ is an element such that $F = k\,\mathsf{P}_\Delta$. The map B is known as the *Iwasawa cocycle*.

For any $\theta \subset \Delta$, note that $\mathsf{P}_\Delta \subset \mathsf{P}_\theta$, so the identity map on G induces a surjection $\Pi_\theta : \mathcal{F}_\Delta \to \mathcal{F}_\theta$. The *partial Iwasawa cocycle* is the map

$$B_\theta : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$$

defined by $B_\theta(g, F) = p_\theta(B(g, F'))$ for some (all) $F' \in \Pi_\theta^{-1}(F)$. By [41, Lem. 6.1 and 6.2], this is a well-defined cocycle, that is

$$B_\theta(gh, F) = B_\theta(g, hF) + B_\theta(h, F)$$

for all $g, h \in \mathsf{G}$ and $F \in \mathcal{F}_\theta$.

We will use two estimates from [41]. In the next two lemmas, let $\|\cdot\|$ denote the norm of the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$.

Lemma 2.4 (Quint [41, Lem. 6.5])**.** *For any $\epsilon > 0$ and distance $\mathrm{d}_{\mathcal{F}_\theta}$ on $\mathcal{F}_\theta$ induced by a Riemannian metric there exists $C = C(\epsilon, \mathrm{d}_{\mathcal{F}_\theta}) > 0$ such that: if $g \in \mathsf{G}$, $g = me^H\ell$ is a KAK-decomposition, $F \in \mathcal{F}_\theta$ and $\mathrm{d}_{\mathcal{F}_\theta}\left(F, \mathcal{Z}_{\ell^{-1}\mathsf{P}_\theta^-}\right) > \epsilon$, then*

$$\|B_\theta(g, F) - \kappa_\theta(g)\| < C.$$

Lemma 2.5 (Quint [41, Lem. 6.6])**.** *For any $\epsilon > 0$ and $g \in \mathsf{G}$ there exists $C = C(\epsilon, g) > 0$ such that: if $h \in \mathsf{G}$ and $\min_{\alpha\in\theta} \alpha(\kappa(h)) > C$, then*

$$\|\kappa_\theta(gh) - \kappa_\theta(h) - B_\theta(g, U_\theta(h))\| < \epsilon.$$

2.2. **When θ is symmetric.** In this section, as in much of the paper, we will consider the case when $\theta \subset \Delta$ is *symmetric*, that is $\iota^*(\theta) = \theta$.

As before, let $k_0 \in \mathsf{N}_\mathsf{K}(\mathfrak{a})$ be a representative of the longest element $w_0 \in W$. Then $k_0\,\mathsf{P}_\theta\,k_0^{-1} = \mathsf{P}_\theta^-$, see Equation (2). So we can identify $\mathcal{F}_\theta$ with $\mathcal{F}_\theta^-$ via the map

$$g\,\mathsf{P}_\theta^- \mapsto gk_0\,\mathsf{P}_\theta\,.$$

Using this identification, we can speak of two elements in $\mathcal{F}_\theta$ being transverse. More explicitly, the flags $g_1\,\mathsf{P}_\theta$ and $g_2\,\mathsf{P}_\theta$ in $\mathcal{F}_\theta$ are *transverse* if and only if there exists $g \in \mathsf{G}$ such that $gg_1 \in \mathsf{P}_\theta$ and $gg_2k_0 \in \mathsf{P}_\theta^-$. With some abuse of the notation, for a flag $F \in \mathcal{F}_\theta$, we now let $\mathcal{Z}_F \subset \mathcal{F}_\theta$ denote the set of flags that are not transverse to F.

Following the notation in [23], we define a map

$$U_\theta : \mathsf{G} \to \mathcal{F}_\theta$$

by fixing a KAK-decomposition $g = m_g e^{\kappa(g)} \ell_g$ for each $g \in \mathsf{G}$ and then letting $U_\theta(g) := m_g \mathsf{P}_\theta$. One can show that if $\alpha(\kappa(g)) > 0$ for all $\alpha \in \theta$, then $U_\theta(g)$ is independent of the choice of KAK-decomposition, see [27, Chap. IX, Thm. 1.1], and hence U_θ is continuous on the set

$$\{g \in \mathsf{G} : \alpha(\kappa(g)) > 0 \text{ for all } \alpha \in \theta\}.$$

Observation 2.2 implies that $\mathrm{Ad}(k_0)(-\kappa(g)) = \kappa(g^{-1})$ and so

$$g^{-1} = \left(\ell_g^{-1} k_0^{-1}\right) e^{\kappa(g^{-1})} \left(k_0 m_g^{-1}\right)$$

is a KAK-decomposition of g^{-1}. So we may assume that $m_{g^{-1}} = \ell_g^{-1} k_0^{-1}$ and $\ell_{g^{-1}} = k_0 m_g^{-1}$ for all $g \in \mathsf{G}$. Then

$$U_\theta(g^{-1}) = \ell_g^{-1} k_0^{-1} \mathsf{P}_\theta,$$

which under our identification $\mathcal{F}_\theta^- = \mathcal{F}_\theta$ coincides with $\ell_g^{-1} \mathsf{P}_\theta^-$.

Then, in the symmetric case, Proposition 2.3 can be restated as follows.

Proposition 2.6 (Proposition 2.3 in the symmetric case)**.** *Suppose $\theta \subset \Delta$ is symmetric, $F^\pm \in \mathcal{F}_\theta$ and $\{g_n\}$ is a sequence in G. The following are equivalent:*

1. *$U_\theta(g_n) \to F^+$, $U_\theta(g_n^{-1}) \to F^-$ and $\lim_{n\to\infty} \alpha(\kappa(g_n)) = \infty$ for every $\alpha \in \theta$,*
2. *$g_n(F) \to F^+$ for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$.*
3. *$g_n^{-1}(F) = F^-$ for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{F^+}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \setminus \mathcal{Z}_{F^+}$.*
4. *There are open sets $\mathcal{U}^\pm \subset \mathcal{F}_\theta$ such that $g_n(F) \to F^+$ for all $F \in \mathcal{U}^+$ and $g_n^{-1}(F) \to F^-$ for all $F \in \mathcal{U}^-$.*

2.3. **Discrete subgroups of semisimple Lie groups.** Next, we discuss some terminology for discrete subgroups of G and their basic properties.

2.3.1. *Critical exponents.* Let $\Gamma \subset \mathsf{G}$ be any discrete subgroup and let $\theta \subset \Delta$. For any $\phi \in \mathfrak{a}_\theta^*$, let $Q_\Gamma^\phi(s)$ denote the Poincaré series

$$Q_\Gamma^\phi(s) = \sum_{\gamma \in \Gamma} e^{-s\phi(\kappa_\theta(\gamma))}.$$

Let $\delta = \delta^\phi(\Gamma)$ be the critical exponent of $Q_\Gamma^\phi(s)$, i.e.

$$\delta = \inf\{s \in \mathbb{R}^+ : Q_\Gamma^\phi(s) < +\infty\}.$$

Equivalently,

$$\delta = \limsup_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \Gamma : \phi(\kappa_\theta(\gamma)) < T\}.$$

The *θ-Benoist limit cone* of Γ is the cone

$$\mathcal{B}_\theta(\Gamma) := \left\{H \in \mathfrak{a}_\theta^+ : \text{ there exists } \{\gamma_n\} \subset \Gamma_n \text{ and } t_n \searrow 0 \text{ such that } t_n \kappa_\theta(\gamma_n) \to H\right\}.$$

Set

$$\mathcal{B}_\theta(\Gamma)^+ := \{\phi \in \mathfrak{a}_\theta^* : \phi > 0 \text{ on } \mathcal{B}_\theta(\Gamma) - \{0\}\}.$$

We observe that for any $\phi \in \mathcal{B}_\theta(\Gamma)^+$, the critical exponent $\delta^\phi(\Gamma)$ is finite.

Proposition 2.7. *Suppose $\Gamma \subset \mathsf{G}$ is a discrete group and $\theta \subset \Delta$. If $\phi \in \mathcal{B}_\theta(\Gamma)^+$, then $\delta^\phi(\Gamma) < +\infty$. In particular, if ϕ is positive on $\mathfrak{a}_\theta^+$, then $\delta^\phi(\Gamma) < +\infty$.*

Proof. Equation (3) implies that ϕ is positive on $\mathcal{B}_\Delta(\Gamma) - \{0\}$. Then, since $\mathcal{B}_\Delta(\Gamma)$ is a cone and ϕ is linear, there exists $A > 0$ such that

$$\phi(H) \geq A \left\| H \right\|$$

for all $H \in \mathcal{B}_\Delta(\Gamma)$.

We claim that there exists $B \geq 0$ such that

$$\phi(\kappa(\gamma)) \geq \frac{A}{2} \left\| \kappa(\gamma) \right\| - B \tag{4}$$

for all $\gamma \in \Gamma$. Suppose not. Then for each $n \geq 1$ there exists $\gamma_n \in \Gamma$ where

$$\phi(\kappa(\gamma_n)) \leq \frac{A}{2} \left\| \kappa(\gamma_n) \right\| - n.$$

This implies that $\left\| \kappa(\gamma_n) \right\| \to +\infty$. Passing to a subsequence we can suppose that $\frac{1}{\left\| \kappa(\gamma_n) \right\|}\kappa(\gamma_n) \to H \in \mathcal{B}_\Delta(\Gamma)$. Then

$$A = A \left\| H \right\| \leq \phi(H) = \lim_{n\to\infty} \phi\left(\frac{1}{\left\| \kappa(\gamma_n) \right\|}\kappa(\gamma_n) \right) \leq \frac{A}{2}.$$

So we have a contradiction and hence such a $B \geq 0$ exists.

Let $X := \mathsf{G}/\mathsf{K}$ and $x_0 := \mathsf{K} \in X$. Then endow X with a G-invariant Riemannian symmetric metric scaled so that

$$\mathrm{d}_X(x_0, gx_0) = \left\| \kappa(g) \right\|$$

for all $g \in \mathsf{G}$. Since $\phi(\kappa_\theta(g)) = \phi(\kappa(g))$ for all $g \in \mathsf{G}$, the inequality (4) implies that

$$\{\gamma \in \Gamma : \phi(\kappa_\theta(\gamma)) < T\} \subset \left\{ \gamma \in \Gamma : \mathrm{d}_X(x_0, \gamma x_0) < \frac{2T}{A} + \frac{2B}{A} \right\}.$$

Thus, $\delta^\phi(\Gamma) \leq \frac{2}{A}\delta_X(\Gamma)$, where

$$\delta_X(\Gamma) := \lim_{T\to\infty} \frac{\log \#\left\{\gamma \in \Gamma : \mathrm{d}_X(x_0, \gamma x_0) < T\right\}}{T}.$$

Recall that the volume growth entropy of M is

$$h(X) := \limsup_{T\to\infty} \frac{\log \mathrm{Vol}_X(B_r(x_0))}{T}$$

where Vol_X is the Riemannian volume on X and $B_r(x_0) \subset X$ is the open ball of radius $r > 0$ centered at x_0. Since X has bounded sectional curvature, volume comparision theorems imply that $h(X) < +\infty$.

Fix $r_0 > 0$ and for $T > 0$ let $\Gamma_T := \{\gamma \in \Gamma : \mathrm{d}_X(x_0, \gamma x_0) < T\}$. Then

$$\#\Gamma_T = \frac{1}{\mathrm{Vol}_X(B_{r_0}(x_0))} \sum_{\gamma \in \Gamma_T} \mathrm{Vol}_X(B_{r_0}(\gamma x_0)) \leq \frac{\#\Gamma_{2r_0}}{\mathrm{Vol}_X(B_{r_0}(x_0))} \mathrm{Vol}(B_{T+r_0}(x_0)).$$

Thus $\delta_X(\Gamma) \leq h(X) < +\infty$. □

2.3.2. P_θ-*divergent groups.* In this subsection we assume that $\theta \subset \Delta$ is symmetric, i.e $\iota^*(\theta) = \theta$.

A subgroup $\Gamma \subset \mathsf{G}$ is P_θ-*divergent* if $\alpha(\kappa(\gamma_n)) \to \infty$ for any $\alpha \in \theta$ and any sequence $\{\gamma_n\}$ in Γ of pairwise distinct elements.

The θ-*limit set* $\Lambda_\theta(\Gamma)$ of Γ is the set of accumulation points in $\mathcal{F}_\theta$ of $\{U_\theta(\gamma) : \gamma \in \Gamma'\}$. Using Proposition 2.6, one can verify that $\Lambda_\theta(\Gamma)$ is a closed, Γ-invariant subset of $\mathcal{F}_\theta$. We will say that Γ is *non-elementary* if $\Lambda_\theta(\Gamma)$ is infinite.

We note that in the literature, divergent groups are sometimes called regular groups (e.g. [29]).

2.3.3. P_θ*-transverse groups.* In this subsection we continue to assume that $\theta \subset \Delta$ is symmetric. A P_θ-divergent subgroup $\Gamma \subset \mathsf{G}$ is P_θ*-transverse* if $\Lambda_\theta(\Gamma)$ is a transverse subset of $\mathcal{F}_\theta$, i.e. distinct pairs of flags in $\Lambda_\theta(\Gamma)$ are transverse. We note that in the literature, transverse groups are sometimes called antipodal groups (e.g. [29]). One crucial feature of P_θ-transverse groups is that Γ acts on $\Lambda_\theta(\Gamma)$ as a convergence group.

We recall that the action, by homeomorphisms, of a group Γ_0 on a compact metric space X is said to be a (discrete) *convergence group action* if whenever $\{\gamma_n\}$ is a sequence of distinct elements in Γ_0, then there are points $x, y \in X$ and a subsequence, still called $\{\gamma_n\}$, so that $\gamma_n(z)$ converges to x for all $z \in X \setminus \{y\}$ (uniformly on compact subsets of $X \setminus \{y\}$).

Proposition 2.8. [30, Prop. 5.38] *If Γ is P_θ-transverse, then Γ acts on $\Lambda_\theta(\Gamma)$ as a convergence group. In particular, if Γ is non-elementary, then Γ acts on $\Lambda_\theta(\Gamma)$ minimally, and $\Lambda_\theta(\Gamma)$ is perfect.*

If a group Γ_0 acts on a metric space X as a convergence group, we say that a point $x \in X$ is a *conical limit point* for the convergence group action if there exist distinct $a, b \in X$ and a sequence $\{\gamma_n\}$ in Γ_0 so that $\gamma_n(x)$ converges to a and $\gamma_n(y)$ converges to b for all $y \in X \setminus \{x\}$.

When $\Gamma \subset \mathsf{G}$ is P_θ-transverse, the set of conical limit points for the action of Γ on $\Lambda_\theta(\Gamma)$ is called the θ*-conical limit set* and is denoted $\Lambda_\theta^{\mathrm{con}}(\Gamma)$.

2.3.4. *Anosov groups.* Anosov groups were introduced by Labourie [33] in his work on Hitchin representations and were further developed by Guchard-Wienhard [25] and others. They are a natural generalization of the notion of a convex cocompact subgroup of a rank one Lie group into the higher rank setting. There are now many different equivalent definitions, and we give a definition which is well-adapted to our setting.

Following [29], a P_θ-transverse subgroup $\Gamma \subset \mathsf{G}$ is said to be P_θ*-Anosov* if Γ is Gromov hyperbolic with Gromov boundary $\partial\Gamma$ and there exists a homeomorphism $\xi : \partial\Gamma \to \Lambda_\theta(\Gamma)$.

2.4. **A helpful reduction.** Since G is semisimple, we may decompose its Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^m \mathfrak{g}_j$ into a product of simple Lie algebras. For each $1 \leq j \leq m$, let $\mathsf{G}_j \subset \mathsf{G}$ denote the connected subgroup with Lie algebra $\mathfrak{g}_j$. The subgroups $\mathsf{G}_1, \ldots, \mathsf{G}_m$ are called the *simple factors* of G. One can verify that each simple factor of G is a closed, normal subgroup and

$$\mathsf{G} = \mathsf{G}_1 \cdots \mathsf{G}_m$$

is an almost direct product, i.e. any distinct pair of simple factors of G commute, and the intersection between G_j and $\mathsf{G}_1 \cdots \mathsf{G}_{j-1}\mathsf{G}_{j+1} \cdots \mathsf{G}_m$ is finite for all j.

In this section we explain why one can often reduce to the case where G has trivial center and the fixed parabolic subgroup contains no simple factors of G. The main construction needed for this reduction is a well-behaved quotient map $p : \mathsf{G} \to \mathsf{G}'$.

Proposition 2.9. *For any $\theta \subset \Delta$ symmetric, there is a semisimple Lie group G' without compact factors and with trivial center, and a quotient $p : \mathsf{G} \to \mathsf{G}'$ with the following properties:*

1. *There exists a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ of the Lie algebra $\mathfrak{g}'$ of G', a Cartan subspace $\mathfrak{a}' \subset \mathfrak{p}'$, and a system of simple restricted roots $\Delta' \subset (\mathfrak{a}')^*$, so that $(\mathrm{d}p)_{\mathrm{id}} : \mathfrak{g} \to \mathfrak{g}'$ sends $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{a}$ to $\mathfrak{k}'$, $\mathfrak{p}'$ and $\mathfrak{a}'$ respectively, and $(\mathrm{d}p)_{\mathrm{id}}^* : (\mathfrak{a}')^* \to \mathfrak{a}^*$ identifies Δ' with a subset of Δ that contains θ.*
2. *The parabolic subgroup $\mathsf{P}'_\theta \subset \mathsf{G}'$ corresponding to $\theta \subset \Delta'$ satisfies $p^{-1}(\mathsf{P}'_\theta) = \mathsf{P}_\theta$, and does not contain any simple factors of G'. Furthermore, if $\mathcal{F}'_\theta := \mathsf{G}' / \mathsf{P}'_\theta$, then the map $\xi : \mathcal{F}_\theta \to \mathcal{F}'_\theta$ given by $\xi : g\,\mathsf{P}_\theta \mapsto p(g)\,\mathsf{P}'_\theta$ is a p-equivariant diffeomorphism which preserves transverality.*

(3) *Let $\kappa_\theta : \mathsf{G} \to \mathfrak{a}_\theta^+$ and $\kappa'_\theta : \mathsf{G}' \to (\mathfrak{a}'_\theta)^+$ be the partial Cartan projections, and let $B_\theta : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ and $B'_\theta : \mathsf{G}' \times \mathcal{F}'_\theta \to \mathfrak{a}'_\theta$ be the partial Iwasawa cocycles. Then $(\mathrm{d}p)_{\mathrm{id}} : \mathfrak{g} \to \mathfrak{g}'$ restricts to an isomorphism from $\mathfrak{a}_\theta$ to $\mathfrak{a}'_\theta$, and satisfies*

$$(\mathrm{d}p)_{\mathrm{id}}(\kappa_\theta(g)) = \kappa'_\theta(p(g)) \quad \textit{and} \quad (\mathrm{d}p)_{\mathrm{id}}(B_\theta(g, F)) = B'_\theta(p(g), \xi(F))$$

for all $g \in \mathsf{G}$ and $F \in \mathcal{F}_\theta$,

Once we have such a Lie group G' and quotient map $p : \mathsf{G} \to \mathsf{G}'$ as in Proposition 2.9, then for any P_θ-transverse subgroup $\Gamma \subset \mathsf{G}$ and any $\phi \in \mathfrak{a}_\theta^*$, we may set $\Gamma' := p(\Gamma)$ and $\phi' := \phi \circ (\mathrm{d}p)_{\mathrm{id}}|_{\mathfrak{a}_\theta}^{-1}$. By Proposition 2.9, it follows that

(I) $p|_\Gamma$ has finite kernel, Γ' is P'_θ-transverse, $\xi(\Lambda_\theta(\Gamma)) = \Lambda_\theta(\Gamma')$ and $\xi(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = \Lambda_\theta^{\mathrm{con}}(\Gamma')$.
(II) $\phi(\kappa_\theta(\gamma)) = \phi'(\kappa'_\theta(p(\gamma)))$ for all $\gamma \in \Gamma$.
(III) $\phi(B_\theta(\gamma, F)) = \phi'(B'_\theta(p(\gamma), \xi(F)))$ for all $\gamma \in \Gamma$ and $F \in \Lambda_\theta(\Gamma)$.

Thus, any result for $\Gamma \subset \mathsf{G}$ and $\phi \in \mathfrak{a}_\theta$ that depends only on $\Lambda_\theta(\Gamma)$, $\Lambda_\theta^{\mathrm{con}}(\Gamma)$, $\phi \circ \kappa_\theta$ and $\phi \circ B_\theta$ will hold if and only if they also hold for $\Gamma' \subset \mathsf{G}'$ and $\phi' \in \mathfrak{a}'_\theta$. In many situations, this allows us to assume without loss of generality that G has trivial center and P_θ does not contain any simple factors of G.

Proof of Proposition 2.9. Let $\mathfrak{p}_\theta \subset \mathfrak{g}$ be the Lie subalgebra corresponding to P_θ. If we set

$$J := \{j : \mathfrak{g}_j \cap \mathfrak{p}_\theta = 0\} \quad \text{and} \quad J^c := \{j : \mathfrak{g}_j \subset \mathfrak{p}_\theta\},$$

then $J \cup J^c = \{1, \dots, m\}$.

Let $\mathsf{H} := Z(\mathsf{G}) \prod_{j \in J^c} \mathsf{G}_j \subset \mathsf{G}$, $\mathsf{G}' := \mathsf{G}/\mathsf{H}$ and $p : \mathsf{G} \to \mathsf{G}'$ be the quotient map. Then observe that via the map $(\mathrm{d}p)_{\mathrm{id}}$, we may identify:

$$\mathfrak{g}' = \bigoplus_{j \in J} \mathfrak{g}_j. \tag{5}$$

In particular, G' is semisimple without compact factors, and has trivial center.

