PRESSURE METRICS FOR CUSPED HITCHIN COMPONENTS

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Abstract. We study the cusped Hitchin component consisting of (conjugacy classes of) cusped Hitchin representations of a torsion-free geometrically finite Fuchsian group Γ into $\mathsf{PSL}(d,\mathbb{R})$. We produce Riemannian pressure metrics associated to the first fundamental weight and the first simple root. We produce a pressure path metric associated to the Hilbert length and describe its degeneracy.

Contents

1. Introduction

In this paper, we construct pressure metrics on the cusped Hitchin component of Hitchin representations of a torsion-free Fuchsian lattice into $\mathsf{PSL}(d,\mathbb{R})$. The first two metrics are mapping class group invariant, analytic Riemannian metrics. These metrics are associated to the first fundamental weight and the first simple root. Our third pressure metric is based on the Hilbert length. It is a mapping class group invariant path metric which is an analytic Riemannian metric off of the self-dual locus. These constructions are based on earlier constructions of Bridgeman, Canary, Labourie and Sambarino [7, 8, 9] in the case of Hitchin components of closed surface groups.

The main new technical difficulties involve the fact that while the geodesic flow of a closed hyperbolic surface may be coded by a finite Markov shift, there is no finite Markov coding of the geodesic flow of a geometrically finite hyperbolic surface. Stadlbauer [44] and Ledrappier-Sarig [27] provide a countable Markov coding of the (recurrent portion of the) geodesic flow of a finite area hyperbolic surface. In a previous paper, we used these codings, work of Canary-Zhang-Zimmer [11] on cusped Hitchin representations, and

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the Thermodynamic Formalism for countable Markov shifts, to establish counting and equidistribution results for cusped Hitchin representations. In this paper, we apply the theory developed in that paper to construct our pressure metrics.

The long-term goal of this project is to realize these metrics as the induced metric on the strata at infinity of the metric completion of the Hitchin component of a closed surface group with its pressure metric. In the classical setting, when $d = 2$, Masur [31] showed that the metric completion of Teichmüller space of a closed surface S, with the Weil-Petersson metric, is the augmented Teichmüller space. The strata at infinity in the augmented Teichmüller space come from Teichmüller space of, possibly disconnected, surfaces obtained from pinching S along a multicurve. We hope that the Hilbert length pressure metric when $d = 3$ may be more natural to study given its connection to Hilbert geometry. When $d = 3$, the Hitchin component of a closed surface is the space of holonomy maps of convex projective structures on the surface. The strata at infinity of the augmented Hitchin component would then be cusped Hitchin components consisting of finite area convex projective structures obtained from pinching the surface along a multicurve. We hope to eventually establish an analogue of Masur's result in the higher rank setting. (See [10] for a more detailed description of the conjectural geometric picture of the augmented Hitchin component.)

We now discuss our results more precisely. We recall that if Γ is a torsion-free, geometrically finite Fuchsian group (i.e. a discrete non-abelian finitely generated subgroup of $\mathsf{PSL}(2,\mathbb{R})$), then a Hitchin representation is a representation $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ which admits a positive equivariant limit map $\xi : \Lambda(\Gamma) \to \mathcal{F}_d$ where $\Lambda(\Gamma) \subset \partial\mathbb{H}^2$ is the limit set of Γ and $\mathcal{F}_d$ is the space of d-dimensional flags. As in the closed case, they all arise as type-preserving deformations of the restriction of an irreducible representation of $\mathsf{PSL}(2,\mathbb{R})$ into $\mathsf{PSL}(d,\mathbb{R})$.

The *Hitchin component* $\mathcal{H}_d(\Gamma)$ is the space of conjugacy classes of Hitchin representations of Γ into $\mathsf{PSL}(d,\mathbb{R})$. Fock and Goncharov, see the discussion in [14, Sec 1.8], show that the Hitchin component is topologically a cell. (When $d = 3$, $\mathcal{H}_3(\Gamma)$ is parameterized by Marquis [30], when Γ is a lattice, and more generally by Loftin and Zhang [28]. Bonahon-Dreyer [2, Thm. 2] and Zhang [47, Prop. 3.5] explicitly describe variations of the Fock-Goncharov parametrization when Γ is cocompact, and their analyses should extend to our setting.) More generally, if G is a real-split Lie subgroup of $\mathsf{PSL}(d,\mathbb{R})$, let $\mathcal{H}(\Gamma,\mathsf{G})$ be the space of Hitchin representations with image in G. (In particular, $\mathcal{H}_d(\Gamma) = \mathcal{H}(\Gamma,\mathsf{PSL}(d,\mathbb{R}))$ in this notation.) Fock-Goncharov [14] and Hitchin [19] (see also [16, §9.3]) show that $\mathcal{H}(\Gamma,\mathsf{G})$ is topologically a cell.