First, we prove part (1). Observe that we may decompose

$$\mathfrak{k} = \bigoplus_{j=1}^m \mathfrak{k}_j, \quad \mathfrak{p} = \bigoplus_{j=1}^m \mathfrak{p}_j, \quad \mathfrak{a} = \bigoplus_{j=1}^m \mathfrak{a}_j, \quad \Sigma = \bigcup_{j=1}^m \Sigma_j, \quad \Delta = \bigcup_{j=1}^m \Delta_j \quad \text{and} \quad \mathfrak{a}^+ = \bigoplus_{j=1}^m \mathfrak{a}_j^+,$$

where $\mathfrak{g}_j = \mathfrak{k}_j \oplus \mathfrak{p}_j$ is a Cartan decomposition of $\mathfrak{g}_j$, $\mathfrak{a}_j \subset \mathfrak{p}_j$ is a Cartan subspace, Σ_j is the set of restricted roots for $\mathfrak{a}_j$ and $\Delta_j \subset \Sigma_j$ is a system of simple restricted roots and $\mathfrak{a}_j^+ \subset \mathfrak{a}_j$ is the positive Weyl chamber relative to Δ_j. Hence, if we set

$$\mathfrak{k}' := \bigoplus_{j \in J} \mathfrak{k}_j, \quad \mathfrak{p}' := \bigoplus_{j \in J} \mathfrak{p}_j, \quad \mathfrak{a}' := \bigoplus_{j \in J} \mathfrak{a}_j, \quad \Sigma' := \bigcup_{j \in J} \Sigma_j, \quad \Delta' := \bigcup_{j \in J} \Delta_j \quad \text{and} \quad (\mathfrak{a}')^+ = \bigoplus_{j \in J} \mathfrak{a}_j^+$$

then via the identification (5), $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ is a Cartan decomposition of $\mathfrak{g}'$, $\mathfrak{a}' \subset \mathfrak{p}'$ is a Cartan subspace, Σ' is the set of restricted roots for $\mathfrak{a}'$, $\Delta' \subset \Sigma'$ is a system of simple restricted roots and $(\mathfrak{a}')^+$ is the positive Weyl chamber relative to Δ'. Furthermore, from the definition of P_θ, if G_j is a simple factor of G that lies in P_θ, then θ does not intersect Δ_j. This proves part (1).

Next, we prove part (2). The fact that $\mathsf{P}_\theta = p^{-1}(\mathsf{P}'_\theta)$ is a straightforward verification from the definition of P_θ and P'_θ. This fact, together with (5) imply that P'_θ does not contain any simple factors of G'. It is clear that ξ is a p-equivariant diffeomorphism. To see that ξ preserves transversality, simply note that the proof that $\mathsf{P}_\theta = p^{-1}(\mathsf{P}'_\theta)$ also verifies that $\mathsf{P}_\theta^- = p^{-1}((\mathsf{P}'_\theta)^-)$. Thus, part (2) holds.

Part (3) holds because with our choice of $\mathfrak{a}'$, p sends the Cartan and Iwasawa decompositions of G to the Cartan and Iwasawa decompositions of G' respectively. $\square$

3. Patterson-Sullivan measures for divergent groups

Patterson-Sullivan measures were first constructed by Patterson [39] for Fuchsian groups. Subsequently they were constructed in many settings where there is a natural boundary at infinity and some amount of Gromov hyperbolic behavior. Almost all these constructions mimic Patterson's original constructions with technical modifications appropriate to the setting.

Given $\theta \subset \Delta$ symmetric, we will now construct Patterson-Sullivan measures for P_θ-divergent subgroups, using the θ-limit set of the group as the natural boundary. More precisely, given $\phi \in \mathfrak{a}_\theta^*$ and a P_θ-divergent group $\Gamma \subset \mathsf{G}$, a probability measure μ on $\mathcal{F}_\theta$ is called a *ϕ-conformal measure for Γ of dimension β* if for any $\gamma \in \Gamma$, the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$\frac{d\gamma_*\mu}{d\mu}(F) = e^{-\beta\phi(B_\theta(\gamma^{-1},F))}.$$

If, in addition, $\mathrm{supp}(\mu) \subset \Lambda_\theta(\Gamma)$, then μ is a *ϕ-Patterson-Sullivan measure.*

Also, recall that $\delta^\phi(\Gamma)$ is the critical exponent of the Poincaré series

$$Q_\Gamma^\phi(s) = \sum_{\gamma\in\Gamma} e^{-s\phi(\kappa_\theta(\gamma))}.$$

Proposition 3.1. *If $\theta \subset \Delta$ is symmetric, $\Gamma \subset \mathsf{G}$ is P_θ-divergent, $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$, then there is a ϕ-Patterson-Sullivan measure μ for Γ of dimension $\delta^\phi(\Gamma)$.*

In the case when Γ is a P_θ-Anosov subgroup, Proposition 3.1 is a consequence of the following theorem of Sambarino [45], who completely classified the linear functionals which admit Patterson-Sullivan measures (see also Lee-Oh [35] for the case when Γ is Zariski dense and Anosov with respect to a minimal parabolic subgroup and Kapovich-Dey [22] for the case when ϕ is symmetric and positive on $\mathfrak{a}_\theta^+$).

Theorem 3.2 (Sambarino [45])**.** *If $\theta \subset \Delta$ is symmetric, $\Gamma \subset \mathsf{G}$ is P_θ-Anosov and $\phi \in \mathfrak{a}_\theta^*$, then the following are equivalent*

(1) $\phi \in \mathcal{B}_\theta^+(\Gamma)$,
(2) $\delta^\phi(\Gamma) < +\infty$, and
(3) Γ admits a ϕ-Patterson-Sullivan measure of dimension $\delta^\phi(\Gamma)$.

Moreover, if $\delta^\phi(\Gamma) < +\infty$, then Q_Γ^ϕ diverges at its critical exponent.

The strategy to prove Proposition 3.1 is to first observe that one can regard $\Gamma \cup \Lambda_\theta(\Gamma)$ as a well-behaved compactification of Γ, see Lemma 3.3. Using this compactification one can simply repeat Patterson's construction verbatim.

Lemma 3.3. *Suppose $\theta \subset \Delta$ is symmetric. If $\Gamma \subset \mathsf{G}$ is P_θ-divergent, then the set $\Gamma \cup \Lambda_\theta(\Gamma)$ has a topology that makes it a compactification of Γ. More precisely:*

(1) $\Gamma \cup \Lambda_\theta(\Gamma)$ is a compact metrizable space.
(2) If Γ has the discrete topology, $\Gamma \hookrightarrow \Gamma \cup \Lambda_\theta(\Gamma)$ is an embedding.
(3) If $\Lambda_\theta(\Gamma)$ has the subspace topology from $\mathcal{F}_\theta$, then $\Lambda_\theta(\Gamma) \hookrightarrow \Gamma \cup \Lambda_\theta(\Gamma)$ is an embedding.
(4) A sequence $\{\gamma_n\}$ in Γ converges to F in $\Lambda_\theta(\Gamma)$ if and only if

$$\min_{\alpha\in\theta} \alpha(\kappa(\gamma_n)) \to \infty \quad \text{and} \quad U_\theta(\gamma_n) \to F.$$

(5) The natural left action of Γ on $\Gamma \cup \Lambda_\theta(\Gamma)$ is by homeomorphisms.

Moreover, for any $\eta \in \Gamma$ *the function* $\bar{B}_\theta(\eta, \cdot) : \Gamma \cup \Lambda_\theta(\Gamma) \to \mathfrak{a}_\theta$ *defined by*

$$\bar{B}_\theta(\eta, x) = \begin{cases} \kappa_\theta(\eta x) - \kappa_\theta(x) & \text{if } x \in \Gamma, \\ B_\theta(\eta, x) & \text{if } x \in \Lambda_\theta(\Gamma), \end{cases}$$

is continuous, where the map $B_\theta : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$ *is the partial Iwasawa cocycle.*

Proof. We will construct an explicit metric on $\Gamma \cup \Lambda_\theta(\Gamma)$. First let d_Γ denote the discrete metric on Γ, that is

$$\mathrm{d}_\Gamma(\gamma_1, \gamma_2) = \begin{cases} 1 & \text{if } \gamma_1 \neq \gamma_2, \\ 0 & \text{if } \gamma_1 = \gamma_2. \end{cases}$$

Second, fix a metric d_θ on $\mathcal{F}_\theta$ which is induced by a Riemannian metric. By scaling we can assume that in the metric d_θ, the diameter of $\mathcal{F}_\theta$ is 1. Finally, define $m_\theta : \Gamma \to (0, 1]$ by

$$m_\theta(\gamma) = \exp\left(-\min_{\alpha \in \theta} \alpha(\kappa(\gamma))\right).$$

We now define a metric d on $\Gamma \cup \Lambda_\theta(\Gamma)$ as follows:

- If $\gamma_1, \gamma_2 \in \Gamma$, then
$$\mathrm{d}(\gamma_1, \gamma_2) = \max\{m_\theta(\gamma_1), m_\theta(\gamma_2)\}\mathrm{d}_\Gamma(\gamma_1, \gamma_2) + \mathrm{d}_\theta(U_\theta(\gamma_1), U_\theta(\gamma_2)).$$
- If $\gamma \in \Gamma$ and $F \in \Lambda_\theta(\Gamma)$, then
$$\mathrm{d}(\gamma, F) = m_\theta(\gamma) + \mathrm{d}_\theta(U_\theta(\gamma), F).$$
- If $F_1, F_2 \in \Lambda_\theta(\Gamma)$, then
$$\mathrm{d}(F_1, F_2) = \mathrm{d}_\theta(F_1, F_2).$$

It is straightforward to check that d defines a metric. Also, from the definition of d, it is clear that the restriction of d to Γ and $\Lambda_\theta(\Gamma)$ induce the discrete topology on Γ and the usual topology on $\mathcal{F}_\theta$ respectively, so (2) and (3) holds. To see that (4) holds, note that $\gamma_n \to F$ if and only if $m_\theta(\gamma_n) \to 0$ and $\mathrm{d}_\theta(U_\theta(\gamma_n), F) \to 0$, which is in turn equivalent to requiring $\min_{\alpha \in \theta} \alpha(\kappa(\gamma_n)) \to \infty$ and $U_\theta(\gamma_n) \to F$.

Next we prove the compactness in (1) by taking a sequence $\{x_n\}$ in $\Gamma \cup \Lambda_\theta(\Gamma)$ and showing that it has a convergent subsequence. Observe that $\{x_n\}$ either has

(i) a subsequence that lies in $\Lambda_\theta(\Gamma)$,
(ii) a subsequence that lies in a finite subset of Γ, or
(iii) a subsequence that lies in Γ, but does not lie in any finite subset of Γ.

If (i) or (ii) holds, then the compactness of $\Lambda_\theta(\Gamma)$ and the compactness of finite subsets of Γ respectively imply $\{x_n\}$ has a convergent subsequence. If (iii) holds, then by taking a further subsequence $\{\gamma_j\}$ of $\{x_n\}$, we may assume that $U_\theta(\gamma_j) \to F$ for some $F \in \mathcal{F}_\theta$. Since the P_θ-divergence of Γ implies that $\min_{\alpha \in \theta} \alpha(\kappa(\gamma_j)) \to \infty$, we may apply (4) to deduce that $\gamma_j \to F$. So $\{x_n\}$ has a convergent subsequence.

Since the left Γ action on Γ and the Γ action on $\Lambda_\theta(\Gamma)$ are both clearly continuous, to prove part (5) it suffices to show: if $\eta \in \Gamma$ and $\{\gamma_n\}$ is a sequence in Γ converging to $F^+ \in \Lambda_\theta(\Gamma)$, then $\eta\gamma_n \to \eta(F^+)$. By compactness, it suffices to consider the case when $\eta\gamma_n \to F'$ and show that $F' = \eta(F^+)$. Notice that (4) implies that $\min_{\alpha \in \theta} \alpha(\kappa(\gamma_n)) \to \infty$ and $U_\theta(\gamma_n) \to F^+$. Then using Proposition 2.3 and passing to a subsequence we can suppose that there exists $F^- \in \mathcal{F}_\theta^-$ such that $\gamma_n(F) \to F^+$ for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$ and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$. Then $\eta\gamma_n(F) \to \eta(F^+)$ for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$ and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$. So Proposition 2.3 implies that $\min_{\alpha \in \theta} \alpha(\kappa(\eta\gamma_n)) \to \infty$ and $U_\theta(\eta\gamma_n) \to \eta(F^+)$. So part (4) implies that $\eta\gamma_n \to \eta(F^+)$. So part (5) is true.

Finally notice that Lemma 2.5 and part (4) of this proposition imply the "moreover" part. $\square$

Proof of Proposition 3.1. Let $\delta := \delta^\phi(\Gamma)$. Endow $\Gamma \cup \Lambda_\theta(\Gamma)$ with the topology from Lemma 3.3 and for $x \in \Gamma \cup \Lambda_\theta(\Gamma)$ let $\mathcal{D}_x$ denote the Dirac measure centered at x. By [39, Lem. 3.1] there exists a continuous non-decreasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

(1) The series

$$\hat{Q}(s) := \sum_{\gamma\in\Gamma} h\left(e^{\phi(\kappa_\theta(\gamma))}\right) e^{-s\phi(\kappa_\theta(\gamma))}$$

converges for $s > \delta$ and diverges for $s \le \delta$.

(2) For any $\epsilon > 0$ there exists $\lambda_0 > 0$ such that: if $s > 1$ and $\lambda > \lambda_0$, then $h(\lambda s) \le s^\epsilon h(\lambda)$.

(In the case when Q^ϕ_Γ diverges at its critical exponent, we can choose $h \equiv 1$.) Then for $s > \delta$ consider the probability measure

$$\mu_s := \frac{1}{\hat{Q}(s)} \sum_{\gamma\in\Gamma} h\left(e^{\phi(\kappa_\theta(\gamma))}\right) e^{-s\phi(\kappa_\theta(\gamma))}\mathcal{D}_\gamma$$

on $\Gamma \cup \Lambda_\theta(\Gamma)$. By compactness, the family of measures $\{\mu_s\}_{s>\delta}$ admits a subsequential weak limit as $s \searrow \delta$, i.e. there exists $\{s_n\} \subset (\delta, \infty)$ so that $\lim s_n = \delta$ and

$$\mu := \lim \mu_{s_n}$$

exists. We will prove that μ is a Patterson-Sullivan measure of dimension δ.

Notice that if $A \subset \Gamma$ is a finite set, then

$$\mu(A) = \lim_{n\to\infty} \frac{1}{\hat{Q}(s_n)} \sum_{\gamma\in A} h\left(e^{\phi(\kappa_\theta(\gamma))}\right) e^{-s_n\phi(\kappa_\theta(\gamma))} = 0 \cdot \sum_{\gamma\in A} h\left(e^{\phi(\kappa_\theta(\gamma))}\right) e^{-\delta\phi(\kappa_\theta(\gamma))} = 0.$$

Hence $\mathrm{supp}(\mu) \subset \Lambda_\theta(\Gamma)$.

To verify the remaining property, fix $\eta \in \Gamma$, let

$$\bar{B}_\theta(\eta^{-1}, \cdot) : \Gamma \cup \Lambda_\theta(\Gamma) \to \mathbb{R}$$

be the continuous function defined in Lemma 3.3, and define the function $g_\eta : \Gamma \cup \Lambda_\theta(\Gamma) \to \mathbb{R}$ by

$$g_\eta(z) = \begin{cases} \frac{h\left(e^{\phi(\kappa_\theta(z))+\phi(\bar{B}_\theta(\eta^{-1},z))}\right)}{h\left(e^{\phi(\kappa_\theta(z))}\right)} & \text{if } z \in \Gamma, \\ 1 & \text{if } z \in \Lambda_\theta(\Gamma). \end{cases}$$

Notice that property (2) of h implies that g_η is continuous.

For any continuous function $f : \Gamma \cup \Lambda_\theta(\Gamma) \to \mathbb{R}$ and $s > \delta$, we have

$$\begin{aligned} \int f(z) d\eta_*\mu_s(z) &= \frac{1}{\hat{Q}(s)} \sum_{\gamma\in\Gamma} h\left(e^{\phi(\kappa_\theta(\gamma))}\right) e^{-s\phi(\kappa_\theta(\gamma))} f(\eta\gamma) \\ &= \frac{1}{\hat{Q}(s)} \sum_{\gamma\in\Gamma} h\left(e^{\phi(\kappa_\theta(\eta^{-1}\gamma))}\right) e^{-s\phi(\kappa_\theta(\eta^{-1}\gamma))} f(\gamma) \\ &= \frac{1}{\hat{Q}(s)} \sum_{\gamma\in\Gamma} h\left(e^{\phi(\kappa_\theta(\gamma))}\right) e^{-s\phi(\kappa_\theta(\gamma))} e^{-s\phi(\bar{B}_\theta(\eta^{-1},\gamma))} \frac{h\left(e^{\phi(\kappa_\theta(\gamma))+\phi(\bar{B}_\theta(\eta^{-1},\gamma))}\right)}{h\left(e^{\phi(\kappa_\theta(\gamma))}\right)} f(\gamma) \\ &= \int f(z) e^{-s\phi(\bar{B}_\theta(\eta^{-1},z))} g_\eta(z) d\mu_s(z). \end{aligned}$$

Then taking limits and recalling that μ is supported on $\Lambda_\theta(\Gamma)$, we obtain

$$\frac{d\eta_*\mu}{d\mu}(F) = e^{-\delta\phi(B_\theta(\eta^{-1},F))}.$$

So μ is a Patterson-Sullivan measure of dimension δ. $\square$

4. Entropy drop

It is natural to conjecture, in analogy with results of Coulon-Dal'bo-Sambusetti [18], that if Γ_0 is a subgroup of a P_θ-Anosov group Γ, $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$, then $\delta^\phi(\Gamma) = \delta^\phi(\Gamma_0)$ if and only if Γ_0 is co-amenable in Γ. (Glorieux and Tapie [24] have studied this conjecture when Γ_0 is normal in Γ and Zariski dense.) We apply an argument of Dal'bo-Otal-Peigné [21] to obtain a criterion guaranteeing entropy drop for subgroups of transverse groups. As a consequence, we obtain generalization of a result of Brooks [10] from the setting of geometrically finite hyperbolic 3-manifolds into the setting of Anosov groups.

Theorem 4.1. *Suppose $\theta \subset \Delta$ is symmetric, $\Gamma \subset \mathsf{G}$ is a non-elementary P_θ-transverse subgroup, $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$. If Γ_0 is a subgroup of Γ such that $Q^\phi_{\Gamma_0}$ diverges at its critical exponent and $\Lambda_\theta(\Gamma_0)$ is a proper subset of $\Lambda_\theta(\Gamma)$, then*

$$\delta^\phi(\Gamma) > \delta^\phi(\Gamma_0).$$

Proof. Let μ be a ϕ-Patterson-Sullivan measure for Γ of dimension $\delta^\phi(\Gamma)$. Let F be a flag in $\Lambda_\theta(\Gamma) \setminus \Lambda_\theta(\Gamma_0)$ which is not fixed by any element of Γ_0. Since Γ is P_θ-tranvserse and $\Lambda_\theta(\Gamma_0) \subset \Lambda_\theta(\Gamma)$ is a proper closed subset, we may choose an open neighborhood W of F where every flag in its closure $\overline{W}$ is transverse to every flag $\Lambda_\theta(\Gamma_0)$ and if $\gamma \in \Gamma_0 \setminus \{\mathrm{id}\}$, then $\gamma(W)$ is disjoint from W.

Fix a distance $\mathrm{d}_{\mathcal{F}_\theta}$ on $\mathcal{F}_\theta$ which is induced by a Riemannian metric. Given $H \in \mathcal{F}_\theta$, recall that $\mathcal{Z}_H \subset \mathcal{F}_\theta$ denotes the set of flags which are not transverse to H. Since $\Lambda_\theta(\Gamma_0)$ is the set of accumulation points of $\{U_\theta(\gamma) : \gamma \in \Gamma_0\}$, there is a finite subset $\mathcal{S} \subset \Gamma_0$ and $\epsilon > 0$ so that

$$\mathrm{d}_{\mathcal{F}_\theta}\left(F, \mathcal{Z}_{U_\theta(\gamma)}\right) \geq \epsilon$$

for all $F \in W$ and $\gamma \in \Gamma_0 \setminus \mathcal{S}$. Then Lemma 2.4 implies that there exists $C > 0$ such that

$$\phi(B_\theta(\gamma, F)) \leq \phi(\kappa_\theta(\gamma)) + C \tag{6}$$

for all $F \in W$ and $\gamma \in \Gamma_0 \setminus \mathcal{S}$ (recall that θ is symmetric and so we can identify $\mathcal{F}_\theta$ and $\mathcal{F}_\theta^-$, see Section 2.2).

Since $\Gamma_0 \subset \Gamma$, it is immediate that $\delta^\phi(\Gamma) \geq \delta^\phi(\Gamma_0)$. Suppose for contradiction that $\delta := \delta^\phi(\Gamma) = \delta^\phi(\Gamma_0)$. Notice that

$$\mu(\gamma(W)) = \gamma_*^{-1}\mu(W) = \int_W e^{-\delta\phi(B_\theta(\gamma,F))} d\mu(F),$$

so (6) implies that

$$\mu(\gamma(W)) \geq e^{-\delta C} e^{-\delta\phi(\kappa_\theta(\gamma))}\mu(W)$$

for all $\gamma \in \Gamma_0 \setminus \mathcal{S}$. Since $Q^\phi_{\Gamma_0}$ diverges at its critical exponent,

$$1 = \mu(\Lambda_\theta(\Gamma)) \geq \sum_{\gamma \in \Gamma_0} \mu(\gamma(W)) \geq e^{-\delta C}\mu(W) \sum_{\gamma \in \Gamma_0 \setminus \mathcal{S}} e^{-\delta\phi(\kappa_\theta(\gamma))} = +\infty$$

which is a contradiction. $\square$

One immediate consequence of our criterion is an entropy gap result for quasiconvex subgroups of Anosov groups. We recall that a subgroup Γ_0 of a hyperbolic group Γ is *quasiconvex* if there exists $K > 0$ such that any geodesic joining two points in Γ_0 in the Cayley graph of Γ (with respect to some finite presentation of Γ) lies within distance K of the vertices associated to Γ_0.

Corollary 4.2. *Suppose $\theta \subset \Delta$ is symmetric, $\Gamma \subset \mathsf{G}$ is a non-elementary P_θ-Anosov subgroup and Γ_0 is an infinite index quasiconvex subgroup of Γ. If $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$, then*

$$\delta^\phi(\Gamma) > \delta^\phi(\Gamma_0).$$

Proof. Since $\Gamma \subset \mathsf{G}$ is a non-elementary P_θ-Anosov subgroup and Γ_0 is a quasiconvex subgroup of Γ, Canary, Lee, Sambarino and Stover observed (see [14, Lem. 2.3]) that $\Gamma_0 \subset \mathsf{G}$ is also a P_θ-Anosov subgroup, and the θ-limit set $\Lambda_\theta(\Gamma_0)$ of Γ_0 lies in the θ-limit set $\Lambda_\theta(\Gamma)$ of Γ. Furthermore, since $\Gamma_0 \subset \Gamma$ is infinite index, it follows that $\Lambda_\theta(\Gamma_0)$ is a proper subset of $\Lambda_\theta(\Gamma)$. Note also that

$$\delta^\phi(\Gamma_0) \leq \delta^\phi(\Gamma) < +\infty,$$

so Theorem 3.2 implies that $Q^\phi_{\Gamma_0}$ diverges at its critical exponent. The corollary now follows from Theorem 4.1. ☐

Remark 4.3.

1. In Corollary 4.2 it is not enough to assume that Γ_0 is infinite index and finitely generated, since the results fails when $\Gamma \subset \mathsf{PO}(3,1)$ uniformizes a closed hyperbolic 3-manifold which fibers over the circle and Γ_0 is the fiber subgroup. In this case, if we set $\theta := \{\alpha_1, \alpha_3\}$, then $\Gamma \subset \mathsf{PGL}(4,\mathbb{R})$ is P_θ-Anosov and $\Gamma_0 \subset \Gamma$ is an infinite index, finitely generated subgroup. However, in this case, $\delta^{\alpha_1}(\Gamma) = \delta^{\alpha_1}(\Gamma_0)$, see [13, Cor. 4.2].
2. Theorem 4.1 also gives a new proof of [15, Prop. 11.5].

5. Projectively visible groups and their geodesic flows

In this mostly expository section, we recall the definition of projectively visible groups from [28] and state some of their basic properties. Projectively visible groups are a class of transverse groups and we will see in the next section that every transverse group can be identified with a projectively visible group in a useful manner

5.1. **Properly convex domains.** We briefly recall some properties of properly convex domains, the Hilbert metric, and the automorphism group of a properly convex domain. For a more detailed discussion we refer the reader to the survey article of Marquis [38].

Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, that is an open set which is convex and bounded in some affine chart of $\mathbb{P}(\mathbb{R}^d)$. Then a *supporting hyperplane* to Ω at a point $x \in \partial\Omega$ is a projective hyperplane $H \subset \mathbb{P}(\mathbb{R}^d)$ (i.e. the projectivization of a codimension one linear subspace) that contains x but does not intersect Ω. By convexity, every boundary point of $\partial\Omega$ is contained in at least one supporting hyperplane and a boundary point which is contained in a unique supporting hyperplane is called a *C^1-smooth point* of $\partial\Omega$. In the case when x is a C^1-smooth point of $\partial\Omega$, we let $T_x\partial\Omega$ denote the unique supporting hyperplane at x.

For any pair of points $x, y \in \overline{\Omega}$, let $[x,y]_\Omega$ denote the closed projective line segment in $\overline{\Omega}$ with x and y as its endpoints. Similarly, $(x,y)_\Omega := [x,y]_\Omega - \{x,y\}$, $[x,y)_\Omega := [x,y]_\Omega - \{y\}$ and $(x,y]_\Omega := [x,y]_\Omega - \{x\}$.

A properly convex domain Ω admits a natural Finsler metric d_Ω, called the *Hilbert metric.* Given a pair of points $p, q \in \Omega$, let $x, y \in \partial\Omega$ be the points such that that x, p, q, y lie along $[x,y]_\Omega$ in that order. Then

$$\mathrm{d}_\Omega(p,q) := \log \frac{|x-q||y-p|}{|x-p||y-q|},$$

where $|\cdot|$ denotes some (any) norm on some (any) affine chart containing x, p, q, y. Observe that all projective line segments in Ω are geodesics of the Hilbert metric.

Although the Hilbert metric is rarely CAT(0), the distance function has the following well known and useful convexity property, for a proof see for instance [28, Prop. 5.3].

Proposition 5.1. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $x \in \overline{\Omega}$ and $q_1, q_2 \in \Omega$. If $p \in [q_1, x)_\Omega$, then*

$$\mathrm{d}_\Omega(p, [q_2, x)_\Omega) \leq \mathrm{d}_\Omega(q_1, q_2).$$

Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, we denote by $\mathrm{Aut}(\Omega) \subset \mathsf{PGL}(d, \mathbb{R})$ the subgroup that leaves Ω invariant. The group $\mathrm{Aut}(\Omega)$ preserves the Hilbert metric and acts properly on Ω.

The *full orbital limit set* of a discrete infinite subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ is

$$\Lambda_\Omega(\Gamma) := \left\{ x \in \partial\Omega : z = \lim_{n \to \infty} \gamma_n(p) \text{ for some } p \in \Omega \text{ and some } \{\gamma_n\} \subset \Gamma \right\}.$$

We also let $\Lambda_\Omega^{\mathrm{con}}(\Gamma) \subset \Lambda_\Omega(\Gamma)$ denote the set of limit points $x \in \Lambda_\Omega(\Gamma)$ where there exist $b_0 \in \Omega$, a sequence $\{\gamma_n\}$ in Γ and some $r > 0$ such that $\gamma_n(b_0) \to x$ and $\mathrm{d}_\Omega(\gamma_n(b_0), [b_0, x)_\Omega) < r$ for all n.

5.2. **Properties of projectively visible groups.** If Ω is a properly convex domain, we say that a discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ is *projectively visible* if

(1) $(x, y)_\Omega \subset \Omega$ for any two points $x, y \in \Lambda_\Omega(\Gamma)$ and
(2) every point in $\Lambda_\Omega(\Gamma)$ is a C^1-smooth point of $\partial\Omega$.

The following proposition collects elementary properties of projectively visible groups and shows, in particular, that they are examples of transverse subgroups.

Proposition 5.2. *If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \subset \mathrm{Aut}(\Omega)$ is a projectively visible subgroup, then the following hold:*

(1) If $b_0 \in \Omega$, then $\Lambda_\Omega(\Gamma) = \overline{\Gamma(b_0)} \cap \partial\Omega$.
(2) If $\theta = \{\alpha_1, \alpha_{d-1}\}$, then $\Gamma \subset \mathsf{PGL}(d, \mathbb{R})$ is a P_θ-transverse subgroup, with

$$\Lambda_\theta(\Gamma_0) = \{(x, T_x \partial\Omega) : x \in \Lambda_\Omega(\Gamma)\}.$$

In particular, Γ acts as a convergence group on $\Lambda_\Omega(\Gamma)$.
(3) If $\{\gamma_n\}$ is a sequence in Γ and there exists $b_0 \in \Omega$ such that $\gamma_n(b_0) \to x \in \Lambda_\Omega(\Gamma_0)$ and $\gamma_n \to T \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$, then T is the projectivization of a rank 1 linear map whose image is x. Furthermore, if $\gamma_n^{-1}(b_0) \to y$, then $\ker(T) = T_y \partial\Omega$.
(4) $x \in \Lambda_\Omega(\Gamma)$ is a conical limit point (in the convergence group sense) if and only if $x \in \Lambda_\Omega^{\mathrm{con}}(\Gamma)$.

Proof. (1): By definition $\overline{\Gamma(b_0)} \cap \partial\Omega \subset \Lambda_\Omega(\Gamma)$. To show the other inclusion, fix $x \in \Lambda_\Omega(\Gamma)$. Then there is a sequence $\{\gamma_n\}$ in Γ and $b_0' \in \Omega$ such that $\lim \gamma_n(b_0') \to x$. Passing to a subsequence we can suppose that $\lim \gamma_n(b_0) \to x'$. Since

$$\lim_{n \to \infty} \mathrm{d}_\Omega(\gamma_n(b_0'), \gamma_n(b_0)) = \mathrm{d}_\Omega(b_0', b_0),$$

the definition of the Hilbert metric implies that $[x, x']_\Omega \subset \partial\Omega$. Then, since Γ is visible, we must have $x = x' \in \overline{\Gamma(b_0)} \cap \partial\Omega$.

(2): This was established as [15, Prop. 3.5].

(3): First, $\ker(T) \cap \Omega = \emptyset$ by [28, Prop. 5.6]. Next, note that $T(\Omega) \subset \Lambda_\Omega(\Gamma)$; indeed, if $b \in \Omega$, then $b \notin \ker(T)$ and hence

$$T(b) = \lim_{n \to \infty} \gamma_n(b) \in \Lambda_\Omega(\Gamma).$$

Thus, if $b \in \Omega$, then

$$[T(b), x]_\Omega = [T(b), T(b_0)]_\Omega = T([b, b_0]_\Omega) \subset \Lambda_\Omega(\Gamma),$$

so $T(b) = x$ because $T(b), x \in \Lambda_\Omega(\Gamma)$ and $\Gamma \subset \mathrm{Aut}(\Omega)$ is projectively visible. Since $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is open and $T(\Omega) = \{x\}$, it follows that T is the projectivization of a rank 1 map whose image is x. By [28, Prop. 5.6], if $\gamma_n^{-1}(b_0) \to y$, then y lies in the kernel of T. Since $\ker(T) \cap \Omega = \emptyset$ and y is a C^1-smooth point, we have $\ker(T) = T_y \partial\Omega$.

(4): This was established as [15, Lem. 3.6]. $\square$

5.3. **The geodesic flow.** Following earlier work of Benoist [3], Bray [8] and Blayac [5, 6], we now develop the theory of the geodesic flow of a projectively visible group.

First given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, let $T^1\Omega \subset T\Omega$ denote the unit tangent bundle with respect to the infinitesimal Hilbert metric. Given $v \in T^1\Omega$, let $\gamma_v : \mathbb{R} \to \Omega$ denote the unique geodesic line with $\gamma_v'(0) = v$ and whose image is a projective line segment. Also, let

$$v^\pm := \lim_{t\to\infty} \gamma_v(t) \in \partial\Omega.$$

The subspace $T^1\Omega$ has a natural flow, called the *geodesic flow*, which is defined by $\varphi_t(v) = \gamma_v'(t)$. Using this flow, we may define a metric $\mathrm{d}_{T^1\Omega}$ on $T^1\Omega$ by

$$\mathrm{d}_{T^1\Omega}(v, w) := \max_{t\in[0,1]} \mathrm{d}_\Omega\big(\pi(\varphi_t(v)), \pi(\varphi_t(w))\big)$$

where $\pi : T^1\Omega \to \Omega$ takes a vector to its basepoint. It is well-known (see [3, Lem. 3.4] for a proof) that two geodesic rays that end at the same C^1-smooth point in the boundary are asymptotic.

Lemma 5.3. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $v, w \in T^1\Omega$ and $v^+ = w^+$. If $v^+ = w^+$ is a C^1-smooth point of $\partial\Omega$, then there exists $T \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \mathrm{d}_{T^1\Omega}\big(\varphi_{t+T}(v), \varphi_t(w)\big) = 0.$$

Next, given a projectively visible subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$, let $\mathsf{U}(\Gamma) \subset T^1\Omega$ denote the space of all unit tangent vectors v where $v^+, v^- \in \Lambda_\Omega(\Gamma)$. Note that $\mathsf{U}(\Gamma)$ is φ_t-invariant and Γ-invariant, further the Γ-action on $\mathsf{U}(\Gamma)$ is properly discontinuous, and the φ_t-action on $\mathsf{U}(\Gamma)$ commutes with the Γ-action. As such, φ_t descends to a flow, still denoted φ_t, on the quotient

$$\widehat{\mathsf{U}}(\Gamma) := \Gamma\backslash\, \mathsf{U}(\Gamma).$$

Since the Hilbert metric on Ω is a length metric, we can define a metric $\mathrm{d}_{\Gamma\backslash\Omega}$ on $\Gamma\backslash\Omega$ by

$$\mathrm{d}_{\Gamma\backslash\Omega}(a, b) = \inf\{\mathrm{d}_\Omega(\tilde{a}, \tilde{b}) : p(\tilde{a}) = a \text{ and } p(\tilde{b}) = b\}$$

where $p : \Omega \to \Gamma\backslash\Omega$ is the natural projection. Then we may define a metric on $\Gamma\backslash T^1\Omega$ by

$$\mathrm{d}_{\Gamma\backslash T^1\Omega}(v, w) := \max_{t\in[0,1]} \mathrm{d}_{\Gamma\backslash\Omega}\big(\pi(\varphi_t(v)), \pi(\varphi_t(w))\big)$$

where $\pi : \Gamma\backslash T^1\Omega \to \Gamma\backslash\Omega$ takes a vector to its basepoint. Notice that if $p : T^1\Omega \to \Gamma\backslash T^1\Omega$ is the natural projection, then

$$\mathrm{d}_{\Gamma\backslash T^1\Omega}(p(v), p(w)) \le \mathrm{d}_{T^1\Omega}(v, w) \tag{7}$$

for all $v, w \in T^1\Omega$.

Let $\Lambda(\Gamma)^{(2)}$ denote the set of distinct pairs in $\Lambda(\Gamma)^2$. Since Γ is a projectively visible group, $\mathsf{U}(\Gamma)$ is homeomorphic to $\Lambda^{(2)}(\Gamma) \times \mathbb{R}$. Using horofunctions, this homeomorphism can be made explicit. Bray [8, Lem. 3.2] showed that if y is a C^1-smooth point of $\partial\Omega$, there is a well-defined *horofunction at* y

$$h_y : \Omega \times \Omega \to \mathbb{R}$$

given by

$$h_y(a,b) := \lim_{x \to y} \mathrm{d}_\Omega(x,a) - \mathrm{d}_\Omega(x,b),$$

where the limit is taken over all sequences of points x in Ω that converge to y. Since $\Gamma \subset \mathrm{Aut}(\Omega)$ is projectively visible, every point in $\Lambda_\Omega(\Gamma)$ is a C^1-smooth point of $\partial\Omega$, so h_y is well-defined for all $y \in \Lambda_\Omega(\Gamma)$.

For every $b_0 \in \Omega$, the *Hopf parameterization* of U_{Γ_0} determined by b_0 is the identification

$$\mathsf{U}(\Gamma) \cong \Lambda_\Omega(\Gamma)^{(2)} \times \mathbb{R},$$

where $v \in \mathsf{U}(\Gamma)$ is identified with $(v^-, v^+, h_{v^+}(b_0, \pi(v)))$. In this parameterization, the flow φ_t on $\mathsf{U}(\Gamma)$ is given by

$$\varphi_t(x,y,s) = (x,y,s+t),$$

and the Γ action on $\mathsf{U}(\Gamma)$ is given by

$$\gamma(x,y,s) = (\gamma(x), \gamma(y), s + h_y(\gamma^{-1}(b_0), b_0)).$$

6. Transverse representations and Bowen-Margulis-Sullivan measures

By results from [15] and Appendix B, we deduce that any P_θ-transverse subgroup $\Gamma \subset \mathsf{G}$ is the image of a well-behaved representation of a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$. Then, given $\phi \in \mathfrak{a}_\theta^*$ with $\delta^\phi(\Gamma) < +\infty$, we produce a geodesic flow-invariant measure m_ϕ on the unit tangent bundle of Ω, which we call the Bowen-Margulis-Sullivan measure. Later, we will use this measure in our proof of the ergodicity properties of the Patterson-Sullivan measure.

6.1. **Transverse representations.** If $\theta \subset \Delta$ is symmetric, $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ is a projectively visible subgroup, a representation $\rho : \Gamma_0 \to \mathsf{G}$ is said to be P_θ*-transverse* if there exists a continuous ρ-equivariant embedding

$$\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$$

with the following properties:

1. (1) ξ is ρ-equivariant,
2. (2) $\xi(\Lambda_\Omega(\Gamma_0))$ is a transverse subset of $\mathcal{F}_\theta$,
3. (3) if $\{\gamma_n\}$ is a sequence in Γ_0 so that $\gamma_n(b_0) \to x \in \Lambda_\Omega(\Gamma_0)$ and $\gamma_n^{-1}(b_0) \to y \in \Lambda_\Omega(\Gamma_0)$ for some (any) $b_0 \in \Omega$, then $\rho(\gamma_n)(F) \to \xi(x)$ for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{\xi(y)}$.

We refer to ξ as the *limit map* of ρ.

The following observation is a consequence of Proposition 2.3.

Observation 6.1. If $\rho : \Gamma_0 \to \mathsf{G}$ is a P_θ-transverse representation, then $\Gamma := \rho(\Gamma_0)$ is a P_θ-transverse subgroup and the limit map ξ induces a homeomorphism $\Lambda_\Omega(\Gamma_0) \to \Lambda_\theta(\Gamma)$. Moreover,

1. (1) $\xi(\Lambda_\Omega^{\mathrm{con}}(\Gamma_0)) = \Lambda_\theta^{\mathrm{con}}(\Gamma_0)$.
2. (2) If $\{\gamma_n\}$ is a sequence in Γ_0 so that $\gamma_n(b_0) \to x \in \Lambda_\Omega(\Gamma_0)$ for some $b_0 \in \Omega$, then $U_\theta(\rho(\gamma_n)) \to \xi(x)$ and $\alpha(\kappa(\rho(\gamma_n))) \to \infty$ for all $\alpha \in \theta$.

Proof. We begin by proving (2). Fix a sequence $\{\gamma_n\}$ in Γ_0 so that $\gamma_n(b_0) \to x \in \Lambda_\Omega(\Gamma_0)$ for some $b_0 \in \Omega$. By compactness it suffices to consider the case where $F^+ := \lim_{n\to\infty} U_\theta(\gamma_n)$ and

$$L := \lim_{n\to\infty} \min_{\alpha\in\theta} \alpha(\kappa(\rho(\gamma_n))) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

both exist, then show that $\xi(x) = F^+$ and $L = +\infty$. Passing to a subsequence we can suppose that $\gamma_n^{-1}(b_0) \to y$. Then by definition $\rho(\gamma_n)(F) \to \xi(x)$ for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{\xi(y)}$ and $\rho(\gamma_n^{-1})(F) \to \xi(y)$ for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{\xi(x)}$. Since $\mathcal{F}_\theta \setminus \mathcal{Z}_{\xi(y)}$ and $\mathcal{F}_\theta \setminus \mathcal{Z}_{\xi(x)}$ are both open, Proposition 2.6 implies that $\xi(x) = F^+$ and $L = +\infty$. Thus (2) is true.

Then $\Gamma := \rho(\Gamma_0)$ is a P_θ-divergent subgroup and ξ induces a homeomorphism $\Lambda_\Omega(\Gamma_0) \to \Lambda_\theta(\Gamma)$. Further, by definition, $\Lambda_\theta(\Gamma) = \xi(\Lambda_\Omega(\Gamma_0))$ is a transverse subset and hence Γ is P_θ-transverse. Finally, Proposition 5.2(4) implies that $\xi(\Lambda_\Omega^{\rm con}(\Gamma_0)) = \Lambda_\theta^{\rm con}(\Gamma_0)$. □

The next two results were established in [15] in the special case when $\mathsf{G} = \mathsf{PSL}(d,\mathbb{R})$. In Appendix B we explain how to reduce the general case to this special case.

The first result states that under mild conditions on G and θ, see Section 2.4, every transverse group is the image of a transverse representation.

Theorem 6.2. *Suppose $Z(\mathsf{G})$ is trivial, $\theta \subset \Delta$ is symmetric and P_θ contains no simple factors of G. If $\Gamma \subset \mathsf{G}$ is P_θ-transverse, then there exist $d \in \mathbb{N}$, a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ and a faithful P_θ-transverse representation $\rho : \Gamma_0 \to \mathsf{G}$ with limit map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$ such that $\rho(\Gamma_0) = \Gamma$ and $\xi(\Lambda_\Omega(\Gamma_0)) = \Lambda_\theta(\Gamma)$.*

It will be useful throughout the paper, to understand how the Cartan projection behaves under multiplication of group elements. The next lemma assures that when two elements translate a basepoint $b_0 \in \Omega$ in roughly the same direction, then the Cartan projection is coarsely additive.

Proposition 6.3. *Suppose $\theta \subset \Delta$ is symmetric, $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ is a projectively visible subgroup and $\rho : \Gamma_0 \to \mathsf{G}$ a P_θ-transverse representation. For any $b_0 \in \Omega$ and $r > 0$, there exist $C > 0$ such that if $\gamma, \eta \in \Gamma_0$ and*

$$\mathrm{d}_\Omega\left(\gamma(b_0), [b_0, \eta(b_0)]_\Omega\right) \le r,$$

then

$$\left\|\kappa_\theta(\rho(\eta)) - \kappa_\theta(\rho(\gamma)) - \kappa_\theta(\rho(\gamma^{-1}\eta))\right\| \le C.$$

6.2. **The Bowen-Margulis-Sullivan measure.** Suppose $\theta \subset \Delta$ is symmetric, $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ is a non-elementary projectively visible subgroup and $\rho : \Gamma_0 \to \mathsf{G}$ is a P_θ-transverse representation with limit map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$. Let $\Gamma := \rho(\Gamma_0)$.

As in Section 2, let $\iota : \mathfrak{a} \to \mathfrak{a}$ denote the opposite involution. Then fix $\phi \in \mathfrak{a}_\theta^*$ with $\delta := \delta^\phi(\Gamma) < +\infty$ and let

$$\bar\phi := \phi \circ \iota \in \mathfrak{a}_\theta^*.$$

Notice that $\bar\phi(\kappa_\theta(g)) = \phi(\kappa_\theta(g^{-1}))$ for all $g \in \mathsf{G}$, and so $\delta^{\bar\phi}(\Gamma) = \delta^\phi(\Gamma) < +\infty$. Finally, suppose μ_ϕ is a ϕ-Patterson-Sullivan measure for Γ and $\mu_{\bar\phi}$ is a $\bar\phi$-Patterson-Sullivan measure for Γ, both with dimension δ.

The goal of this section is to construct, using ρ, μ_ϕ and $\mu_{\bar\phi}$, a measure m on $\widehat{\mathsf{U}}(\Gamma_0)$ that is φ_t-invariant. We will call this measure the *Bowen-Margulis measure associated to ρ, μ_ϕ and $\mu_{\bar\phi}$.*

Let $\mathcal{F}_\theta^{(2)}$ denote the space of pairs of transverse flags in $\mathcal{F}_\theta$. Then there exists a continuous function

$$[\cdot,\cdot]_\theta : \mathcal{F}_\theta^{(2)} \to \mathfrak{a}_\theta,$$

called the *Gromov product* such that

$$[g(F), g(G)]_\theta - [F,G]_\theta = -\iota \circ B_\theta(g,F) - B_\theta(g,G) \tag{8}$$

for all $g \in \mathsf{G}$ and $(F,G) \in \mathcal{F}_\theta^{(2)}$, see [44, Lem. 4.12].

Identify $\mathsf{U}(\Gamma_0) = \Lambda_\Omega(\Gamma_0)^{(2)} \times \mathbb{R}$ via the Hopf parametrization based at a point $b_0 \in \Omega$. Then define a measure $\tilde{m}$ on $\mathsf{U}(\Gamma_0)$ by

$$\mathrm{d}\tilde{m}(x, y, s) = e^{-\delta\phi([\xi(x),\xi(y)]_\theta)}\mathrm{d}\mu_{\bar{\phi}}(x) \otimes \mathrm{d}\mu_\phi(y) \otimes \mathrm{d}t(s)$$

where $\mathrm{d}t$ is the Lebesgue measure on $\mathbb{R}$. This measure is clearly φ_t-invariant. Furthermore, Equation (8) and the quasi-invariance property of μ_ϕ and $\mu_{\bar{\phi}}$, imply that $\tilde{m}$ is Γ_0-invariant. Therefore, $\tilde{m}$ descends to a measure m on $\widehat{\mathsf{U}}(\Gamma_0)$ that is φ_t-invariant.