Theorem 1.1. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite and G is a real-split Lie subgroup of $\mathsf{PSL}(d,\mathbb{R})$, then the cusped Hitchin component $\mathcal{H}(\Gamma,\mathsf{G})$ is an analytic manifold diffeomorphic to $\mathbb{R}^m$ for some $m \in \mathbb{N}$.*

If

$$\mathfrak{a} = \left\{ \vec{x} \in \mathbb{R}^d \mid \sum x_i = 0 \right\}$$

is the standard Cartan algebra for $\mathsf{PSL}(d,\mathbb{R})$, let

$$\Delta = \left\{ \phi = \sum_{i=1}^{d-1} a_i\alpha_i \mid a_i \geq 0 \ \forall\ i,\ \sum a_i > 0 \right\} \subset \mathfrak{a}^*$$

where α_i is the simple root given by $\alpha_i(\vec{x}) = x_i - x_{i-1}$. Notice that Δ is exactly the collection of linear functionals which are strictly positive on the interior of the Weyl chamber

$$\mathfrak{a}^+ = \{\vec{x} \in \mathfrak{a} \mid x_1 \geqslant \ldots \geqslant x_d\}.$$

Consider the *Jordan projection* $\nu : \mathsf{PSL}(d,\mathbb{R}) \to \mathfrak{a}^+$ given by

$$\nu(A) = (\log \lambda_1(A), \ldots, \log \lambda_d(A))$$

where $\lambda_1(A) \geq \cdots \geq \lambda_d(A)$ are the (ordered) moduli of generalized eigenvalues of A.

If $\phi \in \Delta$ and $\rho \in \mathcal{H}_d(\Gamma)$, denote by $\ell^\phi_\rho(\gamma) = \phi(\nu(\rho(\gamma)))$ the ϕ-length of $\gamma \in \Gamma$. We may define the ϕ-entropy of ρ as

$$h^\phi(\rho) = \lim_{T\to\infty} \frac{\# R^\phi_T(\rho)}{T}$$

where $[\Gamma_{hyp}]$ is the set of conjugacy classes of hyperbolic elements in Γ, and

$$R^\phi_T(\rho) = \left\{ [\gamma] \in [\Gamma_{hyp}] \mid \ell^\phi_\rho(\gamma) \leq T \right\}.$$

Moreover, if $\rho, \eta \in \mathcal{H}_d(\Gamma)$, we may define the ϕ-pressure intersection

$$I^\phi(\rho,\eta) = \lim_{T\to\infty} \frac{1}{|R^\phi_T(\rho)|} \sum_{[\gamma]\in R^\phi_T(\rho)} \frac{\ell^\phi_\eta(\gamma)}{\ell^\phi_\rho(\gamma)},$$

and a renormalized ϕ-pressure intersection

$$J^\phi(\rho,\eta) = \frac{h^\phi(\eta)}{h^\phi(\rho)} I^\phi(\rho,\eta).$$

Our key tool in the construction of the pressure metric will be results of Bray, Canary, Kao and Martone [5] and Canary, Zhang and Zimmer [11] which combine to prove that all these quantities vary analytically. See [7, Section 8.1] for the analogous statement when Γ is cocompact.

Theorem 1.2. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite and $\phi \in \Delta$, then $h^\phi(\rho)$ varies analytically over $\mathcal{H}_d(\Gamma)$ and I^ϕ and J^ϕ vary analytically over $\mathcal{H}_d(\Gamma) \times \mathcal{H}_d(\Gamma)$. Moreover, if $\rho, \eta \in \mathcal{H}_d(\Gamma)$, then*

$$J^\phi(\rho,\eta) \geq 1$$

and $J^\phi(\rho,\eta) = 1$ if and only if $\ell^\phi_\rho(\gamma) = \frac{h^\phi(\eta)}{h^\phi(\rho)} \ell^\phi_\eta(\gamma)$ for all $\gamma \in \Gamma$.

Given $\phi \in \Delta$, we define a pressure form on the Hitchin component, by letting

$$\mathbb{P}^\phi\big|_{T_\rho \mathcal{H}_d(\Gamma)} = \mathrm{Hess}(J^\phi(\rho,\cdot)).$$

Since J^ϕ achieves its minimum along the diagonal, $\mathbb{P}^\phi$ will always be non-negative. However, it will not always be non-degenerate. Typically, the most difficult portion of the proof of the construction of a pressure metric is to verify non-degeneracy, or, more generally, to characterize which vectors are degenerate.

We first consider the first fundamental weight $\omega_1 \in \Delta$, given by $\omega_1(\vec{x}) = x_1$. As a consequence of a much more general result, Bridgeman, Canary, Labourie and Sambarino [7] prove that $\mathbb{P}^{\omega_1}$ is non-degenerate on the Hitchin component of a convex cocompact Fuchsian group. We recall that the mapping class group $\mathrm{Mod}(\Gamma)$ is the group of (isotopy classes of) orientation-preserving self-homeomorphisms of $\mathbb{H}^2/\Gamma$.

Theorem 1.3. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite, then the pressure form $\mathbb{P}^{\omega_1}$ is non-degenerate, so gives rise to a mapping class group invariant, analytic Riemannian metric on $\mathcal{H}_d(\Gamma)$.*

Bridgeman, Canary, Labourie and Sambarino [8] later expanded their techniques to show that the first simple root gives rise to a non-degenerate pressure metric on the Hitchin component of a closed surface group. We implement their outline in the cusped setting. We make crucial use of a result of Canary, Zhang and Zimmer [12] which assures us that simple root entropies are constant on the Hitchin components of Fuchsian lattices (which generalizes a result of Potrie and Sambarino [37] for Hitchin components of closed surface groups).