7. A shadow lemma for transverse representations

Sullivan's shadow lemma, originally proven in the setting of convex cocompact Kleinian groups [46], is a central tool in the analysis of Patterson-Sullivan measures in many settings. It gives estimates from above and below on the measure of a shadow in the sphere at infinity of a ball about an orbit point from a light based at the basepoint.

In the setting of properly convex domains, shadows can be defined as follows: If Ω is a properly convex domain, $b, p \in \Omega$ and $r > 0$, one defines the shadow

$$\mathcal{O}_r(b, p) := \{x \in \partial\Omega : \mathrm{d}_\Omega(p, [b, x)_\Omega) < r\}.$$

Our version of Sullivan's shadow then has the following form.

Proposition 7.1. *Suppose $\theta \subset \Delta$ is symmetric, $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ is a non-elementary projectively visible subgroup, $\rho : \Gamma_0 \to \mathsf{G}$ a P_θ-transverse representation with limit map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$, $\Gamma := \rho(\Gamma_0)$, $\phi \in \mathfrak{a}_\theta^*$ and μ is a ϕ-Patterson-Sullivan measure for Γ of dimension β. For any $b_0 \in \Omega$, there exists R_0 such that: if $r > R_0$, then there exists $C = C(b_0, r) > 1$ so that*

$$C^{-1}e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))} \le \mu\Big(\xi\big(\mathcal{O}_r(b_0, \gamma(b_0)) \cap \Lambda_\Omega(\Gamma_0)\big)\Big) \le Ce^{-\beta\phi(\kappa_\theta(\rho(\gamma)))}$$

for all $\gamma \in \Gamma_0$.

Proof. For notational convenience, we let ν be the measure on $\partial\Omega$ defined by

$$\nu(A) = \mu\big(\xi(A \cap \Lambda_\Omega(\Gamma_0))\big).$$

By Proposition 2.8, the action of Γ on $\Lambda_\theta(\Gamma)$ is minimal, so the support of μ is $\Lambda_\theta(\Gamma)$. Also, since $\Lambda_\theta(\Gamma) = \xi(\Lambda_\Omega(\Gamma_0))$, it follows that $\Lambda_\Omega(\Gamma_0)$ is the support of ν. This observation, together with a compactness argument, yields a lower bound on the measure of (large enough) shadows of b_0 based at any point in $\Gamma(b_0)$.

Lemma 7.2. *For any $b_0 \in \Omega$, there exist $\epsilon_0, R_0 > 0$ such that*

$$\nu(\mathcal{O}_{R_0}(z, b_0)) \ge \epsilon_0$$

for all $z \in \Gamma(b_0)$.

Proof. Suppose not. Then for every $n \ge 1$ there exists $z_n \in \Gamma(b_0)$ such that

$$\nu(\mathcal{O}_n(z_n, b_0)) \le 2^{-n}.$$

Passing to a subsequence we can suppose that $z_n \to z \in \Gamma(b_0) \cup \Lambda_\Omega(\Gamma_0)$. If $z \in \Gamma(b_0)$, then

$$\bigcup_{n=N}^{\infty} \mathcal{O}_n(z_n, b_0) = \partial\Omega$$

for every $N \geq 1$. On the other hand, if $z \in \Lambda_\Omega(\Gamma_0)$, then by assumption, $(z,y)_\Omega \subset \Omega$ for every $y \in \Lambda_\Omega(\Gamma_0) \setminus \{z\}$. This implies that $\mathrm{d}_\Omega(b_0,(z,y)_\Omega) < +\infty$, so

$$\bigcup_{n=N}^{\infty} \mathcal{O}_n(z_n, b_0) \supset \Lambda_\Omega(\Gamma_0) - \{z\}$$

for every $N \geq 1$. Thus, in either case

$$\nu(\Lambda_\Omega(\Gamma_0) - \{z\}) \leq \lim_{N\to\infty} \sum_{n\geq N} \nu(\mathcal{O}_n(z_n,b_0)) = 0.$$

Since $\Lambda_\Omega(\Gamma_0) - \{z\}$ is open in $\Lambda_\Omega(\Gamma_0)$, which is the support of ν, this is impossible. $\square$

Next we use Proposition 6.3 to show that if $x \in \Lambda_\Omega(\Gamma_0)$ lies in the shadow $\mathcal{O}_r(b_0,\gamma(b_0))$ for some $\gamma \in \Gamma_0$, then $B_\theta(\rho(\gamma)^{-1},\xi(x))$ can be approximated by $\kappa_\theta(\rho(\gamma))$.

Lemma 7.3. *For any $r > 0$, there exists $C_1 > 0$ such that*

$$\left|\phi\big(B_\theta(\rho(\gamma)^{-1},\xi(x)) + \kappa_\theta(\rho(\gamma))\big)\right| \leq C_1$$

for all $\gamma \in \Gamma_0$ and $x \in \mathcal{O}_r(b_0,\gamma(b_0)) \cap \Lambda_\Omega(\Gamma_0)$.

Proof. Let $\|\phi\|$ denote the operator norm of ϕ (relative to the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$ introduced in Section 2). Then let $C_1 := C\,\|\phi\|$, where C is the constant given by Proposition 6.3 for b_0 and r.

Since $x \in \Lambda_\Omega(\Gamma_0)$, by Proposition 5.2(1), there exists a sequence $\{\eta_n\}$ in Γ_0 such that $\eta_n(b_0) \to x$. Since $x \in \mathcal{O}_r(b_0,\gamma(b_0))$, we have

$$\mathrm{d}_\Omega(\gamma(b_0),[b_0,x)_\Omega) < r$$

and hence

$$\mathrm{d}_\Omega\left(\gamma(b_0),[b_0,\eta_n(b_0)]_\Omega\right) < r$$

for sufficiently large n. So, by Proposition 6.3,

$$\left|\phi\big(\kappa_\theta(\rho(\gamma)) + \kappa_\theta(\rho(\gamma^{-1}\eta_n)) - \kappa_\theta(\rho(\eta_n))\big)\right| \leq C_1$$

for sufficiently large n. Further, Observation 6.1 implies that $U_\theta(\rho(\eta_n)) \to \xi(x)$. So, by the "moreover" part of Lemma 3.3,

$$\begin{aligned}\left|\phi\big(B_\theta(\rho(\gamma)^{-1},\xi(x)) + \kappa_\theta(\rho(\gamma))\big)\right| &= \lim_{n\to\infty}\left|\phi\big(B_\theta(\rho(\gamma)^{-1},U_\theta(\rho(\eta_n))) + \kappa_\theta(\rho(\gamma))\big)\right| \\ &= \lim_{n\to\infty}\left|\phi\big(\kappa_\theta(\rho(\gamma^{-1}\eta_n)) - \kappa_\theta(\rho(\eta_n)) + \kappa_\theta(\rho(\gamma))\big)\right| \leq C_1.\end{aligned}$$

$\square$

Now we can complete the proof of Proposition 7.1.

Let $\epsilon_0, R_0 > 0$ be the constants given by Lemma 7.2 (which depend on b_0). For any $r \geq R_0$ and $\gamma \in \Gamma_0$,

$$\nu\Big(\mathcal{O}_r(\gamma^{-1}(b_0),b_0)\Big) = \gamma_*\nu\Big(\mathcal{O}_r(b_0,\gamma(b_0))\Big) = \int_{\mathcal{O}_r(b_0,\gamma(b_0))} e^{-\beta\phi(B_\theta(\rho(\gamma)^{-1},\xi(x)))}d\nu(x).$$

So Lemma 7.3 implies that there is some $C_1 > 0$ (which depends on r) such that

$$e^{\beta\phi(\kappa_\theta(\rho(\gamma))))-\beta C_1} \leq \frac{\nu\Big(\mathcal{O}_r(\gamma^{-1}(b_0),b_0)\Big)}{\nu\Big(\mathcal{O}_r(b_0,\gamma(b_0))\Big)} \leq e^{\beta\phi(\kappa_\theta(\rho(\gamma))))+\beta C_1}.$$

Since $r \geq R_0$, Lemma 7.2 implies that $\epsilon_0 \leq \nu\Big(\mathcal{O}_r(\gamma^{-1}(b_0), b_0)\Big) \leq 1$, so

$$\epsilon_0 e^{-\beta C_1} e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))} \leq \nu\Big(\mathcal{O}_r(b_0, \gamma(b_0))\Big) \leq e^{\beta C_1} e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))}.$$

Hence the lemma holds with $C := e^{\beta C_1}\epsilon_0^{-1}$. □

8. Consequences of the shadow lemma

In this section, we collect several standard consequences of the shadow lemma. Most importantly, we see that conical limit points cannot be atoms for any Patterson-Sullivan measure and that if the ϕ-Poincaré series converges in the dimension of the measure, then the conical limit set has measure zero. Later, we will see that if the ϕ-Poincaré series diverges at its critical exponent, then the conical limit set has full measure in the ϕ-Patterson-Sullivan measure associated to the critical exponent.

Proposition 8.1. *Suppose $\theta \subset \Delta$ is symmetric, $\Gamma \subset \mathsf{G}$ is a non-elementary P_θ-transverse subgroup, $\phi \in \mathfrak{a}_\theta^*$ and μ is a ϕ-Patterson-Sullivan measure with dimension β.*

(1) $\beta \geq \delta^\phi(\Gamma)$.
(2) If $y \in \Lambda_\theta^{\rm con}(\Gamma)$, then $\mu(\{y\}) = 0$.
(3) If $Q_\Gamma^\phi(\beta) < +\infty$, then $\mu(\Lambda_\theta^{\rm con}(\Gamma)) = 0$.
(4) If $\{\Gamma_n\}$ is a sequence of increasing subgroups with $\Gamma = \cup\Gamma_n$, then

$$\lim_{n\to\infty} \delta^\phi(\Gamma_n) = \delta^\phi(\Gamma).$$

The rest of the section is devoted to the proof of the proposition. Fix a non-elementary, P_θ-transverse group $\Gamma \subset \mathsf{G}$, $\phi \in \mathfrak{a}_\theta^*$ and a ϕ-Patterson-Sullivan measure μ with dimension β.

Using the discussion in Section 2.4 we may assume that G has trivial center and that P_θ does not contain any simple factors of G. By Theorem 6.2, there is a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ and a faithful P_θ-transverse representation $\rho : \Gamma_0 \to \mathsf{G}$ with limit map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$ so that $\rho(\Gamma_0) = \Gamma$ and $\xi(\Lambda_\Omega(\Gamma_0)) = \Lambda_\theta(\Gamma)$. Further, $\xi(\Lambda_\Omega^{\rm con}(\Gamma_0)) = \Lambda_\theta^{\rm con}(\Gamma)$, see Observation 6.1. Define a probability measure ν on $\partial\Omega$ by

$$\nu(A) := \mu\big(\xi(A \cap \Lambda_\Omega(\Gamma_0))\big).$$

Fix $b_0 \in \Omega$. By the Shadow Lemma (Proposition 7.1) there is some $R_0 > 0$ such that for every $r \geq R_0$ there exists a constant $C_1 = C_1(r) \geq 1$ where

$$C_1^{-1} e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))} \leq \nu\big(\mathcal{O}_r(b_0, \gamma(b_0))\big) \leq C_1 e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))} \tag{9}$$

for all $\gamma \in \Gamma$.

Proof of part (1). We will make use of a subdivision of the group into sets of the form

$$\mathcal{A}_n := \{\gamma \in \Gamma_0 : n < \phi(\kappa_\theta(\rho(\gamma))) \leq n+1\}.$$

We observe that if elements in a single $\mathcal{A}_n$ have overlapping shadows, then they are nearby.

Lemma 8.2. *For any $r > 0$, there exists $C_2 = C_2(r) > 0$ such that: if $\gamma_1, \gamma_2 \in \mathcal{A}_n$ and $\mathcal{O}_r(b_0, \gamma_1(b_0)) \cap \mathcal{O}_r(b_0, \gamma_2(b_0)) \neq \emptyset$, then*

$$\mathrm{d}_\Omega(\gamma_1(b_0), \gamma_2(b_0)) \leq C_2.$$

Proof. Fix $x \in \mathcal{O}_r(b_0, \gamma_1(b_0)) \cap \mathcal{O}_r(b_0, \gamma_2(b_0)) \neq \emptyset$. Then for $j = 1, 2$, there exists $p_j \in [b_0, x)$ such that $\mathrm{d}_\Omega(p_j, \gamma_j(b_0)) < r$. After possibly relabelling we may assume that $p_1 \in [b_0, p_2]$. By Proposition 5.1,

$$\mathrm{d}_\Omega(\gamma_1(b_0), [b_0, \gamma_2(b_0)]) \leq \mathrm{d}_\Omega(\gamma_1(b_0), p_1) + \mathrm{d}_\Omega(p_1, [b_0, \gamma_2(b_0)]) \leq r + \mathrm{d}_\Omega(p_2, \gamma_2(p_0)) \leq 2r.$$

Then by Proposition 6.3 there exists a constant $C > 0$ (which depends on r) such that

$$\left|\phi\Big(\kappa_\theta(\rho(\gamma_1)) + \kappa_\theta(\rho(\gamma_1^{-1}\gamma_2)) - \kappa_\theta(\rho(\gamma_2))\Big)\right| \leq C.$$

Since $\gamma_1, \gamma_2 \in \mathcal{A}_n$, it follows that

$$\phi(\kappa_\theta(\rho(\gamma_1^{-1}\gamma_2))) \leq C + 1.$$

Thus, if we choose

$$C_2 := \max\{\mathrm{d}_\Omega(b_0, \gamma(b_0)) : \gamma \in \Gamma_0 \text{ and } \phi(\kappa_\theta(\rho(\gamma))) \leq C + 1\},$$

then

$$\mathrm{d}_\Omega(\gamma_1(b_0), \gamma_2(b_0)) = \mathrm{d}_\Omega(b_0, \gamma_1^{-1}\gamma_2(b_0)) \leq C_2. \qquad \square$$

Fix $r \geq R_0$, and let $C_2 > 0$ be the constant given by Lemma 8.2 for r. For each n, let $\mathcal{A}_n' \subset \mathcal{A}_n$ be a maximal collection of elements such that

$$\mathrm{d}_\Omega(\gamma_1(b_0), \gamma_2(b_0)) > C_2$$

for all distinct $\gamma_1, \gamma_2 \in \mathcal{A}_n'$. Observe that if

$$N := \#\{\gamma \in \Gamma : \mathrm{d}_\Omega(\gamma(b_0), b_0) \leq C_2\},$$

then $\# \mathcal{A}_n' \geq \frac{1}{N} \# \mathcal{A}_n$.

By Lemma 8.2,

$$\mathcal{O}_r(b_0, \gamma_1(b_0)) \cap \mathcal{O}_r(b_0, \gamma_2(b_0)) = \emptyset$$

for all $\gamma_1, \gamma_2 \in \mathcal{A}_n'$. Thus, by (9),

$$1 = \nu\big(\Lambda_\Omega(\Gamma_0)\big) \geq \sum_{\gamma \in \mathcal{A}_n'} \nu\big(\mathcal{O}_r(b_0, \gamma(b_0))\big) \geq \frac{1}{C_1} \sum_{\gamma \in \mathcal{A}_n'} e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))} \geq \frac{1}{C_1} \# \mathcal{A}_n' \, e^{-\beta(n+1)}.$$

This implies that $\# \mathcal{A}_n \leq N \# \mathcal{A}_n' \leq C_1 N e^{\beta(n+1)}$. Then

$$\delta^\phi(\Gamma) = \limsup_{n \to \infty} \frac{1}{n} \log \# \mathcal{A}_n \leq \beta. \qquad \square$$

Proof of part (2). We first observe that β is positive. If this were not the case, then part (1) implies that $\beta = 0$, or equivalently, that μ is a Γ-invariant measure on $\Lambda_\theta(\Gamma)$. However, this is impossible because Γ acts as a non-elementary convergence group on $\Lambda_\theta(\Gamma)$.

Let $y \in \Lambda_\theta^{\mathrm{con}}(\Gamma)$. Then by Observation 6.1(1), $x := \xi^{-1}(y) \in \Lambda_\Omega^{\mathrm{con}}(\Gamma_0)$. By Proposition 5.2(4), there is some $r > 0$ and a sequence $\{\gamma_n\}$ in Γ_0 such that $\gamma_n(b_0) \to x$ and $\mathrm{d}_\Omega(\gamma_n(b_0), [b_0, x)) < r$ for all n. We may assume that $r \geq R_0$.

By part (1), $\delta^\phi(\Gamma) \leq \beta < +\infty$, so $Q_\Gamma^\phi(s)$ converges for s sufficiently large. This implies that

$$\lim_{n \to \infty} \phi(\kappa_\theta(\rho(\gamma_n))) = +\infty.$$

Since $x \in \mathcal{O}_r(b_0, \gamma_n(b_0))$ for all n and $\beta > 0$, it follows from (9) that

$$\mu(\{y\}) \leq \liminf_{n \to \infty} \nu\big(\mathcal{O}_r(b_0, \gamma_n(b_0))\big) \leq C_1 \liminf_{n \to \infty} e^{-\beta\phi(\kappa_\theta(\rho(\gamma_n)))} = 0. \qquad \square$$

Proof of part (3). For $r > 0$ let $\Lambda_{\Omega, b_0, r}(\Gamma_0) \subset \Lambda_\Omega(\Gamma_0)$ denote the set of limit points x where there is a sequence $\{\gamma_n\}$ in Γ_0 such that $\gamma_n(b_0) \to x$ and $\mathrm{d}_\Omega(\gamma_n(b_0), [b_0, x)) < r$ for all n. Notice that $\Lambda_\Omega^{\mathrm{con}}(\Gamma_0) = \bigcup_{n \in \mathbb{N}} \Lambda_{\Omega, b_0, n}(\Gamma_0)$. Therefore, it suffices to show that $\mu(\xi(\Lambda_{\Omega, b_0, r}(\Gamma_0))) = 0$ for all $r \geq R_0$.

Fix $r \geq R_0$, fix an enumeration $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ and let $F_n := \{\gamma_1, \dots, \gamma_n\}$. Then for any n,

$$\Lambda_{\Omega, b_0, r}(\Gamma_0) \subset \bigcup_{\gamma \in \Gamma - F_n} \mathcal{O}_r(b_0, \gamma(b_0)),$$

so by (9),

$$\nu\big(\Lambda_{\Omega,b_0,r}(\Gamma_0)\big) \leq \sum_{\gamma\in\Gamma-F_n} \nu\big(\mathcal{O}_r(b_0,\gamma(b_0))\big) \leq C_1 \sum_{\gamma\in\Gamma-F_n} e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))}.$$

However, since $Q_\Gamma^\phi(\beta) < +\infty$,

$$\lim_{n\to\infty} \sum_{\gamma\in\Gamma-F_n} e^{-\beta\phi(\kappa_\theta(\rho(\gamma)))} = 0.$$

Therefore, $\nu\big(\Lambda_{\Omega,b_0,r}(\Gamma_0)\big) = 0$ for $r \geq R_0$. □

Proof of part (4). Since $\{\Gamma_n\}$ is a sequence of increasing subgroups, $\delta^\phi(\Gamma_1) \leq \delta^\phi(\Gamma_2) \leq \dots$ and hence $\delta := \lim_{n\to\infty} \delta^\phi(\Gamma_n) \in \mathbb{R} \cup \{+\infty\}$ exists. Further, $\delta \leq \delta^\phi(\Gamma)$. If $\delta = +\infty$, then

$$\delta^\phi(\Gamma) = +\infty = \lim_{n\to\infty} \delta^\phi(\Gamma_n).$$

If $\delta < +\infty$, then for each n there exists a ϕ-Patterson-Sullivan measure μ_n for Γ_n with dimension $\delta^\phi(\Gamma_n)$. If μ is a weak-$*$ limit point of $\{\mu_n\}$, then μ is a ϕ-Patterson-Sullivan measure for Γ with dimension δ. Hence by part (1) we have $\delta \geq \delta^\phi(\Gamma)$. □

9. The conical limit set has full measure in the divergent case

In this section we show that the Patterson-Sullivan measure is supported on the conical limit set in case when the associated Poincaré series diverges at its critical exponent. The proof is similar to Roblin's [42] argument for the analogous result in CAT(-1) spaces – in that we use a variant of the Borel-Cantelli Lemma. However, we use a different variant of the lemma and apply it to a different collection of sets. This seems to simplify the argument and this approach was developed during discussions between the authors and Pierre-Louis Blayac.

Proposition 9.1. *Suppose $\theta \subset \Delta$ is symmetric, $\Gamma \subset \mathsf{G}$ is a non-elementary P_θ-transverse subgroup, $\phi \in \mathfrak{a}_\theta^*$, $\delta^\phi(\Gamma) < +\infty$ and μ is a ϕ-Patterson-Sullivan measure for Γ with dimension $\delta := \delta^\phi(\Gamma)$. If $Q_\Gamma^\phi(\delta) = +\infty$, then $\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 1$. In particular, μ has no atoms.*

We will use the following variant of the Borel-Cantelli Lemma, sometimes called the Kochen-Stone Lemma.

Lemma 9.2 (Kochen-Stone Lemma [32])**.** *Let (X,μ) be a finite measure space. If $\{A_n\}$ is a sequence of measurable sets where $\sum_{n=1}^\infty \mu(A_n) = +\infty$ and*

$$\liminf_{N\to\infty} \frac{\sum_{1\leq m,n\leq N} \mu(A_n\cap A_m)}{\left(\sum_{n=1}^N \mu(A_n)\right)^2} < +\infty,$$

then the set $\{x \in X : x$ is in infinitely many of $A_1, A_2, \dots\}$ has positive μ measure.

For the rest of the section fix Γ, ϕ and μ as in the statement of Proposition 9.1. Using the discussion in Section 2.4 we may assume that G has trivial center and that P_θ does not contain any simple factors of G. Then by Theorem 6.2, there is a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ and a faithful P_θ-transverse representation $\rho : \Gamma_0 \to \mathsf{G}$ with limit map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$ so that $\rho(\Gamma_0) = \Gamma$ and $\xi(\Lambda_\Omega(\Gamma_0)) = \Lambda_\theta(\Gamma)$. Define a measure ν on $\partial\Omega$ by

$$\nu(A) = \mu\big(\xi(A\cap\Lambda_\Omega(\Gamma_0))\big).$$

Fix $b_0 \in \Omega$. Then using Proposition 7.1 we may fix $C, r > 0$ such that

$$\frac{1}{C} e^{-\delta\phi(\kappa_\theta(\rho(\gamma)))} \leq \nu\left(\mathcal{O}_r(b_0,\gamma(b_0))\right) \leq C e^{-\delta\phi(\kappa_\theta(\rho(\gamma)))} \tag{10}$$

for all $\gamma \in \Gamma$. Fix an enumeration $\Gamma_0 = \{\gamma_1, \gamma_2, \dots\}$ and let $T_n := \mathrm{d}_\Omega(b_0, \gamma_n(b_0))$. By reordering we may assume that

$$T_1 \leq T_2 \leq T_3 \leq \cdots.$$

Then let $A_n := \mathcal{O}_r(b_0, \gamma_n(b_0))$. We will verify that the sets $\{A_n\}$ satisfy the hypotheses of Lemma 9.2.