Theorem 1.4. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is a torsion-free lattice, then the pressure form $\mathbb{P}^{\alpha_1}$ is non-degenerate, so gives rise to a mapping class group invariant, analytic Riemannian metric on $\mathcal{H}_d(\Gamma)$.*

Finally, we consider the functional ω_H associated to the Hilbert length given by $\omega_H(\vec{x}) = x_1 - x_d$. It is easy to see that if $C : \mathcal{H}_d(\Gamma) \to \mathcal{H}_d(\Gamma)$ is the contragredient involution and $\vec{v} \in T\mathcal{H}_d(\Gamma)$ is *anti-self-dual*, i.e. $\mathrm{D}C(\vec{v}) = -\vec{v}$, then $\mathbb{P}^{\omega_H}(\vec{v},\vec{v}) = 0$ (see [9, Lem. 5.22]). In particular, $\mathbb{P}^{\omega_H}$ is not globally non-degenerate. However, one can still show that the pressure form gives rise to a path metric. Bridgeman, Canary and Sambarino [9, Sec. 5.8] previously remarked that this is the case when Γ is a closed surface group.

Theorem 1.5. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite, then $\mathbb{P}^{\omega_H}$ gives rise to a mapping class group invariant path metric on $\mathcal{H}_d(\Gamma)$ which is an analytic Riemannian metric off of the self-dual locus.*

When $d = 3$, cusped Hitchin representations of a torsion-free lattice are holonomy maps of finite area convex projective surfaces and the Hilbert length is the translation length with respect to the Hilbert metric. In this case, the analogy with the augmented Teichmüller space is most compelling and we expect that this case may be the easiest case in which to begin the analysis of the augmented Hitchin component. Notice that our proposed augmented Hitchin component would be a proper subspace of the augmented Hitchin component introduced and studied in [28].

Theorems 1.3 and 1.5 are derived by generalizing the main result of [7, Thm. 1.4] into the cusped setting.

Let $\widetilde{\mathcal{P}}^{irr}_{\{1,d-1\}}(\Gamma,d)$ be the set of irreducible $P_{\{1,d-1\}}$-Anosov representations into $\mathsf{PSL}(d,\mathbb{R})$ and let $\mathcal{P}^{irr}_{\{1,d-1\}}(\Gamma,d) = \widetilde{\mathcal{P}}^{irr}_{\{1,d-1\}}(\Gamma,d)/\mathsf{PSL}(d,\mathbb{R})$. If H is a reductive subgroup of $\mathsf{PSL}(d,\mathbb{R})$, then an element $h \in \mathsf{H}$ is H-generic if its centralizer is a maximal torus in H. If $\mathsf{H} = \mathsf{PSL}(d,\mathbb{R})$, then an element is H-generic if and only if it is diagonalizable over $\mathbb{C}$ with distinct eigenvalues. A representation into H is said to be H-generic if its image contains an H-generic element. In particular, all Hitchin representations are $\mathsf{PSL}(d,\mathbb{R})$-generic.

Theorem 1.6. *Suppose that $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite. If W is an analytic submanifold of $\mathcal{P}^{irr}_{\{1,d-1\}}(\Gamma,d)$, H is a reductive subgroup of $\mathsf{PSL}(d,\mathbb{R})$ and every representation in W has image in H and is H-generic, then $\mathbb{P}^{\omega_1}|_{\mathsf{T}W}$ is an analytic Riemannian metric on W. Moreover, if W is invariant under a subgroup M of the mapping class group, then $\mathbb{P}^{\omega_1}|_{\mathsf{T}W}$ is M-invariant.*

Finally, we remark that if Γ is geometrically finite but has torsion, then it has a finite index normal subgroup Γ_0 which is torsion-free. One may identify Γ/Γ_0 with a finite index subgroup G of the mapping class group of $\mathbb{H}^2/\Gamma_0$ and then identify $\mathcal{H}_d(\Gamma)$ with the submanifold of $\mathcal{H}_d(\Gamma_0)$ which is stabilized by G. It follows that one obtains mapping class group invariant analytic Riemannian metrics $\mathbb{P}^{\omega_1}$ and $\mathbb{P}^{\alpha_1}$ on $\mathcal{H}_d(\Gamma)$ and a mapping class group invariant path metric on $\mathcal{H}_d(\Gamma)$ which is analytic Riemannian off of the self-dual locus.

Historical remarks. Thurston described a metric on Teichmüller space which was the "Hessian of the length of a random geodesic." Wolpert [46] showed that this metric gives a scalar multiple of the classical Weil-Petersson metric. Bonahon [1] reinterpreted Thurston's metric in terms of geodesic currents. McMullen [33] showed that one may interpret Thurston's metric in terms of Thermodynamic Formalism, as the Hessian of a pressure intersection function. Bridgeman [6] generalized McMullen's construction to the setting of quasifuchsian space. Bridgeman, Canary, Labourie and Sambarino [7] then showed how to use his construction to produce analytic Riemannian metrics at "generic" smooth points of deformation spaces of projective Anosov representations, and in particular on Hitchin components. Pollicott and Sharp [36] gave an alternate interpretation of this metric.

Kao [20] used countable Markov codings to construct pressure metrics on Teichmüller spaces of punctured surfaces. Bray, Canary and Kao [4] generalized this to the setting of cusped quasifuchsian groups.

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2. Background

2.1. **Linear algebra.** The *Jordan projection* $\nu\colon \mathsf{SL}(d,\mathbb{R}) \to \mathfrak{a}^+$ is the map which associates to $A \in \mathsf{SL}(d,\mathbb{R})$ the list $(\log\lambda_1(A),\dots,\log\lambda_d(A))$ of logarithms of moduli of generalized eigenvalues of A in decreasing order.