The first hypothesis in Lemma 9.2 is easy to check. Directly from Equation (10) we obtain

$$\sum_{n=1}^{\infty} \nu(A_n) \geq \frac{1}{C} \sum_{n=1}^{\infty} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} = \frac{1}{C} Q_\Gamma^\phi(\delta) = +\infty.$$

Verifying the second hypothesis in Lemma 9.2 is slightly more involved. We require the following technical result, which informally says that the "boundaries" of sums of the form $\sum_{n=1}^{N} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))}$ are controlled by their "interiors."

For $N \in \mathbb{N}$, set

$$N' := \max\{n \in \mathbb{N} : T_n \leq T_N + 2r\}.$$

Lemma 9.3. *There exists $C_1 > 1$ such that: if $N \geq 1$, then*

$$\sum_{n=1}^{N'} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} \leq C_1 \sum_{n=1}^{N} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))}.$$

Proof. Note that if $T_n, T_m \in [T_N, T_N + 2r]$ and $\mathcal{O}_r(b_0, \gamma_n(b_0)) \cap \mathcal{O}_r(b_0, \gamma_m(b_0)) \neq \emptyset$, then

$$\mathrm{d}_\Omega(b_0, \gamma_n^{-1}\gamma_m(b_0)) = \mathrm{d}_\Omega(\gamma_n(b_0), \gamma_m(b_0)) \leq 6r.$$

Thus, if we set

$$M := \#\{\gamma \in \Gamma_0 : \mathrm{d}_\Omega(b_0, \gamma(b_0)) \leq 6r\},$$

then every point in $\partial\Omega$ lies in at most M different sets of the form $\mathcal{O}_r(b_0, \gamma_n(b_0))$ such that $T_n \in [T_N, T_N + 2r]$. This implies that

$$\sum_{n=N+1}^{N'} \nu\left(A_n\right) = \sum_{n=N+1}^{N'} \nu\left(\mathcal{O}_r(b_0, \gamma_n(b_0))\right) \leq M\nu(\partial\Omega) = M.$$

Then by Equation (10),

$$\begin{aligned}\sum_{n=1}^{N'} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} &= \sum_{n=1}^{N} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} + \sum_{n=N+1}^{N'} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} \leq \sum_{n=1}^{N} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} + CM \\ &\leq \left(1 + \frac{CM}{e^{-\delta\phi(\kappa_\theta(\rho(\gamma_1)))}}\right) \sum_{n=1}^{N} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))}\end{aligned}$$

for all $N \geq 1$. The lemma now holds with $C_1 := 1 + \frac{CM}{e^{-\delta\phi(\kappa_\theta(\rho(\gamma_1)))}}$. $\square$

The next lemma verifies that the sequence $\{A_n\}$ satisfy the second hypothesis of Lemma 9.2.

Lemma 9.4. *There exists $C_2 > 0$ such that: if $N \geq 1$, then*

$$\sum_{1 \leq n,m \leq N} \nu(A_n \cap A_m) \leq C_2 \left(\sum_{n=1}^{N} \nu(A_n)\right)^2.$$

Proof. Let

$$\Delta_N := \{(m,n) : 1 \leq n \leq m \leq N \text{ and } A_m \cap A_n \neq \emptyset\}.$$

One can show (see the proof of Lemma 8.2) that if $(m,n) \in \Delta_N$, then

$$\mathrm{d}_\Omega(\gamma_n(b_0), [b_0, \gamma_m(b_0)]_\Omega) \leq 2r.$$

Then Proposition 6.3 implies

$$\sup_{(m,n)\in\Delta_N} \left\| \kappa_\theta(\rho(\gamma_n)) + \kappa_\theta(\rho(\gamma_n^{-1}\gamma_m)) - \kappa_\theta(\rho(\gamma_m)) \right\| < +\infty, \tag{11}$$

and so by Equation (10), there exists a constant $C' > 0$ such that

$$\nu(A_n \cap A_m) \leq \nu(\mathcal{O}_r(b_0, \gamma_m(b_0))) \leq C' e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n^{-1}\gamma_m)))}$$

for all $(m,n) \in \Delta_N$. Also,

$$\begin{aligned} \mathrm{d}_\Omega(b_0, \gamma_n^{-1}\gamma_m(b_0)) = \mathrm{d}_\Omega(\gamma_n(b_0), \gamma_m(b_0)) &\leq \mathrm{d}_\Omega(\gamma_n(b_0), [b_0, \gamma_m(b_0)]_\Omega) + \mathrm{d}_\Omega(b_0, \gamma_m(b_0)) \\ &\leq 2r + T_m \leq 2r + T_N \end{aligned}$$

for all $(m,n) \in \Delta_N$. In particular, if $(m,n) \in \Delta_N$, then $\gamma_n^{-1}\gamma_m = \gamma_k$ for some $k \leq N'$.

These observations, Lemma 9.3 and Equation (10) imply that if $N \geq 1$, then

$$\begin{aligned} \sum_{1\leq n,m\leq N} \nu(A_n \cap A_m) &\leq 2 \sum_{(m,n)\in\Delta_N} \nu(A_n \cap A_m) \leq 2C' \sum_{(m,n)\in\Delta_N} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n^{-1}\gamma_m)))} \\ &\leq 2C' \sum_{k=1}^{N'} \sum_{n=1}^{N} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_k)))} \leq 2C'C_1 \left(\sum_{n=1}^{N} e^{-\delta\phi(\kappa_\theta(\rho(\gamma_n)))} \right)^2 \\ &\leq 2C'C_1C^2 \left(\sum_{n=1}^{N} \nu(A_n) \right)^2. \end{aligned}$$

□

We may now apply Lemma 9.2 to the finite measure space $(\partial\Omega, \nu)$ and the sequence $\{A_n\}$ to finish the proof of Proposition 9.1.

Proof of Proposition 9.1. We first show that $\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) > 0$. By Lemma 9.2, if we set

$$Y := \{x \in \partial\Omega : x \text{ is in infinitely many of } A_1, A_2, \dots\},$$

then $\nu(Y) > 0$. Notice that if $x \in Y$, then there is a sequence $\{\gamma_n\}$ in Γ such that $\gamma_n(b_0) \to x$ and

$$\mathrm{d}_\Omega(\gamma_n(b_0), [b_0, x)) < r$$

for all $n \geq 1$. Thus $Y \subset \Lambda_\Omega^{\mathrm{con}}(\Gamma_0)$. By Observation 6.1(1), $\xi^1(Y) \subset \Lambda_\theta^{\mathrm{con}}(\Gamma)$, so

$$\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) \geq \mu(\xi^1(Y)) = \nu(Y) > 0.$$

Now suppose for contradiction that $\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) < 1$. If we set $S := \Lambda_\theta(\Gamma) - \Lambda_\theta^{\mathrm{con}}(\Gamma)$, then $\mu(S) > 0$, so we may define a probability measure μ_S on $\Lambda_\theta(\Gamma)$ by

$$\mu_S(A) := \frac{1}{\mu(S)} \mu(A \cap S).$$

By definition, $\mu_S(\Lambda_\theta(\Gamma)) = 0$. On the other hand, since S is Γ-invariant, μ_S is a ϕ-Patterson-Sullivan measure for Γ of dimension δ, so the above argument implies that $\mu_S(\Lambda_\theta^{\mathrm{con}}(\Gamma)) > 0$, which is a contradiction. Therefore, $\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 1$.

By Proposition 8.1, μ has no atoms in $\Lambda_\theta^{\mathrm{con}}(\Gamma)$. Since $\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 1$, we conclude that μ has no atoms. □

10. Ergodicity of the flow in the divergent case

In this section, we prove that the geodesic flow of a transverse representation is ergodic if its image is in the divergent case of our Hopf-Sullivan-Tsuji dichotomy.

Theorem 10.1. *Let $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ be a properly convex domain, let $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ be a non-elementary projectively visible subgroup and let $\rho : \Gamma_0 \to \mathsf{G}$ be a P_θ-transverse representation for some symmetric $\theta \subset \Delta$. Let $\phi \in \mathfrak{a}_\theta^*$, let μ and $\bar{\mu}$ respectively be ϕ and $\bar{\phi}$-Patterson-Sullivan measures for $\rho(\Gamma_0)$ of dimension $\delta := \delta^\phi(\rho(\Gamma_0))$ and let m be the Bowen-Margulis measure on $\widehat{\mathsf{U}}(\Gamma_0)$ associated to ρ, μ and $\bar{\mu}$. If $Q^\phi_{\rho(\Gamma_0)}(\delta) = +\infty$, then the geodesic flow $\{\varphi_t\}$ on $\left(\widehat{\mathsf{U}}(\Gamma_0), m\right)$ is ergodic.*

Before starting the proof of Theorem 10.1, we recall a result of Coudène. Suppose $\{\varphi_t\}$ is a continuous flow on a metric space X which preserves a Borel measure m. The *strong stable manifold* of $x \in X$ is

$$W^{ss}(x) := \left\{ y \in X : \lim_{t\to\infty} \mathrm{d}(\varphi_t(x), \varphi_t(y)) = 0 \right\}$$

and the *strong unstable manifold* of $x \in X$ is

$$W^{su}(x) := \left\{ y \in X : \lim_{t\to-\infty} \mathrm{d}(\varphi_t(x), \varphi_t(y)) = 0 \right\}.$$

A measurable function $f : X \to \mathbb{R}$ is W^{ss}*-invariant* if there exists a full measure set $X' \subset X$ where $f(x) = f(y)$ whenever $x, y \in X'$ and $y \in W^{ss}(x)$. Similarly, a measurable function $f : X \to \mathbb{R}$ is W^{su}*-invariant* if there exists a full measure set $X' \subset X$ where $f(x) = f(y)$ whenever $x, y \in X'$ and $y \in W^{su}(x)$.

Theorem 10.2 (Coudène [17])**.** *Let X be a metric space, $\{\varphi_t\}$ a flow on X and m a $\{\varphi_t\}$-invariant Borel measure on X such that $(X, m, \{\varphi_t\})$ is conservative. Suppose that there is a full measure subset of X that is covered by a countable family of open sets with finite m-measure. Then every flow-invariant, m-measurable function on X is W^{ss}-invariant and W^{su}-invariant.*

Proof of Theorem 10.1. Notice that by definition, $\widehat{\mathsf{U}}(\Gamma_0)$ is is covered by a countable family of open sets of with finite m-measure. We first prove that $(\widehat{\mathsf{U}}(\Gamma_0), m, \{\varphi_t\})$ is conservative, i.e. for some (every) m-integrable, strictly positive, continuous function $\sigma : \widehat{\mathsf{U}}(\Gamma_0) \to \mathbb{R}$ and for m-almost-every $v \in \widehat{\mathsf{U}}(\Gamma_0)$, we have

$$\int_{\mathbb{R}} \sigma(\varphi_t(v))dt = +\infty,$$

see for instance [5, Fact 2.27]).

Recall that $p : \mathsf{U}(\Gamma_0) \to \widehat{\mathsf{U}}(\Gamma_0)$ is the quotient map. By Proposition 9.1 and Observation 6.1(1), the set

$$\mathcal{R} := \left\{ v \in \widehat{\mathsf{U}}(\Gamma_0) : v = p(v^-, v^+, t) \text{ and } v^+ \in \Lambda^{\mathrm{con}}_\Omega(\Gamma_0) \right\}$$

has full m-measure. Further, if $v \in \mathcal{R}$, then there is some $r > 0$ and a sequence $\{\gamma_n\}$ in Γ such that $\gamma_n(\pi(v)) \to v^+$ and $\mathrm{d}_\Omega\big(\gamma_n(\pi(v)), [\pi(v), v^+)_\Omega\big) < r$ for all n, where $\pi : T^1\Omega \to \Omega$ is the projection. Let

$$K := \pi|^{-1}_{\mathsf{U}(\Gamma_0)}\left(\overline{B_{2r}(\pi(v))}\right) \subset \mathsf{U}(\Gamma_0),$$

where $B_{2r}(\pi(v)) \subset \Omega$ is the ball of radius $2r$ centered at $\pi(v)$. Observe that K is compact, and has the property that for all n, the set $\{t \in \mathbb{R} : \varphi_t(v) \in \gamma_n(K)\}$ has Lebesgue measure at least

$2r$. Thus, $p(K)$ is compact and

$$\{t \in \mathbb{R} : \varphi_t(v) \in p(K)\}$$

has infinite Lebesgue measure, so $\int_{\mathbb{R}} \sigma(\varphi_t(v))dt = +\infty$. This proves that $(\widehat{\mathsf{U}}(\Gamma_0), m, \{\varphi_t\})$ is conservative.

Next, let $\tilde{m}$ be the lift of m to $\mathsf{U}(\Gamma_0)$, and endow $\mathsf{U}(\Gamma_0)$ and $\widehat{\mathsf{U}}(\Gamma_0)$ with the metrics $\mathrm{d}_{T^1\Omega}$ and $\mathrm{d}_{\Gamma\backslash T^1\Omega}$ defined in Section 5.3. Notice that Equation (7) implies that

$$p(W^{ss}(v)) \subset W^{ss}(p(v)) \quad \text{and} \quad p(W^{su}(v)) \subset W^{su}(p(v))$$

for all $v \in \mathsf{U}(\Gamma_0)$, so the lift of a W^{ss}-invariant (respectively W^{su}-invariant) function on $\widehat{\mathsf{U}}(\Gamma_0)$ is a W^{ss}-invariant (respectively W^{su}-invariant) function on $\mathsf{U}(\Gamma_0)$. So by Theorem 10.2, it suffices to show that if $f : \mathsf{U}(\Gamma_0) \to \mathbb{R}$ is a $\tilde{m}$-measurable, Γ-invariant, $\{\varphi_t\}$-invariant, W^{ss}-invariant and W^{su}-invariant function, then f is constant on a set of full $\tilde{m}$-measure.

Since f is W^{ss}-invariant and W^{su}-invariant, by definition there exists a full $\tilde{m}$-measure set $Y_0 \subset \mathsf{U}(\Gamma_0)$ such that $f(v) = f(w)$ whenever $v, w \in Y_0$ and $v \in W^{ss}(w) \cup W^{su}(w)$. Since f is $\{\varphi_t\}$-invariant, we can assume that Y_0 is also $\{\varphi_t\}$-invariant. Let ν and $\bar{\nu}$ be measures on $\partial\Omega$ given by

$$\nu(A) = \mu\left(\xi(A \cap \Lambda_\Omega(\Gamma_0))\right) \quad \text{and} \quad \bar{\nu}(A) = \bar{\mu}\left(\xi(A \cap \Lambda_\Omega(\Gamma_0))\right),$$

where ξ is the limit map of ρ. By the definition of m, we see that $Y_0 = Y_0' \times \mathbb{R}$ for some set $Y_0' \subset \Lambda_\Omega(\Gamma_0)^{(2)}$ of full $\bar{\nu} \otimes \nu$-measure. Set

$$X^+ := \{y \in \Lambda_\Omega(\Gamma_0) : (x,y) \in Y_0' \text{ for } \bar{\nu}\text{-almost every } x \in \Lambda_\Omega(\Gamma_0)\},$$

and note that $\nu(X^+) = 1$ by Fubini's theorem. Hence, if we fix $(v_0^-, v_0^+) \in (\Lambda_\Omega(\Gamma_0) \times X^+) \cap Y_0'$, then the set

$$Y' := \left\{(x,y) \in Y_0' : (x, v_0^+) \in Y_0'\right\}$$

has full $\bar{\nu} \otimes \nu$-measure, so $Y := Y' \times \mathbb{R} \subset \mathsf{U}(\Gamma_0)$ has full $\tilde{m}$-measure.

Let $(x,y,t) \in Y$. By Lemma 5.3, there is some $s \in \mathbb{R}$ such that $(x,y,t) \in W^{su}(x, v_0^+, s)$, and there is some $r \in \mathbb{R}$ such that $(x, v_0^+, s) \in W^{ss}(v_0^-, v_0^+, r)$. By definition, (x,y,t), (x, v_0^+, s), and (v_0^-, v_0^+, r) lie in Y_0, so

$$f(x,y,t) = f(x, v_0^+, s) = f(v_0^-, v_0^+, r) = f(v_0^-, v_0^+, 0).$$

This proves that f is constant on Y. □

11. Consequences of ergodicity

In this section we record some consequences of Theorem 10.1. The first finishes our Hopf-Sullivan-Tsuji dichotomy in the divergent case.

Corollary 11.1. *Suppose $\Gamma \subset \mathsf{G}$ is a non-elementary P_θ-transverse subgroup for some symmetric $\theta \subset \Delta$, $\phi \in \mathfrak{a}_\theta^*$ and $\delta := \delta^\phi(\Gamma) < +\infty$. Let μ and $\bar{\mu}$ respectively be ϕ and $\bar{\phi}$-Patterson-Sullivan measures for Γ of dimension δ. If $Q_\Gamma^\phi(\delta) = +\infty$, then the Γ actions on $(\Lambda_\theta(\Gamma)^{(2)}, \bar{\mu} \otimes \mu)$ and on $(\Lambda_\theta(\Gamma), \mu)$ are ergodic.*

Proof. Using the discussion in Section 2.4 we may assume that G has trivial center and that P_θ does not contain any simple factors of G. Then by Theorem 6.2, there is a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ and a P_θ-transverse representation $\rho : \Gamma_0 \to \mathsf{G}$ such that $\rho(\Gamma_0) = \Gamma$. Let $\xi : \Lambda_\Omega(\Gamma_0) \to \Lambda_\theta(\Gamma)$ be the ρ-equivariant boundary map and let m be the Bowen-Margulis measure on $\widehat{\mathsf{U}}(\Gamma_0)$ associated to ρ, μ and $\bar{\mu}$.

Theorem 10.1 implies that the geodesic flow on $(\widehat{\mathsf{U}}(\Gamma_0), m)$ is ergodic. Any Γ-invariant subset of either $(\Lambda_\theta(\Gamma)^{(2)}, \bar{\mu} \otimes \mu)$ or $(\Lambda_\theta(\Gamma), \mu)$ of positive, but not full measure, gives rise to a flow-invariant subset of $(\widehat{\mathsf{U}}(\Gamma_0), m)$ of positive but not full measures. Therefore, the actions of Γ on $(\Lambda_\theta(\Gamma)^{(2)}, \bar{\mu} \otimes \mu)$ and $(\Lambda_\theta(\Gamma), \mu)$ are both ergodic. $\square$

Next, using a standard argument (see for instance [46, pg. 181]), we deduce that in the divergent case, the uniqueness of Patterson-Sullivan measures whose dimension is the critical exponent.

Corollary 11.2. *Suppose $\Gamma \subset \mathsf{G}$ is a non-elementary P_θ-transverse subgroup for some symmetric $\theta \subset \Delta$, $\phi \in \mathfrak{a}_\theta^*$ and $\delta := \delta^\phi(\Gamma) < +\infty$. If $Q_\Gamma^\phi(\delta) = +\infty$, then there is a unique ϕ-Patterson-Sullivan measure μ_ϕ for Γ of dimension δ.*

For the next two results let $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ be a properly convex domain, let $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ be a projectively visible subgroup and let $b_0 \in \Omega$. For any $R > 0$, we denote by $\Lambda_{\Omega,b_0,R}(\Gamma_0)$ the set of points $x \in \Lambda_\Omega^{\mathrm{con}}(\Gamma_0)$ for which there exists a sequence $\{\gamma_n\}$ in Γ_0 such that $\gamma_n(b_0) \to x$ and

$$\mathrm{d}_\Omega(\gamma_n(b_0), [b_0, x)_\Omega) < R$$

for all n. The next corollary proves that if the image of a transverse representation is in the divergent case, then there is an $R > 0$ such that the set of R-conical limit points have full measure.

Corollary 11.3. *Suppose $\rho : \Gamma_0 \to \mathsf{G}$ is a P_θ-transverse representation for some symmetric $\theta \subset \Delta$, $\phi \in \mathfrak{a}_\theta^*$, $\delta := \delta^\phi(\rho(\Gamma_0)) < +\infty$ and μ is the ϕ-Patterson-Sullivan measure for $\rho(\Gamma_0)$ of dimension δ. If $Q_{\rho(\Gamma_0)}^\phi(\delta) = +\infty$, then for any $b_0 \in \Omega$, there exists $R > 0$ such that*

$$\mu\left(\xi(\Lambda_{\Omega,b_0,R}(\Gamma_0))\right) = 1.$$

Proof. The following argument is standard, see for instance [46, pg. 190]. Define a measure ν on $\partial\Omega$ by

$$\nu(A) = \mu\big(\xi(A \cap \Lambda_\Omega(\Gamma_0))\big).$$

Since $Q_{\rho(\Gamma_0)}^\phi(\delta) = +\infty$, by Proposition 9.1,

$$1 = \nu\left(\Lambda_\Omega^{\mathrm{con}}(\Gamma_0)\right) = \lim_{n\to\infty} \nu\left(\Lambda_{\Omega,b_0,n}(\Gamma_0)\right).$$

Hence there exists $R_0 > 0$ such that $\nu\left(\Lambda_{\Omega,b_0,R_0}(\Gamma_0)\right) > 0$.

Let L be the set of points $x \in \Lambda_\Omega(\Gamma_0)$ for which there exist $b \in \Gamma_0(b_0)$ and a sequence $\{\gamma_n\}$ in Γ_0 such that $\gamma_n(b) \to x$ and

$$\mathrm{d}_\Omega(\gamma_n(b), [b, x)_\Omega) \leq R_0$$

for all n. Observe that L is Γ_0-invariant, and $\nu(L) > 0$ because $\Lambda_{\Omega,b_0,R_0}(\Gamma_0) \subset L$. Hence by Corollary 11.1, $\nu(L) = 1$.

It now suffices to show that $L \subset \Lambda_{\Omega,b_0,R_0+1}(\Gamma_0)$. Fix $x \in L$. Then there exist $b \in \Gamma_0(b_0)$, a sequence $\{\gamma_n\}$ in Γ_0, and a sequence $\{b_n\}$ in $[b, x)_\Omega$ where $\gamma_n(b) \to x$ and

$$\mathrm{d}_\Omega(\gamma_n(b), b_n) \leq R_0$$

for all n. By Lemma 5.3, there exists a sequence $\{b_n'\}$ in $[b_0, x)_\Omega$ such that

$$\lim_{n\to\infty} \mathrm{d}_\Omega(b_n, b_n') = 0.$$

Since $b \in \Gamma_0(b_0)$, we can write $b = \gamma(b_0)$ for some $\gamma \in \Gamma_0$. Then

$$\mathrm{d}_\Omega(\gamma_n\gamma(b_0), [b_0, x)_\Omega) \leq R_0 + 1$$

for all n sufficiently large. So $x \in \Lambda_{\Omega,b_0,R_0+1}(\Gamma_0)$. $\square$

Finally, we prove the following rigidity result for length functions which have non-singular Bowen-Margulis-Sullivan measures.