The *Cartan projection* $\kappa\colon \mathsf{SL}(d,\mathbb{R}) \to \mathfrak{a}^+$ is

$$\kappa(A) = (\log\sigma_1(A),\dots,\log\sigma_d(A))$$

where $\{\sigma_i(A)\}_{i=1}^d$ are the singular values of A labelled in decreasing order. Recall that each element of $\mathsf{SL}(d,\mathbb{R})$ may be written as $A = KDL$ where $K, L \in \mathsf{SO}(d)$ and D is the diagonal matrix with (i,i)-entry given by $\sigma_i(A)$. If $\alpha_k(\kappa(A)) > 0$, then $U_k(A) = K(\langle e_1,\dots,e_k\rangle)$ is well-defined and is the k-plane spanned by the first k major axes of $A(S^{d-1})$.

Suppose that θ is a symmetric subset of $\{1,\ldots,d-1\}$, i.e. $k\in\theta$ if and only if $d-k\in\theta$. Define the *θ-Cartan subspace* as

$$\mathfrak{a}_\theta = \{\vec{a}\in\mathfrak{a} : \alpha_j(\vec{a}) = 0 \text{ if } j\notin\theta\}$$

and let Δ_θ denote the set of functionals which are positive on $\mathfrak{a}_\theta^+ = \mathfrak{a}^+\cap\mathfrak{a}_\theta$.

The *θ-Cartan projection* $\kappa_\theta\colon \mathsf{SL}(d,\mathbb{R})\to\mathfrak{a}_\theta$ is the unique map so that $\omega_k(\kappa_\theta(A)) = \omega_k(\kappa(A))$ for all $A\in\mathsf{SL}(d,\mathbb{R})$ and all $k\in\theta$.

If $\theta=\{k_1,\ldots,k_n\}$ we define the *θ-flag variety*

$$\mathcal{F}_\theta = \{(F^{k_1},F^{k_2},\ldots,F^{k_n}) : F^{k_1}\subset F^{k_2}\subset\cdots\subset F^{k_n}\}$$

where each F^{k_i} is a vector subspace of $\mathbb{R}^d$ of dimension k_i. In particular, the full flag variety $\mathcal{F}_d$ is the same as $\mathcal{F}_{\{1,2,\ldots,d-1\}}$ in this notation.

Quint [38] introduced a vector valued smooth cocycle, called the *θ-Iwasawa cocycle*,

$$B_\theta : \mathsf{SL}(d,\mathbb{R})\times\mathcal{F}_\theta\to\mathfrak{a}_\theta$$

with the defining property that if $k\in\theta$, $A\in\mathsf{SL}(d,\mathbb{R})$, $F\in\mathcal{F}_\theta$, $\vec{v}_k$ is a non-trivial vector in $E^k(F^k)\subset E^k(\mathbb{R}^d)$, where E^k denotes the k^{th} exterior power, then

$$\omega_k(B_\theta(A,F)) = \log\frac{||E^kA(\vec{v}_k)||}{||\vec{v}_k||}.$$

Note that the Jordan and Cartan projections (resp. θ-Iwasawa cocycle) descend to well-defined functions on $\mathsf{PSL}(d,\mathbb{R})$ (resp. $\mathsf{PSL}(d,\mathbb{R})\times\mathcal{F}_\theta$).

2.2. **Thermodynamic Formalism.** In this section, we recall the background results we will need from the Thermodynamic Formalism for countable Markov shifts as developed by Gurevich-Savchenko [18], Mauldin-Urbanski [32] and Sarig [43].

Given a countable alphabet $\mathcal{A}$ and a transition matrix $\mathbb{T}=(t_{ab})\in\{0,1\}^{\mathcal{A}\times\mathcal{A}}$ a one-sided Markov shift is

$$\Sigma^+ = \{x=(x_i)\in\mathcal{A}^{\mathbb{N}} \mid t_{x_ix_{i+1}} = 1 \text{ for all } i\in\mathbb{N}\}$$

equipped with a shift map $\sigma:\Sigma^+\to\Sigma^+$ which takes $(x_i)_{i\in\mathbb{N}}$ to $(x_{i+1})_{i\in\mathbb{N}}$. One says that (Σ^+,σ) is *topologically mixing* if for all $a,b\in\mathcal{A}$, there exists $N=N(a,b)$ so that if $n\geq N$, then there exists $x\in\Sigma^+$ so that $x_1=a$ and $x_n=b$. The shift (Σ^+,σ) has the big images and pre-images property (BIP) if there exists a finite subset $\mathcal{B}\subset\mathcal{A}$ so that if $a\in\mathcal{A}$, then there exists $b_0,b_1\in\mathcal{B}$ so that $t_{b_0,a}=1=t_{a,b_1}$.

Given a one-sided countable Markov shift (Σ^+,σ) and a function $g:\Sigma^+\to\mathbb{R}$, we say that g is *locally Hölder continuous* if there exists $C>0$ and $\eta\in(0,1)$ so that if $x,y\in\Sigma^+$ and $x_i=y_i$ for all $1\leq i\leq n$, then

$$|g(x)-g(y)|\leq C\eta^n.$$

If $n\in\mathbb{N}$, the *n^{th}-ergodic sum* of g at $x\in\Sigma^+$ is

$$S_ng(x) = \sum_{i=1}^n g(\sigma^{i-1}(x))$$

and $\text{Fix}^n = \{x\in\Sigma^+ \mid \sigma^n(x) = x\}$ is the set of periodic words with period n.