Corollary 11.4. *For $j=1,2$, suppose $\rho_j:\Gamma_0\to\mathsf{G}_j$ is a P_{θ_j}-transverse representation for some symmetric $\theta_j\subset\Delta_j$, $\phi_j\in\mathfrak{a}_{\theta_j}^*$ and $\delta_j:=\delta^{\phi_j}(\rho_j(\Gamma_0))<+\infty$. For $\psi\in\{\phi_j,\bar\phi_j\}$, let μ_ψ be the ψ-Patterson-Sullivan measure for $\rho_j(\Gamma_0)$ of dimension δ_j and let m_j denote the Bowen-Margulis-Sullivan measure associated to ρ_j, μ_{ϕ_j} and $\mu_{\bar\phi_j}$. If $Q^{\phi_j}_{\rho_j(\Gamma_0)}(\delta_j)=+\infty$ and m_1 is non-singular with respect to m_2, then there exists $c>0$ so that $m_1=cm_2$ and*

$$\delta^{\phi_1}(\rho_1(\Gamma_0))\ell^{\phi_1}(\rho_1(\gamma))=\delta^{\phi_2}(\rho_2(\Gamma_0))\ell^{\phi_2}(\rho_2(\gamma)) \tag{12}$$

for all $\gamma\in\Gamma_0$. If, in addition, G_j is simple, $Z(\mathsf{G}_j)$ is trivial and ρ_j has Zariski-dense image for $j=1,2$, then there is an isomorphism $\Psi:\mathsf{G}_1\to\mathsf{G}_2$ such that $\rho_2=\Psi\circ\rho_1$.

The proof of Corollary 11.4 requires the following lemma.

Lemma 11.5. *Suppose $\Omega\subset\mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma\subset\mathrm{Aut}(\Omega)$ is a projectively visible subgroup. Let $\mathrm{d}_{\mathbb{P}}$ be a distance on $\mathbb{P}(\mathbb{R}^d)$ induced by a Riemannian metric. If $r>0$, $b_0\in\Omega$ and $\{\gamma_n\}$ is a sequence of distinct elements in Γ, then*

$$\lim_{n\to\infty}\mathrm{diam}\left(\mathcal{O}_r(b_0,\gamma_n(b_0))\right)=0,$$

where the diameter is computed using $\mathrm{d}_{\mathbb{P}}$.

Proof. Fix a subsequence $\{\gamma_{n_j}\}$ such that

$$\limsup_{n\to\infty}\mathrm{diam}\left(\mathcal{O}_r(b_0,\gamma_n(b_0))\right)=\lim_{j\to\infty}\mathrm{diam}\left(\mathcal{O}_r(b_0,\gamma_{n_j}(b_0))\right).$$

Passing to a further subsequence we can suppose that $\gamma_{n_j}(b_0)\to x\in\Lambda_\Omega(\Gamma)$ and $\gamma_{n_j}\to T\in\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. To show that $\mathrm{diam}\left(\mathcal{O}_r(b_0,\gamma_{n_j}(b_0))\right)$ converges to 0, it suffices to fix a sequence $\{y_j\}$ where $y_j\in\mathcal{O}_r(b_0,\gamma_{n_j}(b_0))$ for all $j\ge1$ and show that $y_j\to x$. By definition, for each $j\ge1$ there exists $y_j'\in[b_0,y_j)$ such that $\mathrm{d}_\Omega(y_j',\gamma_{n_j}(b_0))<r$. Then the sequence $\{\gamma_{n_j}^{-1}(y_j')\}$ is relatively compact in Ω. So by Proposition 5.2(3)

$$x=T\left(\lim_{j\to\infty}\gamma_{n_j}^{-1}(y_j')\right)=\lim_{j\to\infty}\gamma_{n_j}\gamma_{n_j}^{-1}(y_j')=\lim_{j\to\infty}y_j'.$$

Since $y_j'\in[b_0,y_j)$ for all $j\ge1$, this implies that $y_j\to x$. □

Proof of Corollary 11.4. By the ergodicity of the flow $\{\varphi_t\}$ (see Theorem 10.1) and the assumption that m_1 is non-singular with respect to m_2, there exists $c>0$ such that $m_1=cm_2$.

Note that for $j=1,2$ and $\gamma\in\Gamma_0$,

$$\ell^{\phi_j}(\rho_j(\gamma))=\lim_{n\to\infty}\frac{1}{n}\phi_j(\kappa_{\theta_j}(\rho_j(\gamma^n))).$$

Thus, to prove Equation (12), it suffices to prove that

$$\sup_{\gamma\in\Gamma_0}\left|\delta_1\phi_1(\kappa_{\theta_1}(\rho_1(\gamma)))-\delta_2\phi_2(\kappa_{\theta_2}(\rho_2(\gamma)))\right|<+\infty.$$

For all $\psi\in\left\{\phi_1,\phi_2,\bar\phi_1,\bar\phi_2\right\}$, let ν_ψ be the measure on $\partial\Omega$ given by

$$\nu_\psi(A)=\mu_\psi\big(\xi(A\cap\Lambda_\Omega(\Gamma_0))\big).$$

Fix $r > 0$ sufficiently large so that the Shadow lemma (Proposition 7.1) holds for the probability measures ν_{ϕ_1} and ν_{ϕ_2}. Then there is some $C > 0$ such that

$$\frac{1}{C}\frac{e^{\delta_2\phi_2(\kappa_{\theta_2}(\rho_2(\gamma)))}}{e^{\delta_1\phi_1(\kappa_{\theta_1}(\rho_1(\gamma)))}} \leq \frac{\nu_{\phi_1}(\mathcal{O}_r(b_0,\gamma(b_0)))}{\nu_{\phi_2}(\mathcal{O}_r(b_0,\gamma(b_0)))} \leq C\frac{e^{\delta_2\phi_2(\kappa_{\theta_2}(\rho_2(\gamma)))}}{e^{\delta_1\phi_1(\kappa_{\theta_1}(\rho_1(\gamma)))}} \tag{13}$$

for all $\gamma \in \Gamma_0$.

Fix a distance $\mathrm{d}_{\mathbb{P}}$ on $\mathbb{P}(\mathbb{R}^d)$ induced by a Riemannian metric, fix $x_1, x_2 \in \Lambda_\Omega(\Gamma_0)$ distinct and let $\epsilon := \frac{1}{6}\mathrm{d}_{\mathbb{P}}(x_1, x_2)$. Lemma 11.5 implies that there exists a finite set $S \subset \Gamma_0$ such that

$$\operatorname{diam}\left(\mathcal{O}_r(b_0,\gamma(b_0))\right) \leq \epsilon$$

for all $\gamma \in \Gamma_0 - S$. Hence, for each $\gamma \in \Gamma_0 - S$, there is some $i \in \{1,2\}$ so that

$$B_i \times \mathcal{O}_r(b_0,\gamma(b_0)) \subset \left\{(x,y) \in \Lambda_\Omega(\Gamma_0)^2 : \mathrm{d}_{\mathbb{P}}(x,y) \geq \epsilon\right\} =: K,$$

where $B_i := \{y \in \Lambda_\Omega(\Gamma_0) : \mathrm{d}_{\mathbb{P}}(y, x_j) \leq \epsilon\}$. From the definitions of $\mathrm{d}m_1$ and $\mathrm{d}m_2$, and the fact that $m_1 = cm_2$, we see that if we set

$$C_0 := c \max_{(x,y)\in K} \frac{e^{\phi_1([\xi_1(x),\xi_1(y)]_{\theta_1})}}{e^{\phi_2([\xi_2(x),\xi_2(y)]_{\theta_2})}},$$

then

$$\frac{1}{C_0}(\nu_{\bar\phi_2}\otimes\nu_{\phi_2})(A) \leq (\nu_{\bar\phi_1}\otimes\nu_{\phi_1})(A) \leq C_0(\nu_{\bar\phi_2}\otimes\nu_{\phi_2})(A)$$

for all Borel measurable sets $A \subset K$. Hence, if we set

$$C_1 := C_0 \max\left\{\frac{\nu_{\bar\phi_2}(B_1)}{\nu_{\bar\phi_1}(B_1)}, \frac{\nu_{\bar\phi_2}(B_2)}{\nu_{\bar\phi_1}(B_2)}\right\},$$

then

$$\frac{1}{C_1} \leq \frac{\nu_{\phi_1}(\mathcal{O}_r(b_0,\gamma(b_0)))}{\nu_{\phi_2}(\mathcal{O}_r(b_0,\gamma(b_0)))} \leq C_1 \tag{14}$$

for all $\gamma \in \Gamma_0 - S$.

Since S is finite, (13) and (14) imply that

$$\sup_{\gamma\in\Gamma_0} |\delta_1\phi_1(\kappa_{\theta_1}(\rho_1(\gamma))) - \delta_2\phi_2(\kappa_{\theta_2}(\rho_2(\gamma)))| < +\infty.$$

To prove the last claim of the corollary, we use the following argument of Dal'bo and Kim [20]. Consider the product representation $\rho_1 \times \rho_2 : \Gamma \to \mathsf{G}_1 \times \mathsf{G}_2$, let Δ denote the set of simple roots of $\mathsf{G}_1 \times \mathsf{G}_2$, and let $\mathfrak{a}$ denote the Cartan subspace of $\mathsf{G}_1 \times \mathsf{G}_2$. Equation (12) implies that the Δ-Benoist limit cone $\mathcal{B}(\rho_1 \times \rho_2)$ lies in a hyperplane in $\mathfrak{a}$. A theorem of Benoist [2] then implies that the Zariski closure Z of $(\rho_1 \times \rho_2)(\Gamma)$ is properly contained in $\mathsf{G}_1 \times \mathsf{G}_2$.

Let $\pi_j : \mathsf{G}_1 \times \mathsf{G}_2 \to \mathsf{G}_j$ be the projection map. Then the kernel $\pi_{3-j}|_Z$ is a normal subgroup of G_j, which is not all of G_j. Since G_j is simple and $Z(\mathsf{G}_j)$ is trivial, we conclude that $\pi_{3-j}|_Z$ is injective. Since ρ_j has Zariski dense image, $\pi_{3-j}|_Z$ is also surjective. Hence, $\Psi := \pi_2|_Z \circ \pi_1|_Z^{-1}$ is an isomorphism such that $\rho_2 = \Psi \circ \rho_1$. $\square$

12. A Manhattan curve theorem

Sambarino [45] showed that when Γ is Anosov, the entropy functional is concave and characterizes when it is not strictly concave (see also Potrie-Sambarino [40]). One may view this as an analogue of Burger's Manhattan Curve Theorem [11], since in this setting both are consequences of the convexity of the pressure function and rigidity results for equilibrium measures. However, in our setting we do not have access to thermodynamic formalism, so we must adapt other methods.

Theorem 12.1. *Suppose $\theta \subset \Delta$ is symmetric, Γ is a non-elementary P_θ-transverse subgroup of G and $\phi_1, \phi_2 \in \mathfrak{a}_\theta^*$ satisfy $\delta^{\phi_1}(\Gamma) = \delta^{\phi_2}(\Gamma) = 1$. If $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$ for some $\lambda \in (0,1)$, then*

$$\delta := \delta^{\phi}(\Gamma) \leq 1.$$

Moreover, if $Q_\Gamma^\phi(\delta) = +\infty$, then equality occurs if and only if $\ell^{\phi_1}(\gamma) = \ell^{\phi_2}(\gamma)$ for all $\gamma \in \Gamma$.

As a consequence of Theorem 12.1, we use a result of Benoist [2] to show that equality never occurs when Γ is Zariski dense.

Corollary 12.2. *Suppose $\theta \subset \Delta$ is symmetric, Γ is a Zariski dense P_θ-transverse subgroup of G, and $\phi_1, \phi_2 \in \mathfrak{a}_\theta^*$ are distinct and satisfy $\delta^{\phi_1}(\Gamma) = \delta^{\phi_2}(\Gamma) = 1$. If $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$ for some $\lambda \in (0,1)$ and $Q_\Gamma^\phi(\delta) = +\infty$, then $\delta^\phi(\Gamma) < 1$.*

Proof. For $g \in \mathsf{G}$ define

$$\nu(g) := \lim_{n\to\infty} \frac{1}{n}\kappa(g^n) \in \mathfrak{a}^+ \quad \text{and} \quad \nu_\theta(g) := \lim_{n\to\infty} \frac{1}{n}\kappa_\theta(g^n) \in \mathfrak{a}_\theta^+$$

(these limit exists by Fekete's Subadditive Lemma). Note that via the identification of $\mathfrak{a}_\theta^*$ as a subspace of $\mathfrak{a}^*$ described in Section 2, we have

$$\ell^{\phi_j}(\gamma) = \phi_j(\nu_\theta(\gamma)) = \phi_j(\nu(\gamma))$$

for both $j = 1, 2$ and all $\gamma \in \Gamma$.

Suppose for a contradiction that $\delta^\phi(\Gamma) = 1$. By Theorem 12.1, $\phi_1(\nu(\gamma)) = \phi_2(\nu(\gamma))$ for all $\gamma \in \Gamma$, which implies that $\phi_1 = \phi_2$ on

$$\mathcal{C} := \overline{\cup_{\gamma\in\Gamma} \mathbb{R}_{>0}\, \nu(\gamma)}$$

Since Γ is Zariski dense, a result of Benoist [2] implies that $\mathcal{C}$ is a convex subset of $\mathfrak{a}$ with non-empty interior, so $\phi_1 = \phi_2$, and we obtain a contradiction. $\square$

Proof of Theorem 12.1. The general strategy of our proof is inspired by the proof of Theorem 1(a) in [11].

The first part follows immediately from the definition and Hölder's inequality which gives that, for all s,

$$Q_\Gamma^\phi(s) \leq Q_\Gamma^{\phi_1}(s)^\lambda Q_\Gamma^{\phi_2}(s)^{1-\lambda}. \tag{15}$$

So our main work is to establish the "moreover" part of the theorem. Suppose that $\delta^\phi(\Gamma) = 1$ and $Q_\Gamma^\phi(1) = +\infty$. Then Equation (15) implies that $Q_\Gamma^{\phi_1}(1) = +\infty$ and $Q_\Gamma^{\phi_2}(1) = +\infty$. For $\psi \in \{\phi_1, \phi_2, \phi, \bar\phi_1, \bar\phi_2, \bar\phi\}$, let μ_ψ denote the unique ψ-Patterson-Sullivan measure for Γ of dimension 1.

Using the discussion in Section 2.4 we may assume that G has trivial center and that P_θ does not contain any simple factors of G. Then by Theorem 6.2, there is a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ and a faithful P_θ-transverse representation $\rho : \Gamma_0 \to \mathsf{G}$ with limit map $\xi : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$ so that $\rho(\Gamma_0) = \Gamma$ and $\xi(\Lambda_\Omega(\Gamma_0)) = \Lambda_\theta(\Gamma)$.

For $\psi \in \{\phi_1, \phi_2, \phi, \bar{\phi}_1, \bar{\phi}_2, \bar{\phi}\}$, define a measure ν_ψ on $\partial\Omega$ by

$$\nu_\psi(A) = \mu_\psi\left(\xi(A \cap \Lambda_\Omega(\Gamma_0))\right).$$

Fix $b_0 \in \Omega$. Recall, from Section 11, that $\Lambda_{\Omega,b_0,R}(\Gamma_0) \subset \Lambda_\Omega(\Gamma_0)$ denotes the set of limit points which are R-conical. By Corollary 11.3 we can fix $R > 0$ sufficiently large so that

$$\nu_\psi\left(\Lambda_{\Omega,b_0,R}(\Gamma_0)\right) = 1$$

for all $\psi \in \{\phi_1, \phi_2, \phi, \bar{\phi}_1, \bar{\phi}_2, \bar{\phi}\}$. Using the Shadow Lemma (Proposition 7.1) and possibly increasing R, we can also assume that for every $r \geq R$ there exists a constant $C_r \geq 1$ such that

$$C_r^{-1} e^{-\psi(\kappa_\theta(\rho(\gamma)))} \leq \nu_\psi\big(\mathcal{O}_r(b_0, \gamma(b_0))\big) \leq C_r e^{-\psi(\kappa_\theta(\rho(\gamma)))} \tag{16}$$

for all $\gamma \in \Gamma$ and $\psi \in \{\phi_1, \phi_2, \phi, \bar{\phi}_1, \bar{\phi}_2, \bar{\phi}\}$.

For all $\alpha, \beta \in \Gamma_0$ and $r > 0$, let

$$\mathcal{R}_r(\alpha, \beta) := \mathcal{O}_r(b_0, \alpha(b_0)) \times \mathcal{O}_r(b_0, \beta(b_0)).$$

The following lemma is the crucial place where we use the fact that $\delta^\phi(\Gamma) = \delta^{\phi_1}(\Gamma) = \delta^{\phi_2}(\Gamma)$.

Lemma 12.3. *If $r \geq R$ and $\alpha, \beta \in \Gamma_0$, then*

$$(\nu_{\bar{\phi}} \otimes \nu_\phi)\big(\mathcal{R}_r(\alpha, \beta)\big) \leq C_r^4 \left(\nu_{\bar{\phi}_1} \otimes \nu_{\phi_1} + \nu_{\bar{\phi}_2} \otimes \nu_{\phi_2}\right)\big(\mathcal{R}_r(\alpha, \beta)\big). \tag{17}$$

Proof. By repeated applications of the Shadow Lemma (16), we see that if $\alpha, \beta \in \Gamma_0$, then

$$\begin{aligned}
(\nu_{\bar{\phi}} \otimes \nu_\phi)\big(\mathcal{R}_r(\alpha, \beta)\big) &\leq C_r^2 e^{-\bar{\phi}(\kappa_\theta(\rho(\alpha)))} e^{-\phi(\kappa_\theta(\rho(\beta)))} \\
&= C_r^2 e^{-\left(\lambda\bar{\phi}_1(\kappa_\theta(\rho(\alpha))) + (1-\lambda)\bar{\phi}_2(\kappa_\theta(\rho(\alpha))) + \lambda\phi_1(\kappa_\theta(\rho(\beta))) + (1-\lambda)\phi_2(\kappa_\theta(\rho(\beta)))\right)} \\
&\leq C_r^4 \nu_{\bar{\phi}_1}\left(\mathcal{O}_r(b_0, \alpha(b_0))\right)^\lambda \nu_{\bar{\phi}_2}\left(\mathcal{O}_r(b_0, \alpha(b_0))\right)^{1-\lambda} \nu_{\phi_1}\left(\mathcal{O}_r(b_0, \beta(b_0))\right)^\lambda \nu_{\phi_2}\left(\mathcal{O}_r(b_0, \beta(b_0))\right)^{1-\lambda} \\
&= C_r^4 (\nu_{\bar{\phi}_1} \otimes \nu_{\phi_1})\big(\mathcal{R}_r(\alpha, \beta)\big)^\lambda (\nu_{\bar{\phi}_2} \otimes \nu_{\phi_2})\big(\mathcal{R}_r(\alpha, \beta)\big)^{1-\lambda}.
\end{aligned}$$

We may then apply the weighted Arithmetic Mean-Geometric Mean Inequality to see that

$$(\nu_{\bar{\phi}} \otimes \nu_\phi)\big(\mathcal{R}_r(\alpha, \beta)\big) \leq C_r^4 \left(\nu_{\bar{\phi}_1} \otimes \nu_{\phi_1} + \nu_{\bar{\phi}_2} \otimes \nu_{\phi_2}\right)\big(\mathcal{R}_r(\alpha, \beta)\big)$$

for all $\alpha, \beta \in \Gamma_0$. □

Our goal is to upgrade the inequality in Equation (17) to all Borel measurable sets in $\Lambda_\Omega(\Gamma_0)^2$. We first show that Shadows form a neighborhood basis of every point in $\Lambda_{\Omega,b_0,R}(\Gamma_0)$.

Lemma 12.4. *If $x \in \Lambda_{\Omega,b_0,R}(\Gamma_0)$ and U is a neighborhood of x in $\partial\Omega$, then there exists $\gamma \in \Gamma$ such that*

$$x \in \mathcal{O}_R(b_0, \gamma(b_0)) \subset U.$$

Proof. Fix a sequence $\{\gamma_n\}$ in Γ such that $\gamma_n(b_0) \to x$ and $\mathrm{dist}_\Omega(\gamma_n(b_0), [b_0, x)) < R$ for all $n \geq 1$. Then $x \in \mathcal{O}_R(b_0, \gamma_n(b_0))$ for all $n \geq 1$ and Lemma 11.5 implies that $\mathcal{O}_R(b_0, \gamma_n(b_0)) \subset U$ when n is sufficiently large. □

Next, by the argument in [42, pg. 23], we observe that the shadows satisfy a version of the Vitali covering lemma.

Lemma 12.5. *If $I \subset \Gamma_0$ and $r > 0$, then there exists $J \subset I$ such that the sets $\{\mathcal{O}_r(b_0, \gamma(b_0)) : \gamma \in J\}$ are pairwise disjoint and*

$$\bigcup_{\gamma \in I} \mathcal{O}_r(b_0, \gamma(b_0)) \subset \bigcup_{\gamma \in J} \mathcal{O}_{5r}(b_0, \gamma(b_0)).$$

We now leverage our covering lemma to upgrade Equation (17) to all measurable subsets of $\Lambda(\Gamma_0)^2$.