The *pressure* of a locally Hölder continuous function $g : \Sigma^+ \to \mathbb{R}$ is defined to be

$$P(g) = \sup \left\{ h_\sigma(m) + \int_{\Sigma^+} g \, dm : m \in \mathcal{M}_\sigma \text{ and } -\int_{\Sigma^+} g \, dm < \infty \right\}$$

where $\mathcal{M}_\sigma$ is the space of σ-invariant probability measures on Σ^+ and $h_\sigma(m)$ is the measure-theoretic entropy of σ with respect to the measure m.

A σ-invariant Borel probability measure m on Σ^+ is an *equilibrium measure* for a locally Hölder continuous function $g : \Sigma^+ \to \mathbb{R}$ if

$$P(g) = h_\sigma(m) + \int_{\Sigma^+} g \, dm.$$

We remark that there are several different but equivalent definitions of pressure and equilibrium measure in the current setting. Readers can find a more detailed discussion of this in Bray-Canary-Kao-Martone [5, pg. 11]. Mauldin-Urbanski ([32, Thm. 2.6.12, Prop. 2.6.13 and 2.6.14]) and Sarig ([42, Cor. 4], [43, Thm 5.10 and 5.13]) prove that the pressure function is real analytic in our setting and compute its derivatives. Recall that $\{g_u : \Sigma^+ \to \mathbb{R}\}_{u \in M}$ is a *real analytic family* if M is a real analytic manifold and for all $x \in \Sigma^+$, $u \to g_u(x)$ is a real analytic function on M.

Theorem 2.1 (Mauldin-Urbanski, Sarig)**.** *Suppose that* (Σ^+, σ) *is a one-sided countable Markov shift which has (BIP) and is topologically mixing. If* $\{g_u : \Sigma^+ \to \mathbb{R}\}_{u \in M}$ *is a real analytic family of locally Hölder continuous functions such that* $P(g_u) < \infty$ *for all* u*, then* $u \to P(g_u)$ *is real analytic.*

Moreover, if $\vec{v} \in T_{u_0}M$ *and there exists a neighborhood* U *of* u_0 *in* M *so that if* $u \in U$*, then* $-\int_{\Sigma^+} g_u dm_{g_{u_0}} < \infty$*, then*

$$D_{\vec{v}} P(g_u) = \int_{\Sigma^+} D_{\vec{v}}(g_u(x)) \, dm_{g_{u_0}}.$$

In the case of finite Markov shifts, the assumption that $P(g_u) < \infty$ is automatically satisfied and Theorem 2.1 is due to Ruelle [39] and Parry-Pollicott [35].

Bowen and Series [3] constructed a finite Markov coding for the action of a convex cocompact group Γ on its limit set $\Lambda(\Gamma)$. Dal'bo and Peigné [13], when Γ is geometrically finite but not a lattice, and Stadlbauer [44] and Ledrappier-Sarig [27], when Γ is a lattice, constructed a countable Markov coding for the action of Γ on its conical limit set $\Lambda_c(\Gamma)$. We summarize their crucial properties below (see [4] for a more complete description in our language). If $a \in \mathcal{A}$, then $G(a)$ is the associated element of Γ and $\log r(a)$ is "coarsely" the translation distance (of some fixed basepoint) of $G(a)$.

Theorem 2.2 (Bowen-Series [3], Dal'bo-Peigné [13], Ledrappier-Sarig [27], Stadlbauer [44])**.** *Suppose that* Γ *is a torsion-free geometrically finite Fuchsian group. There exists a topologically mixing Markov shift* (Σ^+, σ) *with countable alphabet* $\mathcal{A}$ *with (BIP) and maps*

$$G : \mathcal{A} \to \Gamma, \quad \omega : \Sigma^+ \to \Lambda(\Gamma), \quad \text{and} \quad r : \mathcal{A} \to \mathbb{N}$$

with the following properties.

(1) *ω is locally Hölder continuous, finite-to-one and $\omega(\Sigma^+) = \Lambda_c(\Gamma)$, i.e. the complement in $\Lambda(\Gamma)$ of the set of fixed points of parabolic elements of Γ. Moreover, $\omega(x) = G(x_1)\omega(\sigma(x))$ for every $x \in \Sigma^+$.*
(2) *If $x \in \mathrm{Fix}^n$, then $\omega(x)$ is the attracting fixed point of $G(x_1)\cdots G(x_n)$. Moreover, if $\gamma \in \Gamma$ is hyperbolic, then there exists $x \in \mathrm{Fix}^n$ (for some n) so that γ is conjugate to $G(x_1)\cdots G(x_n)$ and x is unique up to cyclic permutation.*
(3) *There exists $D \in \mathbb{N}$ so that $1 \le \#(r^{-1}(n)) \le D$ for all $n \in \mathbb{N}$.*

2.3. **Anosov representations of geometrically finite Fuchsian groups.** We next recall the definition of a P_θ-Anosov representation of a geometrically finite Fuchsian group and the results of Bray-Canary-Kao-Martone [5] and Canary-Zhang-Zimmer [11] which will play a crucial role in our work.