Lemma 12.6. *There exists $C > 0$ such that: if $A \subset \Lambda_\Omega(\Gamma_0)^2$ is a Borel measurable set, then*

$$(\nu_{\bar\phi} \otimes \nu_\phi)(A) \le C \left(\nu_{\bar\phi_1} \otimes \nu_{\phi_1} + \nu_{\bar\phi_2} \otimes \nu_{\phi_2}\right)(A).$$

Proof. It suffices to prove the lemma in the case when $A = A_1 \times A_2$ for some $A_1, A_2 \subset \Lambda_\Omega(\Gamma_0)$. Fix $\epsilon > 0$. By the outer regularity of the measures, for both $j = 1, 2$, there exists an open set $U_j \supset A_j$ with

$$\left(\nu_{\bar\phi_1} \otimes \nu_{\phi_1} + \nu_{\bar\phi_2} \otimes \nu_{\phi_2}\right)\left(U_1 \times U_2\right) \le \left(\nu_{\bar\phi_1} \otimes \nu_{\phi_1} + \nu_{\bar\phi_2} \otimes \nu_{\phi_2}\right)\left(A_1 \times A_2\right) + \epsilon.$$

If we let $I_j := \{\alpha \in \Gamma_0 : \mathcal{O}_R(b_0, \alpha(b_0)) \subset U_j\}$, then by Lemma 12.4

$$(A_1 \times A_2) \cap \Lambda_{\Omega, b_0, R}(\Gamma_0)^2 \subset \bigcup_{(\alpha,\beta) \in I_1 \times I_2} \mathcal{R}_R(\alpha, \beta) \subset U_1 \times U_2.$$

By Lemma 12.5, we can find a subset $J_j \subset I_j$ such that the sets $\{\mathcal{O}_R(b_0, \alpha(b_0)) : \alpha \in I_j\}$ are pairwise disjoint and

$$\bigcup_{\alpha \in I_j} \mathcal{O}_R(b_0, \alpha(b_0)) \subset \bigcup_{\alpha \in J_j} \mathcal{O}_{5R}(b_0, \alpha(b_0)).$$

Since we chose $R > 0$ such that $\nu_{\bar\phi}\big(\Lambda_{\Omega, b_0, R}(\Gamma_0)\big) = 1$, it follows that

$$\begin{aligned}(\nu_{\bar\phi} \otimes \nu_\phi)(A_1 \times A_2) &= (\nu_{\bar\phi} \otimes \nu_\phi)\big((A_1 \times A_2) \cap \Lambda_{\Omega, b_0, R}(\Gamma_0)^2\big) \\ &\le \sum_{(\alpha,\beta) \in J_1 \times J_2} (\nu_{\bar\phi} \otimes \nu_\phi)\left(\mathcal{R}_{5R}(\alpha, \beta)\right).\end{aligned}$$

Then by repeated applications of Equations (16) and (17),

$$\begin{aligned}\sum_{(\alpha,\beta) \in J_1 \times J_2} (\nu_{\bar\phi} \otimes \nu_\phi)\big(\mathcal{R}_{5R}(\alpha, \beta)\big) &\le C_R^4 C_{5R}^4 \sum_{(\alpha,\beta) \in J_1 \times J_2} (\nu_{\bar\phi} \otimes \nu_\phi)\big(\mathcal{R}_R(\alpha, \beta)\big) \\ &\le C_R^8 C_{5R}^4 \sum_{(\alpha,\beta) \in J_1 \times J_2} \left(\nu_{\bar\phi_1} \otimes \nu_{\phi_1} + \nu_{\bar\phi_2} \otimes \nu_{\phi_2}\right)\big(\mathcal{R}_R(\alpha, \beta)\big) \\ &\le C_R^8 C_{5R}^4 \left(\nu_{\bar\phi_1} \otimes \nu_{\phi_1} + \nu_{\bar\phi_2} \otimes \nu_{\phi_2}\right)\left(U_1 \times U_2\right) \\ &\le C_R^8 C_{5R}^4 \left(\nu_{\bar\phi_1} \otimes \nu_{\phi_1} + \nu_{\bar\phi_2} \otimes \nu_{\phi_2}\right)\left(A_1 \times A_2\right) + C_R^6 C_{5R}^2 \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$\left(\nu_{\bar\phi} \otimes \nu_\phi\right)(A_1 \times A_2) \le C_R^6 C_{5R}^2 \left(\nu_{\bar\phi_1} \otimes \nu_{\phi_1} + \nu_{\bar\phi_2} \otimes \nu_{\phi_2}\right)(A_1 \times A_2). \qquad \square$$

Lemma 12.6 implies that $\nu_{\bar\phi} \otimes \nu_\phi$ is absolutely continuous with respect to $\nu_{\bar\phi_1} \otimes \nu_{\phi_1} + \nu_{\bar\phi_2} \otimes \nu_{\phi_2}$. So by relabelling we can suppose that $\nu_{\bar\phi} \otimes \nu_\phi$ is non-singular with respect to $\nu_{\bar\phi_1} \otimes \nu_{\phi_1}$. Therefore, m_ϕ is non-singular with respect to m_{ϕ_1}. Then by the divergence assumption and Corollary 11.4 we have $\ell^\phi(\gamma) = \ell^{\phi_1}(\gamma)$ for all $\gamma \in \Gamma$. Hence $\ell^{\phi_1}(\gamma) = \ell^{\phi_2}(\gamma)$ for all $\gamma \in \Gamma$. $\square$

Notice that the Hölder inequality similarly proves a statement which is of the same form as Burger's Manhattan Curve Theorem. However, we are not able to give an analogous characterization of when equality occurs.

Theorem 12.7. *Suppose $\theta \subset \Delta$ is symmetric, $\Gamma_1, \Gamma_2 \subset \mathsf{G}$ are P_θ-transverse subgroups and there exists an isomorphism $\rho : \Gamma_1 \to \Gamma_2$. If $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma_1) = \delta^\phi(\Gamma_2) = 1$, then for any $\lambda \in (0,1)$, then the weighted Poincaré series.*

$$\sum_{\gamma \in \Gamma_1} e^{-s\left(\lambda\phi(\kappa_\theta(\gamma)) + (1-\lambda)\phi(\kappa_\theta(\rho(\gamma)))\right)}$$

has critical exponent $\delta \leq 1$.

Appendix A. Proof of Proposition 2.3

In this section we prove Proposition 2.3 which we restate here.

Proposition A.1. *Suppose $F^\pm \in \mathcal{F}_\theta^\pm$, $\{g_n\}$ is a sequence in G and $g_n = m_n e^{\kappa(g_n)} \ell_n$ is a KAK-decomposition for each $n \geq 1$. The following are equivalent:*

1. *$m_n \mathsf{P}_\theta \to F^+$, $\ell_n^{-1} \mathsf{P}_\theta^- \to F^-$ and $\alpha(\kappa(g_n)) \to +\infty$ for every $\alpha \in \theta$,*
2. *$g_n(F) \to F^+$ for all $F \in \mathcal{F}_\theta^+ \setminus \mathcal{Z}_{F^-}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta^+ \setminus \mathcal{Z}_{F^-}$.*
3. *$g_n^{-1}(F) \to F^-$ for all $F \in \mathcal{F}_\theta^- \setminus \mathcal{Z}_{F^+}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta^- \setminus \mathcal{Z}_{F^+}$.*
4. *There are open sets $\mathcal{U}^\pm \subset \mathcal{F}_\theta^\pm$ such that $g_n(F) \to F^+$ for all $F \in \mathcal{U}^+$ and $g_n^{-1}(F) \to F^-$ for all $F \in \mathcal{U}^-$.*

It is well-known that

$$\exp : \mathfrak{u}_\theta^- \to \mathsf{U}_\theta^- := \exp(\mathfrak{u}_\theta^-)$$

is a diffeomorphism. Furthermore, the *Langlands decomposition* (see for instance [48, Thm. 1.2.4.8]) of parabolic subgroups states that the map

$$(u, \ell) \in \mathsf{U}_\theta^- \times L_\theta \mapsto u\ell \in \mathsf{P}_\theta^-$$

is a diffeomorphism, where $\mathsf{L}_\theta := \mathsf{P}_\theta \cap \mathsf{P}_\theta^-$. It follows that U_θ^- acts simply transitively on $\mathcal{F}_\theta \setminus \mathcal{Z}_{\mathsf{P}_\theta^-}$. Thus, the map

$$T : \mathfrak{u}_\theta^- \to \mathcal{F}_\theta \setminus \mathcal{Z}_{\mathsf{P}_\theta^-}$$

given by $T(X) = e^X \mathsf{P}_\theta$ is a diffeomorphism.

Note that if $H \in \mathfrak{a}$ and $X \in \mathfrak{u}_\theta^-$, then

$$e^H T(X) = e^H e^X \mathsf{P}_\theta = e^H e^X e^{-H} \mathsf{P}_\theta = e^{\mathrm{Ad}(e^H)(X)} \mathsf{P}_\theta = T\left(\mathrm{Ad}(e^H)(X)\right). \tag{18}$$

Furthermore, if we decompose

$$X = \sum_{\alpha \in \Sigma_\theta^+} X_{-\alpha} \in \mathfrak{u}_\theta^-,$$

where $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ for all $\alpha \in \Sigma_\theta^+$, then

$$\mathrm{Ad}(e^H)(X) = \sum_{\alpha \in \Sigma_\theta^+} \mathrm{Ad}(e^H)(X_{-\alpha}) = \sum_{\alpha \in \Sigma_\theta^+} e^{-\alpha(H)} X_{-\alpha}. \tag{19}$$

Together, Equations (18) and (19) imply the following observation.

Lemma A.2. *Let $\{H_n\}$ be a sequence in $\mathfrak{a}^+$. If $\alpha(H_n) \to +\infty$ for all $\alpha \in \theta$, then $e^{H_n} F \to \mathsf{P}_\theta$ for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{\mathsf{P}_\theta^-}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \setminus \mathcal{Z}_{\mathsf{P}_\theta^-}$.*

Using Equations (18) and (19), we can also prove the following lemma.

Lemma A.3. *Let $g_n = m_n e^{\kappa(g_n)} \ell_n$ be as in the statement of Proposition A.1.*

(1) If there is an open set $\mathcal{U} \subset \mathcal{F}_\theta^+$ such that $g_n(F) \to F^+$ for all $F \in \mathcal{U}$, then $m_n \mathsf{P}_\theta \to F^+$ and $\alpha(\kappa(g_n)) \to +\infty$ for every $\alpha \in \theta$.

(2) If there is an open set $\mathcal{U} \subset \mathcal{F}_\theta^-$ such that $g_n^{-1}(F) \to F^-$ for all $F \in \mathcal{U}$, then $\ell_n^{-1} \mathsf{P}_\theta^- \to F^-$ and $\alpha(\kappa(g_n)) \to +\infty$ for every $\alpha \in \theta$.

Proof. By compactness, it suffices to consider the case where $m_n \to m \in \mathsf{K}$ and $\ell_n \to \ell \in \mathsf{K}$.

(1): We first prove that $\alpha(\kappa(g_n)) \to +\infty$ for all $\alpha \in \theta$. If this is not the case, then by taking a subsequence, we may assume that there is some $\alpha_0 \in \theta$ such that $\alpha_0(\kappa(g_n)) \to c \in [0, \infty)$. Choose $F, F' \in \mathcal{U}$ such that $\ell(F), \ell(F') \in \mathcal{F}_\theta \setminus \mathcal{Z}_{\mathsf{P}_\theta^-}$, and if we decompose

$$T^{-1}(\ell(F)) = \sum_{\alpha \in \Sigma_\theta^+} X_{-\alpha} \quad \text{and} \quad T^{-1}(\ell(F')) = \sum_{\alpha \in \Sigma_\theta^+} X'_{-\alpha},$$

where $X_{-\alpha}, X'_{-\alpha} \in \mathfrak{g}_{-\alpha}$ for all $\alpha \in \Sigma_\theta^+$, then $X_{-\alpha_0} \neq X'_{-\alpha_0}$. Then by (18) and (19),

$$\lim_{n\to\infty} T^{-1}(e^{\kappa(g_n)}\ell_n(F)) = \lim_{n\to\infty} \mathrm{Ad}(e^{\kappa(g_n)}) T^{-1}(\ell_n(F)) = e^{-c} X_{-\alpha_0} + \lim_{n\to\infty} \sum_{\alpha \in \Sigma_\theta^+ - \{\alpha_0\}} e^{-\alpha(\kappa(g_n))} X_{-\alpha}.$$

Similarly,

$$\lim_{n\to\infty} T^{-1}(e^{\kappa(g_n)}\ell_n(F')) = e^{-c} X'_{-\alpha_0} + \lim_{n\to\infty} \sum_{\alpha \in \Sigma_\theta^+ - \{\alpha_0\}} e^{-\alpha(\kappa(g_n))} X'_{-\alpha},$$

so $\lim_{n\to\infty} m e^{\kappa(g_n)} \ell_n(F) \neq \lim_{n\to\infty} m e^{\kappa(g_n)} \ell_n(F')$, which implies that

$$\lim_{n\to\infty} g_n(F) \neq \lim_{n\to\infty} g_n(F').$$

This is a contradiction because $F, F' \in \mathcal{U}$.

Next, we prove that $m_n \mathsf{P}_\theta \to F^+$, or equivalently, $m\mathsf{P}_\theta = F^+$. Let $F \in \mathcal{F}_\theta$ such that F is transverse to F^- and $\ell(F)$ is transverse to P_θ^-. Then there is some compact subset $K \subset \mathcal{F}_\theta \setminus \mathcal{Z}_{\mathsf{P}_\theta^-}$ such that $\ell_n(F) \in K$ for all sufficiently large n. Since $\alpha(\kappa(g_n)) \to +\infty$ for all $\alpha \in \theta$, Lemma A.2 implies that

$$e^{\kappa(g_n)} \ell_n(F) \to \mathsf{P}_\theta,$$

which implies that

$$g_n(F) = m_n e^{\kappa(g_n)} \ell_n(F) \to m\mathsf{P}_\theta.$$

It follows that $m\mathsf{P}_\theta = F^+$.

(2): As in Section 2, let $k_0 \in \mathsf{N}_\mathsf{K}(\mathfrak{a})$ be a representative of the longest element $w_0 \in W$. Observation 2.2 implies that $\mathrm{Ad}(k_0)(-\kappa(g)) = \kappa(g^{-1})$ for all $g \in \mathsf{G}$, and so

$$g_n^{-1} = \left(\ell_n^{-1} k_0^{-1}\right) e^{\kappa(g_n^{-1})} \left(k_0 m_n^{-1}\right)$$

is a KAK-decomposition of g_n^{-1}.

Further, $\mathsf{P}_{\iota^*(\theta)} = k_0 \mathsf{P}_\theta^- k_0^{-1}$, see Equation (2), so we can define a G-equivariant diffeomorphism

$$\Phi_\theta : \mathcal{F}_\theta^- \to \mathcal{F}_{\iota^*(\theta)}$$

by $\Phi_\theta(g \mathsf{P}_\theta^-) = g k_0 \mathsf{P}_{\iota^*(\theta)}$. Then $g_n^{-1}(F) \to \Phi_\theta(F^-)$ for all $F \in \Phi_\theta(\mathcal{U})$. So by part (1), we see that $\ell_n^{-1} k_0^{-1} \mathsf{P}_{\iota^*(\theta)} \to \Phi_\theta(F^-)$ and $\alpha(\kappa(g_n^{-1})) \to +\infty$ for all $\alpha \in \iota^*(\theta)$. Since $\Phi_\theta(\ell_n^{-1} \mathsf{P}_\theta^-) = \ell_n^{-1} k_0^{-1} \mathsf{P}_{\iota^*(\theta)}$ this implies that $\ell_n^{-1} \mathsf{P}_\theta^- \to F^-$. Further, by Observation 2.2,

$$\alpha(\kappa(g)) = \iota^*(\alpha)(\kappa(g^{-1}))$$

for all $g \in \mathsf{G}$ and all $\alpha \in \theta$. So we see that $\alpha(\kappa(g_n)) \to +\infty$ for all $\alpha \in \theta$. $\square$

Proof of Proposition A.1. It follows immediately from Lemma A.3 that (4) implies (1), and it is obvious that (2) and (3) together imply (4). It thus suffices to show that (2) and (3) are both individually equivalent to (1). By compactness, it suffices to consider the case where $m_n \to m \in \mathsf{K}$ and $\ell_n \to \ell \in \mathsf{K}$.

We first prove that (1) implies (2). Since $\alpha(\kappa(g_n)) \to +\infty$ for all $\alpha \in \theta$, Lemma A.2 implies

$$\lim_{n\to\infty} e^{\kappa(g_n)} F = \mathsf{P}_\theta$$

for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{\mathsf{P}_\theta^-}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \setminus \mathcal{Z}_{\mathsf{P}_\theta^-}$. Since, $m\,\mathsf{P}_\theta = F^+$ and $\ell^{-1}\,\mathsf{P}_\theta^- = F^-$, it follows that

$$\lim_{n\to\infty} g_n(F) = F^+$$

for all $F \in \mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$.

Next, we prove (2) implies (1). By Lemma A.3, $m_n\,\mathsf{P}_\theta \to F^+$ and $\alpha(\kappa(g_n)) \to +\infty$ for every $\alpha \in \theta$, so it suffices to show that $\ell_n^{-1}\,\mathsf{P}_\theta^- \to F^-$, or equivalently, that $\ell F^- = \mathsf{P}_\theta^-$. If this were not the case, then there exists some $F \in \mathcal{Z}_{\mathsf{P}_\theta^-} \setminus \mathcal{Z}_{\ell F^-}$. Then there is a compact set $K \subset \mathcal{F}_\theta \setminus \mathcal{Z}_{F^-}$ such that $\ell_n^{-1}(F) \in K$ for all sufficiently large n. Then by assumption,

$$m \lim_{n\to\infty} e^{\kappa(g_n)} F = \lim_{n\to\infty} g_n \ell_n^{-1} F = F^+ = m\,\mathsf{P}_\theta,$$

so $e^{\kappa(g_n)} F \to \mathsf{P}_\theta$. However, $\{e^{\kappa(g_n)}\} \subset \mathsf{P}_\theta^-$, so each $e^{\kappa(g_n)}$ preserves the closed set $\mathcal{Z}_{\mathsf{P}_\theta^-}$, which implies that

$$\mathsf{P}_\theta = \lim_{n\to\infty} e^{\kappa(g_n)} F \in \mathcal{Z}_{\mathsf{P}_\theta^-}.$$

Since P_θ and P_θ^- are transverse, we have a contradiction.

Finally, we prove that (1) and (3) are equivalent. Let $k_0 \in N_{\mathsf{K}}(\mathfrak{a})$ be a representative of the longest element $w_0 \in W$, and let

$$\Phi_\theta : \mathcal{F}_\theta^- \to \mathcal{F}_{\iota^*(\theta)}$$

be the G-equivariant homeomorphism given by $\Phi_\theta(g\,\mathsf{P}_\theta^-) = g k_0\,\mathsf{P}_{\iota^*(\theta)}$. Observe that

$$\Phi_\theta(\mathcal{F}_\theta^- \setminus \mathcal{Z}_{F^+}) = \mathcal{F}_{\iota^*(\theta)} \setminus \mathcal{Z}_{\Phi^{-1}_{\iota^*(\theta)}(F^+)},$$

so (3) can be rewritten as:

(3') $g_n^{-1}(F) \to \Phi_\theta(F^-)$ for all $F \in \mathcal{F}_{\iota^*(\theta)} \setminus \mathcal{Z}_{\Phi^{-1}_{\iota^*(\theta)}(F^+)}$, and this convergence is uniform on compact subsets of $\mathcal{F}_{\iota^*(\theta)} \setminus \mathcal{Z}_{\Phi^{-1}_{\iota^*(\theta)}(F^+)}$.

By Observation 2.2, $\alpha(\kappa(g_n)) = \iota^*(\alpha)(\kappa(g_n^{-1}))$ for all $n \in \mathbb{N}$ and all $\alpha \in \Delta$. Thus, (1) can be rewritten as:

(1') $m_n k_0^{-1}\,\mathsf{P}^-_{\iota^*(\theta)} \to \Phi^{-1}_{\iota^*(\theta)}(F^+)$, $\ell_n^{-1} k_0^{-1}\,\mathsf{P}_{\iota^*(\theta)} \to \Phi_\theta(F^-)$ and $\alpha(\kappa(g_n^{-1})) \to +\infty$ for every $\alpha \in \iota^*(\theta)$.

We also saw in the proof of Lemma A.3(2) that if $g_n = m_n e^{\kappa(g_n)} \ell_n$ is a KAK-decomposition of $g \in \mathsf{G}$, then

$$g_n^{-1} = (\ell_n^{-1} k_0^{-1}) e^{\kappa(g_n^{-1})} (k_0 m_n^{-1})$$

is a KAK-decomposition of g_n^{-1}. Thus, the equivalence between (1) and (2) implies the equivalence between (1') and (3'). □

Appendix B. Proofs of Theorem 6.2 and Proposition 6.3

In this appendix we prove Theorem 6.2 and Proposition 6.3.

When $\mathsf{G} = \mathsf{PSL}(d, \mathbb{K})$, where $\mathbb{K}$ is either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, recall from the introduction that $\Delta := \{\alpha_1, \ldots, \alpha_{d-1}\} \subset \mathfrak{a}^*$ denotes the standard system of simple restricted roots, i.e.

$$\alpha_j(\mathrm{diag}(a_1, \ldots, a_d)) = a_j - a_{j+1}$$

for all $\mathrm{diag}(a_1, \ldots, a_d) \in \mathfrak{a}$. To simplify notation, we replace subscripts of the form $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ with $i_1, \ldots, i_k$. For instance,

$$\mathcal{F}_{1,d-1} = \mathcal{F}_{\{\alpha_1, \alpha_{d-1}\}} \quad \text{and} \quad U_{1,d-1}(g) = U_{\{\alpha_1, \alpha_{d-1}\}}(g).$$

As mentioned before, in the case when $\mathsf{G} = \mathsf{PSL}(d, \mathbb{K})$, Theorem 6.2 and Proposition 6.3 were proven in [15]. We will use results from [23] to prove the following proposition, which allows us to generalize these results in [15] to general G.

Proposition B.1. *For any symmetric $\theta \subset \Delta$ and $\chi \in \sum_{\alpha \in \theta} \mathbb{N} \cdot \omega_\alpha$ there exist $d \in \mathbb{N}$, an irreducible linear representation $\Phi : \mathsf{G} \to \mathsf{SL}(d, \mathbb{R})$ and a Φ-equivariant smooth embedding*

$$\xi : \mathcal{F}_\theta \to \mathcal{F}_{1,d-1}(\mathbb{R}^d)$$

such that:

(1) *$F_1, F_2 \in \mathcal{F}_\theta$ are transverse if and only if $\xi(F_1)$ and $\xi(F_2)$ are transverse.*
(2) *There exists $N \in \mathbb{N}$ such that*
$$\log \sigma_1(\kappa(\Phi(g))) = N\chi(\kappa(g))$$
for all $g \in \mathsf{G}$.
(3) *$\alpha_1(\kappa(\Phi(g))) = \min_{\alpha \in \theta} \alpha(\kappa(g))$ for all $g \in \mathsf{G}$.*
(4) *If $\min_{\alpha \in \theta} \alpha(\kappa(g)) > 0$, then*
$$\xi(U_\theta(g)) = U_{1,d-1}(\Phi(g)).$$
(5) *$\Gamma \subset \mathsf{G}$ is P_θ-divergent (respectively P_θ-transverse) if and only if $\Phi(\Gamma)$ is $\mathsf{P}_{1,d-1}$-divergent (respectively $\mathsf{P}_{1,d-1}$-transverse). Moreover, in this case*
$$\xi(\Lambda_\theta(\Gamma)) = \Lambda_{1,d-1}(\Phi(\Gamma)).$$
(6) *If $\rho : \Gamma_0 \to \mathsf{G}$ is a P_θ-transverse representation with boundary map $\xi_\rho : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_\theta$, then $\Phi \circ \rho$ is a $\mathsf{P}_{1,d-1}$-transverse representation with boundary map $\xi \circ \xi_\rho$.*

Delaying the proof of Proposition B.1 for a moment, we prove Theorem 6.2 and Proposition 6.3.

B.1. **Proof of Theorem 6.2.** Let $\Phi : \mathsf{G} \to \mathsf{PSL}(d, \mathbb{R})$ and $\xi_\Phi : \mathcal{F}_\theta \to \mathcal{F}_{1,d-1}(\mathbb{R}^d)$ satisfy Proposition B.1 for some $\chi \in \sum_{\alpha \in \theta} \mathbb{N} \cdot \omega_\alpha$.

Then $\Phi(\Gamma)$ is $\mathsf{P}_{1,d-1}$-transverse and so by [15, Thm. 4.2] there exist $d_0 \in \mathbb{N}$, a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$, a projectively visible subgroup $\Gamma_0 \subset \mathrm{Aut}(\Omega)$ and a faithful $\mathsf{P}_{1,d-1}$-transverse representation $\rho_0 : \Gamma_0 \to \mathsf{PSL}(d, \mathbb{R})$ with limit map $\xi_0 : \Lambda_\Omega(\Gamma_0) \to \mathcal{F}_{1,d-1}(\mathbb{R}^d)$ so that $\rho_0(\Gamma_0) = \Phi(\Gamma)$ and

$$\xi_0(\Lambda_\Omega(\Gamma_0)) = \Lambda_{1,d-1}(\Phi(\Gamma)) = \xi_\Phi(\Lambda_\theta(\Gamma)).$$

We claim that Φ is injective. Since G is semisimple, $\ker \Phi$ is either discrete or contains a simple factor of G. Since $\xi : \mathcal{F}_\theta \to \mathcal{F}_{1,d-1}(\mathbb{R}^d)$ is a Φ-equivariant embedding, $\ker \Phi$ must act trivially on $\mathcal{F}_\theta$. So $\ker \Phi \subset \mathsf{P}_\theta$. By assumption P_θ contains no simple factors of G, so $\ker \Phi$ is discrete. However then, since $\ker \Phi$ is also normal, we see that $\ker \Phi$ is contained in the center of G which by assumption is trivial. Hence Φ is injective.

Then $\rho := \Phi^{-1} \circ \rho_0$ and $\xi := \xi_\Phi^{-1} \circ \xi_0$ are well defined and have the desired properties.

B.2. **Proof of Proposition 6.3.** We start by recalling a result in [15] about transverse representations into $\mathsf{PSL}(d, \mathbb{K})$. Let $\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}$ be a distance on $\mathbb{P}(\mathbb{R}^{d_0})$ induced by a Riemannian metric. Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$ and $b_0 \in \Omega$ let

$$\iota_{b_0} : \Omega \setminus \{b_0\} \to \partial\Omega$$

denote the *radial projection* map obtained by letting $\iota_{b_0}(z) \in \partial\Omega$ be the unique point so that $z \in (b_0, \iota_{b_0}(z))_\Omega$. The following lemma was proven as Lemma 6.2 and Observation 6.3 in [15].

Lemma B.2. *Suppose $\theta \subset \{\alpha_1, \dots, \alpha_{d-1}\}$ is symmetric. Let $\rho : \Gamma_0 \to \mathsf{PSL}(d, \mathbb{K})$ be a P_θ-transverse representation, where Γ_0 is a projectively visible subgroup of $\mathrm{Aut}(\Omega)$ for some properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$. For any $b_0 \in \Omega$ and $\epsilon > 0$, there exist $C > 0$ such that if $\gamma, \eta \in \Gamma_0$ and*

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\left(\iota_{b_0}(\gamma^{-1}(b_0)), \iota_{b_0}(\eta(b_0))\right) \geq \epsilon,$$

then

$$\left|\omega_{\alpha_k}\Big(\kappa(\rho(\gamma\eta)) - \kappa(\rho(\gamma)) - \kappa(\rho(\eta))\Big)\right| \leq C$$

for all $\alpha_k \in \theta$.

Lemma B.2 can be restated as follows.

Lemma B.3. *Suppose $\theta \subset \{\alpha_1, \dots, \alpha_{d-1}\}$ is symmetric. Let $\rho : \Gamma_0 \to \mathsf{PSL}(d, \mathbb{K})$ be a P_θ-transverse representation where Γ_0 is a projectively visible subgroup of $\mathrm{Aut}(\Omega)$ for some properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d_0})$. For any $b_0 \in \Omega$ and $r > 0$, there exist $C > 0$ such that if $\gamma, \eta \in \Gamma_0$ and*

$$\mathrm{d}_\Omega\left(\gamma(b_0), [b_0, \eta(b_0)]_\Omega\right) \leq r,$$

then

$$\left|\omega_{\alpha_k}\Big(\kappa(\rho(\eta)) - \kappa(\rho(\gamma)) - \kappa(\rho(\gamma^{-1}\eta))\Big)\right| \leq C$$

for all $\alpha_k \in \theta$.

Proof. Suppose not. Then there exist $\alpha_k \in \theta$ and sequences $\{\gamma_n\}$, $\{\eta_n\}$ in Γ such that

$$\mathrm{d}_\Omega\left(\gamma_n(b_0), [b_0, \eta_n(b_0)]_\Omega\right) \leq r \quad \text{but} \quad \left|\omega_{\alpha_k}\Big(\kappa_\theta(\rho(\eta_n)) - \kappa_\theta(\rho(\gamma_n)) - \kappa_\theta(\rho(\gamma_n^{-1}\eta_n))\Big)\right| \geq n.$$

Since

$$\left\|\kappa(\rho(\eta_n)) - \kappa(\rho(\gamma_n^{-1}\eta_n))\right\| \leq \sqrt{d}\max\left\{\log\sigma_1(\rho(\gamma_n^{-1})), \log\sigma_1(\rho(\gamma_n))\right\},$$

$\{\gamma_n\}$ is a diverging sequence. A similar argument also shows that $\{\gamma_n^{-1}\eta_n\}$ is diverging.

Since both $\{\gamma_n\}$ and $\{\gamma_n^{-1}\eta_n\}$ are diverging and

$$\mathrm{d}_\Omega\left(b_0, [\gamma_n^{-1}(b_0), \gamma_n^{-1}\eta_n(b_0)]_\Omega\right) = \mathrm{d}_\Omega\left(\gamma_n(b_0), [b_0, \eta_n(b_0)]_\Omega\right) \leq r,$$

it follows that there is some $\epsilon > 0$ so that

$$\mathrm{d}_{\mathbb{P}(\mathbb{R}^{d_0})}\left(\iota_{b_0}(\gamma_n^{-1}(b_0)), \iota_{b_0}(\gamma_n^{-1}\eta_n(b_0))\right) \geq \epsilon$$

for all n. Thus, Lemma B.2 implies that $\left|\omega_{\alpha_k}\Big(\kappa(\rho(\eta_n)) - \kappa(\rho(\gamma_n)) - \kappa(\rho(\gamma_n^{-1}\eta_n))\Big)\right|$ has a uniform upper bound, which is a contradiction. ☐

Proof of Proposition 6.3. Since $\{\omega_\alpha|_{\mathfrak{a}_\theta}\}_{\alpha\in\theta}$ is a basis for $\mathfrak{a}_\theta^*$, it suffices to fix $\beta \in \theta$ and find $C > 0$ such that: if $\gamma, \eta \in \Gamma_0$ and

$$\mathrm{d}_\Omega\left(\gamma(b_0), [b_0, \eta(b_0)]_\Omega\right) \leq r,$$

then

$$\left|\omega_\beta\left(\kappa_\theta(\rho(\gamma\eta)) - \kappa_\theta(\rho(\gamma)) - \kappa_\theta(\rho(\eta))\right)\right| \leq C.$$

Let $\chi_1 := \sum_{\alpha\in\theta}\omega_\alpha$ and $\chi_2 := \omega_\beta + \sum_{\alpha\in\theta}\omega_\alpha$. For $j = 1, 2$, let $\Phi_j : \mathsf{G} \to \mathsf{PSL}(d_j, \mathbb{R})$ satisfy Proposition B.1 for χ_j, and let $\rho_j := \Phi_j \circ \rho$. Then ρ_j is a $\mathsf{P}_{1,d-1}$-transverse representation and there exists $N_j \in \mathbb{N}$ such that

$$\left|\chi_j\left(\kappa_\theta(\rho(\gamma\eta)) - \kappa_\theta(\rho(\gamma)) - \kappa_\theta(\rho(\eta))\right)\right| = \frac{1}{N_j}\left|\omega_{\alpha_1}\left(\kappa(\rho_j(\gamma\eta)) - \kappa(\rho_j(\gamma)) - \kappa(\rho_j(\eta))\right)\right|$$

for all $\gamma, \eta \in \Gamma$. Applying Lemma B.3 to ρ_j, there exists $C_j > 0$ such that: if $\gamma, \eta \in \Gamma_0$ and

$$\mathrm{d}_\Omega\left(\gamma(b_0), [b_0, \eta(b_0)]_\Omega\right) \leq r,$$

then

$$\left|\omega_{\alpha_1}\left(\kappa(\rho_j(\gamma\eta)) - \kappa(\rho_j(\gamma)) - \kappa(\rho_j(\eta))\right)\right| \leq C_j.$$

Since $\chi_2 - \chi_1 = \omega_\beta$, we then have: if $\gamma, \eta \in \Gamma_0$ and

$$\mathrm{d}_\Omega\left(\gamma(b_0), [b_0, \eta(b_0)]_\Omega\right) \leq r,$$

then

$$\left|\omega_\beta\left(\kappa_\theta(\rho(\gamma\eta)) - \kappa_\theta(\rho(\gamma)) - \kappa_\theta(\rho(\eta))\right)\right| \leq \frac{C_1}{N_1} + \frac{C_2}{N_2}. \qquad \square$$

B.3. **The proof of Proposition B.1.** Fix a symmetric set $\theta \subset \Delta$ and $\chi \in \sum_{\alpha\in\theta}\mathbb{N}\,\omega_\alpha$. By [23, Lem. 3.2, Prop. 3.3, Rem. 3.6 and Lem. 3.7] there exist $N, d \in \mathbb{N}$, an irreducible linear representation $\Phi : \mathsf{G} \to \mathsf{SL}(d, \mathbb{R})$ and a Φ-equivariant smooth embedding

$$\xi : \mathcal{F}_\theta \to \mathcal{F}_{1,d-1} := \mathcal{F}_{1,d-1}(\mathbb{R}^d)$$

such that:

(a) Φ is proximal and has highest weight $N\chi$, that is: if $H \in \mathrm{int}(\mathfrak{a}^+)$, then $\Phi(e^H)$ is proximal and the eigenvalue with largest modulus is $e^{N\chi(H)}$.
(b) $\Phi(\mathsf{K}) \subset \mathsf{SO}(d, \mathbb{R})$ and $\Phi(e^{\mathfrak{a}})$ is a subgroup of the diagonal matrices in $\mathsf{SL}(d, \mathbb{R})$.
(c) $\alpha_1(\kappa(\Phi(g))) = \min_{\alpha\in\theta}\alpha(\kappa(g))$ for all $g \in \mathsf{G}$.
(d) $F_1, F_2 \in \mathcal{F}_\theta$ are transverse if and only if $\xi(F_1), \xi(F_2) \in \mathcal{F}_{1,d-1}$ are transverse.

In the statement of Proposition B.1, parts (1) and (3) are restatements of properties (d) and (c) of Φ respectively, while part (2) is a consequence of properties (a) and (b) of Φ. Part (5) follows immediately from parts (1), (3), and (4), while part (6) follows immediately from part (5) and Proposition 2.5. Thus, it suffices to prove part (4).

Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. Using properties (a) and (b), we can conjugate Φ by a permutation matrix and assume that

$$\Phi(e^H)e_1 = e^{N\chi(H)}e_1 \quad \text{and} \quad \Phi(e^H)e_d = e^{N\bar{\chi}(H)}e_d \tag{20}$$

when $H \in \mathfrak{a}$ (where as usual $\bar{\chi} = \chi \circ \iota$). We first observe that the value of $\xi(\mathsf{P}_\theta)$ is determined.

Lemma B.4. $\xi(\mathsf{P}_\theta) = \big(\langle e_1\rangle, \langle e_1, \ldots, e_{d-1}\rangle\big)$.

Proof. Let $\hat{F}_0^+ := \big(\langle e_1\rangle, \langle e_1,\dots,e_{d-1}\rangle\big)$ and $\hat{F}_0^- := \big(\langle e_d\rangle, \langle e_2,\dots,e_d\rangle\big)$. Fix $H \in \mathrm{int}(\mathfrak{a}^+)$. Then by property (b) and Equation (20), $\Phi(e^H) = \mathrm{diag}(a_1,\dots,a_d)$ is a diagonal matrix with

$$|a_1| > \max\{|a_j| : 2 \le j \le d\} \quad \text{and} \quad |a_d| < \min\{|a_j| : 1 \le j \le d-1\}.$$

So

$$\Phi(e^{nH})F \to \hat{F}_0^+$$

for all $\hat{F} \in \mathcal{F}_{1,d-1}$ transverse to $\hat{F}_0^-$. Since Φ is irreducible, there exists some $F \in \mathcal{F}_\theta$ such that $\xi(F)$ is transverse to $\hat{F}_0^-$. Using Lemma A.2 and perturbing F we may also assume that $e^{nH}(F) \to \mathsf{P}_\theta$. Then

$$\xi(\mathsf{P}_\theta) = \lim_{n\to\infty} \xi(e^{nH}F) = \lim_{n\to\infty} \Phi(e^{nH})\xi(F) = \hat{F}_0^+. \qquad \square$$

Now we prove (4).

Lemma B.5. *If* $\min_{\alpha\in\theta}\alpha(\kappa(g)) > 0$, *then* $\xi(U_\theta(g)) = U_{1,d-1}(\Phi(g))$.

Proof. Fix a KAK-decomposition $g = me^H\ell$. By properties (a) and (b), there exists a permutation matrix $k \in \mathsf{O}(d)$ such that

$$\Phi(g) = \big(\Phi(m)k^{-1}\big)\big(k\Phi(e^H)k^{-1}\big)\big(k\Phi(\ell)\big)$$

is a singular value decomposition of $\Phi(g)$. By Equation (20), $k(e_1) = e_1$ and $k(e_d) = e_d$. Further, by property (c), we have

$$\alpha_j(\Phi(g)) > 0 \quad \text{for } j = 1, d-1,$$

so by Lemma B.4,

$$\begin{aligned} U_{1,d-1}(\Phi(g)) &= \big(\Phi(m)k^{-1}\big)\big(\langle e_1\rangle, \langle e_1,\dots,e_{d-1}\rangle\big) = \Phi(m)\big(\langle e_1\rangle, \langle e_1,\dots,e_{d-1}\rangle\big) \\ &= \Phi(m)\xi(\mathsf{P}_\theta) = \xi(m\,\mathsf{P}_\theta) = \xi(U_\theta(g)). \end{aligned} \qquad \square$$

References

[1] P. Albuquerque, "Patterson-Sullivan theory in higher rank symmetric spaces," *G.A.F.A.* **9**(1999), 1–28.
[2] Y. Benoist, "Propriétés asymptotiques des groupes linéaires," *G.A.F.A.* **7**(1997), 1–47.
[3] Y. Benoist, "Convexes divisibles I," in *Algebraic groups and arithmetic*, Tata Inst. Fund. Res. Stud. Math. **17**(2004), 339–374.
[4] Y. Benoist and J.F. Quint, *Random Walks on Reductive Groups*, Springer-Verlag, 2016.
[5] P. Blayac, "Patterson-Sullivan densities in convex projective geometry," preprint, arxiv:2106.08089.
[6] P. Blayac, "Topological mixing of the geodesic flow on convex projective manifolds," preprint, arxiv:2009.05035.
[7] P. Blayac and F. Zhu, "Ergodicity and equidistribution in Hilbert geometry," preprint, arXiv:2106.08079.
[8] H. Bray, "Ergodicity of Bowen-Margulis measure for the Benoist 3-manifolds," *J. Mod. Dyn.* **16**(2020), 205–329.
[9] M. Bridgeman, R. Canary, F. Labourie and A. Sambarino, "The pressure metric for Anosov representations," *G.A.F.A.* **25**(2015), 1089–1179.
[10] R. Brooks, "The bottom of the spectrum of a Riemannian covering," *Crelle's Journal* **357**(1985), 101–114.
[11] M. Burger, "Intersection, the Manhattan curve and Patterson-Sullivan theory in rank 2," *I.M.R.N.* **7**(1993), 217–225.
[12] M. Burger, O. Landesberg, M. Lee and H. Oh, "The Hopf-Tsuji-Sullivan dichotomy in higher rank and applications to Anosov subgroups," *J. Mod. Dyn.*, to appear, arXiv:2105.13930.
[13] R. Canary, "On the Laplacian and the geometry of hyperbolic 3-manifolds," *J. Diff. Geom.* **36**(1992), 349–367.
[14] R. Canary, M. Lee, A. Sambarino and M. Stover, "Amalgam Anosov representations," *Geom. Top.* **21**(2017), 215–251.
[15] R. Canary, T. Zhang and A. Zimmer, "Entropy rigidity for cusped Hitchin representations," preprint, arXiv:2201.04859.
[16] Y. Coudene, "On invariant distributions and mixing," *Erg. Thy. Dyn. Sys.* **27**(2007), 109–112.
[17] Y. Coudene, "The Hopf Argument," *J. Mod. Dyn.* **1**(2007), 147–153.
[18] R. Coulon, F. Dal'bo and A. Sambusetti, "Growth gap in hyperbolic groups and amenability," *G.A.F.A.* **28**(2018), 1260–1320 .
[19] R. Coulon, S. Dougall, B. Schapira and S. Tapie, "Twisted Patterson-Sullivan measures and applications to amenability and coverings," preprint, arxiv:1809.10881.
[20] F. Dal'Bo and I. Kim, "A criterion of conjugacy for Zariski dense subgroups," *C. R. Math.* **330**(2000), 647–650.

[21] F. Dal'bo, J.-P. Otal, and M. Peigné, "Séries de Poincaré des groupes géométriquement finis," *Israel J. Math.* **118**(2000), 109–124.

[22] S. Dey and M. Kapovich, "Patterson-Sullivan theory for Anosov subgroups," *Trans. A.M.S.* **375**(2022), 8687–8737.

[23] F. Guéritaud, O. Guichard, F. Kassel and A. Wienhard, "Anosov representations and proper actions," *Geom. Top.* **21**(2017), 485–584.

[24] O. Glorieux and S. Tapie, "Critical exponents of normal subgroups in higher rank," preprint, arXiv:2006.05730.

[25] O. Guichard and A. Wienhard, "Anosov representations: Domains of discontinuity and applications," *Invent. Math.* **190**(2012), 357–438.

[26] J.W. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Springer-Verlag New York Inc. (1972).

[27] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Crm Proceedings & Lecture Notes, American Mathematical Soc. (2001).

[28] M. Islam and A. Zimmer, "A flat torus theorem for convex co-compact actions of projective linear groups," *Jour. L.M.S.* **103**(2021), 470–489.

[29] M. Kapovich, B. Leeb and J. Porti, "Anosov subgroups: Dynamical and geometric characterizations," *Eur. Math. J.* **3**(2017), 808–898.

[30] M. Kapovich, B. Leeb and J. Porti, "Morse actions of discrete groups on symmetric spaces," preprint, arXiv:1403.7671.

[31] A.W. Knapp, *Lie Groups Beyond an Introduction*, Progress in Mathematics, Birkhäuser Boston Inc. (1996).

[32] S. Kochen and C. Stone, "A note on the Borel-Cantelli lemma," *Illinois J. Math.* **8**(1964), 248–251.

[33] F. Labourie, "Anosov flows, surface groups and curves in projective space," *Invent. Math.* **165**(2006), 51–114.

[34] F. Ledrappier, "Structure au bord des variétés à courbure négative," in *Séminaire de théorie spectrale et géométrie de Grenoble* 13, 1994-1995, Université de Grenoble I, Institut Fourier, Saint-Martin-d'Héres, 1995, 97–122.

[35] M. Lee and H. Oh, "Invariant measures for horospherical actions and Anosov groups," *I.M.R.N.*, to appear, arXiv: 2008.05296.

[36] M. Lee and H. Oh, "Dichotomy and measures on limit sets of Anosov groups," preprint, arXiv:2203.06794.

[37] G. Link, "Ergodicity of generalised Patterson-Sullivan measures in higher rank symmetric spaces," *Math. Z.* **254**(2006), 611–625.

[38] L. Marquis, "Around groups in Hilbert geometry," in *Handbook of Hilbert Geometry*, European Mathematical Society Publishing House, 2014, 207–261.

[39] S.J. Patterson, "The limit set of a Fuchsian group," *Acta Math.* **136**(1976), 241–273.

[40] R. Potrie and A. Sambarino, "Eigenvalues and entropy of a Hitchin representation," *Invent. Math.* **209**(2017), 885–925.

[41] J.F. Quint, "Mesures de Patterson-Sullivan en rang supérieur," *G.A.F.A.* **12**(2002), 776–809.

[42] T. Roblin, "Ergodicité et équidistribution en courbure négative," *Mem. Soc. Math. Fr.* No. 95 (2003).

[43] A. Sambarino, "Quantitative properties of convex representations," *Comment. Math. Helv.* **89**(2014), 443–488.

[44] A. Sambarino, "The orbital counting problem for hyperconvex representations," *Ann. Inst. Fourier* **65**(2015), 1755–1797.

[45] A. Sambarino, "A report on an ergodic dichotomy," *Ergod. Theory Dyn. Syst.* (2023) 1–54. doi:10.1017/etds.2023.13

[46] D. Sullivan, "The density at infinity of a discrete group of hyperbolic motions," *Publ. I.H.E.S.* **50**(1979), 171–202.

[47] D. Sullivan, "The ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions," in *Riemann surfaces and related topics: proceedings of the 1978 Stony Brook conference*, I. Kra and B. Maski, ed., Princeton University Press, 1981, 465–496.

[48] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I*, Grundlehren der mathematischen Wissenschaften, Springer Berlin (1972).

[49] F. Zhu, "Ergodicity and equidistribution in strictly convex Hilbert geometry," preprint, arXiv:2008.00328.

University of Michigan

National University of Singapore

University of Wisconsin-Madison