Let Γ be a geometrically finite Fuchsian group and let θ be a symmetric subset of $\{1, \ldots, d-1\}$. We say that a representation $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ is *P_θ-Anosov*, if there exists a continuous ρ-equivariant map $\xi_\rho : \Lambda(\Gamma) \to \mathcal{F}_\theta$ so that

(1) ξ_ρ is *transverse*, i.e. if $x \neq y \in \Lambda(\Gamma)$ and $k \in \theta$, then

$$\xi_\rho^k(x) \oplus \xi_\rho^{d-k}(y) = \mathbb{R}^d,$$

(2) ξ_ρ is *strongly dynamics preserving*, i.e. if $\{\gamma_n\}$ is a sequence in Γ so that $\gamma_n(b_0) \to x \in \Lambda(\Gamma)$ and $\gamma_n^{-1}(b_0) \to y \in \Lambda(\Gamma)$ for some basepoint $b_0 \in \mathbb{H}^2$, then if $F \in \mathcal{F}_\theta$ is transverse to $\xi_\rho(y)$, then $\rho(\gamma_n)(F) \to \xi_\rho(x)$.

We denote the space of P_θ-Anosov representations of Γ into $\mathsf{PSL}(d, \mathbb{R})$ by $\widetilde{\mathcal{P}}_\theta(\Gamma, d)$. We will need the following observation, which follows immediately from the above definition.

Lemma 2.3. *If $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ is in $\widetilde{\mathcal{P}}_{\{1,d-1\}}(\Gamma, d)$ and Γ_0 is a Schottky subgroup of Γ, then $\rho|_{\Gamma_0}$ is a projective Anosov representation of the convex cocompact subgroup Γ_0.*

Canary, Zhang and Zimmer establish fundamental properties of P_θ-Anosov representations of geometrically finite Fuchsian groups which generalize the properties of classical Anosov representations.

Theorem 2.4 (Canary-Zhang-Zimmer [11])**.** *Suppose that Γ is a geometrically finite Fuchsian group, $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ is a P_θ-Anosov representation.*

(1) *If $\gamma \in \Gamma$ is hyperbolic and $k \in \theta$, then $\rho(\gamma)$ is P_k-proximal.*
(2) *If $\alpha \in \Gamma$ is parabolic, then $\rho(\alpha)$ is weakly unipotent in $\mathsf{PSL}(d, \mathbb{R})$, i.e. its Jordan-Chevalley decomposition has elliptic semi-simple part and non-trivial unipotent part.*
(3) *There exist $A, a > 0$ so that if $\gamma \in \Gamma$ and $k \in \theta$, then*

$$A e^{a d(b_0, \gamma(b_0))} \ge e^{\alpha_k(\kappa_\theta(\rho(\gamma)))} \ge \frac{1}{A} e^{\frac{d(b_0, \gamma(b_0))}{a}}$$

where b_0 is a basepoint for $\mathbb{H}^2$.
(4) *ρ has the P_θ-*Cartan property*, i.e. whenever $\{\gamma_n\}$ is a sequence of distinct elements of Γ such that $\gamma_n(b_0)$ converges to $z \in \Lambda(\Gamma)$, then $\xi_\rho^k(z) = \lim U_k(\rho(\gamma_n))$ for all $k \in \theta$.*

They also show that limit maps of Anosov representations vary analytically.

Theorem 2.5 (Canary-Zhang-Zimmer [11]). *If $\{\rho_u : \Gamma \to \mathsf{PSL}(d,\mathbb{R})\}_{u\in M}$ is a real analytic family of P_θ-Anosov representations of a geometrically finite Fuchsian group and $z \in \Lambda(\Gamma)$, then the map from M to $\mathcal{F}_\theta$ given by $u \to \xi_{\rho_u}(z)$ is real analytic.*

If $\rho \in \widetilde{\mathcal{P}}_\theta(\Gamma, d)$, the θ-*Benoist limit cone* of ρ is

$$\mathcal{B}_\theta(\rho) = \bigcap_{n\geqslant 0} \overline{\bigcup_{\|\kappa_\theta(\rho(\gamma))\|\geqslant n} \mathbb{R}_+\kappa_\theta(\rho(\gamma))} \subset \mathfrak{a}_\theta^+.$$

The positive dual to the θ-Benoist limit cone is given by

$$\mathcal{B}_\theta(\rho)^+ = \left\{\phi \in \mathfrak{a}_\theta^* \mid \phi\left(\mathcal{B}_\theta(\rho) - \{\vec{0}\}\right) \subset (0,\infty)\right\}. \tag{1}$$

In previous work [5], we constructed potentials on the Markov shift which encode the spectral properties of Anosov representations of geometrically finite, torsion-free Fuchsian groups. First we use the θ-Iwasawa cocycle to define a vector-valued roof function $\tau_\rho : \Sigma^+ \to \mathfrak{a}_\theta$ by

$$\tau_\rho(x) = B_\theta\big(\rho(G(x_1)), \rho(G(x_1))^{-1}(\xi_\rho(\omega(x)))\big)$$

If $\phi \in \mathcal{B}_\theta(\rho)^+$ one defines the roof function $\tau_\rho^\phi = \phi\circ\tau_\rho$. Notice that since $\mathcal{B}_\theta(\rho)$ is contained in the interior of the positive Weyl chamber $\mathfrak{a}_\theta^+$, the set Δ_θ is contained in $\mathcal{B}_\theta(\rho)^+$.

We use the Thermodynamic Formalism for countable Markov shifts to analyze these potentials. In particular, we use a renewal theorem of Kesseböhmer and Kombrink [24] to generalize arguments of Lalley [26] to establish counting and equidistribution results in our setting. We summarize the results we will need from our work below.

Theorem 2.6 (Bray-Canary-Kao-Martone [5]). *Suppose that Γ is a torsion-free, geometrically finite Fuchsian group which is not convex cocompact, $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a P_θ-Anosov representation and $\phi \in \mathcal{B}_\theta(\rho)^+$. Then, there exists a locally Hölder continuous function $\tau_\rho^\phi = \phi\circ\tau_\rho : \Sigma^+ \to \mathbb{R}$ such that*

1. *τ_ρ^ϕ is eventually positive, i.e. there exist $N \in \mathbb{N}$ and $B > 0$ such that $S_n\tau_\rho^\phi(x) > B$ for all $n \geqslant N$ and $x \in \Sigma^+$.*
2. *There exists $d(\phi) > 0$, so that $h \to P(-h\tau_\rho^\phi)$ is finite, proper and strictly monotone on $(d(\phi),\infty)$ and infinite otherwise.*
3. *There exists $C_\rho > 0$, and for all $x_1 \in \mathcal{A}$, $c(\rho,\phi,x_1) \geq 1/d(\phi)$ so that if $x \in \Sigma^+$, then*
$$\left|\tau_\rho^\phi(x) - c(\rho,\phi,x_1)\log r(x_1)\right| \leq C_\rho.$$
4. *If $x = \overline{x_1\cdots x_n}$ is a periodic element of Σ^+, then*
$$S_n\tau_\rho^\phi(x) = \ell_\rho^\phi\big(G(x_1)\cdots G(x_n)\big).$$
5. *The ϕ-entropy $h^\phi(\rho)$ of ρ is the unique solution of $P(-h\tau_\rho^\phi) = 0$. Moreover,*
$$\lim_{T\to\infty} \frac{h^\phi(\rho)TR_T^\phi(\rho)}{e^{h^\phi(\rho)T}} = 1.$$
6. *There is a unique equilibrium measure m_ρ^ϕ for $-h^\phi(\rho)\tau_\rho^\phi$.*

We also established a rigidity theorem for renormalized pressure intersection and use our equidistribution result to give a thermodynamical reformulation of the pressure intersection.

Theorem 2.7 (Bray-Canary-Kao-Martone [5])**.** *If $\rho, \eta : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ are P_θ-Anosov representations of a geometrically finite Fuchsian group and $\phi \in \Delta_\theta$, then*

$$J^\phi(\rho, \eta) \geq 1$$

with equality if and only if

$$\ell^\phi_\rho(\gamma) = \frac{h_\phi(\eta)}{h_\phi(\rho)} \ell^\phi_\eta(\gamma)$$

for all $\gamma \in \Gamma$. Moreover,

$$I^\phi(\rho, \eta) = \frac{\int_{\Sigma^+} \tau^\phi_\eta \; dm^\phi_\rho}{\int_{\Sigma^+} \tau^\phi_\rho \; dm^\phi_\rho}$$

and $-I^\phi(\rho, \eta)$ is the slope of the tangent line at $(h^\phi(\rho), 0)$ to

$$\mathcal{C}^\phi(\rho, \eta) = \{(a, b) \in \mathbb{R}^2 \mid P(-a\tau^\phi_\rho - b\tau^\phi_\eta) = 0, a \geqslant 0, b \geqslant 0, a + b > 0\}.$$

2.4. **Hitchin representations.** We say that a basis $b = (b_1, \ldots, b_d)$ is *consistent* with a pair (F, G) of transverse flags if $\langle b_i \rangle = F^i \cap G^{d-i+1}$ for all i. We denote by $U(b)_{>0} \subset \mathsf{SL}(d, \mathbb{R})$ the subsemigroup of upper triangular unipotent matrices which are totally positive with respect to b, i.e. $A \in U(b)_{>0}$ if, in the basis b, A is upper triangular unipotent and the determinants of all the minors of A are positive, unless they are forced to be zero by the fact that A is upper triangular.

Then, a k-tuple $(F_1, \ldots, F_k)$ in $\mathcal{F}_d$ is *positive* if there exists a basis b consistent with (F_1, F_k) and there exists $\{u_2, \ldots, u_{k-1}\} \in U(b)_{>0}$ so that $F_i = u_{k-1} \cdots u_i F_k$ for all $i = 2, \ldots, k-1$. If X is a subset of S^1, we say that a map $\xi : X \to \mathcal{F}_d$ is *positive* if whenever $(x_1, \ldots, x_k)$ is a consistently ordered k-tuple in X (ordered either clockwise or counter-clockwise), then $(\xi(x_1), \ldots, \xi(x_k))$ is a positive k-tuple of flags.

Let Γ be a geometrically finite Fuchsian group and let $\Lambda(\Gamma) \subset \partial \mathbb{H}^2$ be its limit set. Following Fock and Goncharov [14], a *Hitchin representation* $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ is a representation such that there exists a ρ-equivariant positive map $\xi_\rho : \Lambda(\Gamma) \to \mathcal{F}_d$. If S is closed, Hitchin representations are just the traditional Hitchin representations introduced by Hitchin [19] and further studied by Labourie [25]. When Γ contains a parabolic element, we sometimes refer to these Hitchin representations as cusped Hitchin representations to distinguish them from the traditional Hitchin representations.

Canary, Zhang and Zimmer [11] proved the following important structural results. (Sambarino [41] independently showed that Hitchin representations are strongly irreducible.)

Theorem 2.8. *If $\rho \in \mathcal{H}_d(\Gamma)$, then ρ is $\{1, \ldots, d-1\}$-Anosov and strongly irreducible.*

We recall that Sambarino [41, Theorem A] classified the possible Zariski closures of images of Hitchin representations.

Theorem 2.9 (Sambarino [41])**.** *Suppose that $\Gamma \subset \mathsf{PSL}(2, \mathbb{R})$ is a geometrically finite Fuchsian group, and $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ is a Hitchin representation. Then the Zariski closure* *of* $\rho(\Gamma)$ *either lies in an irreducible image of* $\mathsf{PSL}(2,\mathbb{R})$ *or is conjugate to either* $\mathsf{PSL}(d,\mathbb{R})$, $\mathsf{PSp}(2n,\mathbb{R})$ *when* $d=2n$, $\mathsf{PSO}(n,n-1)$ *when* $d=2n-1$, *or* G_2 *when* $d=7$.

Historical remarks: The results in this subsection generalize earlier results in the case when Γ is convex cocompact. More precisely, when Γ is convex cocompact, Theorem 2.4 follows from work of Labourie [25], Fock and Goncharov [14], Guichard and Wienhard [17], Kapovich, Leeb and Porti [22, 23], Guérituad, Guichard, Kassel, and Wienhard [15] and Tsouvalas [45], Theorems 2.5 and 2.7 are due to Bridgeman, Canary, Labourie, and Sambarino [7] and Theorem 2.6 is due to Sambarino [40]. Finally, Theorem 2.8 is due to Labourie [25].

Anosov representations of geometrically finite Fuchsian groups are also relatively Anosov in the sense of Kapovich and Leeb [21] and relatively dominated in the sense of Zhu [48]. In particular, one can derive Theorem 2.4 in either of their settings. The approach in [11] was motivated by the need to prove Theorem 2.5, whose proof was not clear from either pre-existing viewpoint. Zhu and Zimmer [49] have now generalized the techniques of [11] to establish a generalization of Theorem 2.5 to the setting of all relatively Anosov representations.

3. Entropy, intersection and the pressure form

Our pressure form is defined as the Hessian of a renormalized intersection function, so it is crucial to show that this function is analytic (or at least C^2). Let $\widetilde{\mathcal{P}}_\theta(\Gamma,d)$ be the space of P_θ-Anosov representations of Γ into $\mathsf{PSL}(d,\mathbb{R})$.

Theorem 3.1. *If* $\tilde W$ *is an analytic submanifold of* $\widetilde{\mathcal{P}}_\theta(\Gamma,d)$ *and* $\phi\in\Delta_\theta$, *then* $h^\phi(\rho)$ *varies analytically over* $\tilde W$ *and* I^ϕ *and* J^ϕ *vary analytically over* $\tilde W\times\tilde W$. *Moreover, if* $\rho,\eta\in\tilde W$, *then*

$$J^\phi(\rho,\eta)\geq 1$$

and $J^\phi(\rho,\eta)=1$ *if and only if* $\ell^\phi_\rho(\gamma)=\frac{h^\phi(\eta)}{h^\phi(\rho)}\ell^\phi_\eta(\gamma)$ *for all* $\gamma\in\Gamma$.

Proof. If Γ is convex cocompact, then this result is established in [7]. So we will assume that Γ is geometrically finite but not convex cocompact for the rest of this proof.

Theorem 2.5 implies that the limit map ξ_ρ varies analytically over $\tilde W$. Since $\tau_\rho(x)=B_\theta(\rho(G(x_1)),\rho(G(x_1))^{-1}(\xi_\rho(\omega(x))))$ and B_θ is analytic we see that τ_ρ, and hence $\tau^\phi_\rho=\phi\circ\tau_\rho$, varies analytically over $\tilde W$. It then follows from Theorem 2.6 and Theorem 2.1 that $(h,\rho)\to P(-h\tau^\phi_\rho)$ is analytic on $(d(\phi),\infty)\times\tilde W$. Since $P(-h^\phi(\rho)\tau^\phi_\rho)=0$ and

$$\frac{d}{dt}P(-t\tau^\phi_\rho)\Big|_{t=h^\phi(\rho)}=-\int\tau^\phi_\rho\ dm^\phi_\rho<0$$

for all $\rho\in\tilde W$, the Implicit Function Theorem implies that $h^\phi(\rho)$ varies analytically over $\tilde W$.

Let

$$R=\tilde W\times\tilde W\times\hat D_\phi\qquad\text{where}\qquad \hat D_\phi=\left\{(a,b)\in\mathbb{R}^2\ |a+b>d(\phi)\right\}$$

and let $P_R : R \to \mathbb{R}$ be given by $P_R(\rho,\eta,a,b) = P(-a\tau^\phi_\rho - b\tau^\phi_\eta)$. Mauldin and Urbanski [32, Thm. 2.1.9] show that if f is locally Hölder continuous, then $P(f)$ is finite if and only if $Z_1(f) < +\infty$, where

$$Z_1(f) = \sum_{s\in\mathcal{A}} e^{\sup\{f(x)\ :\ x_1=s\}} < +\infty.$$

By grouping the terms so that $r(s) = n$, Theorem 2.6 implies that

$$Z_1(-a\tau^\phi_\rho - b\tau^\phi_\eta) \le \sum_{n=1}^{\infty} e^{-(a+b)\left(\frac{1}{d(\phi)}\log n - \max\{C_\rho, C_\eta\}\right)}$$

so $P(-a\tau^\phi_\rho - b\tau^\phi_\eta) < +\infty$ if $a+b > d(\phi)$. Therefore, P_R is finite on R, and hence, by Theorem 2.1, analytic on R. As above, P_R is a submersion on $P_R^{-1}(0)$, so $P_R^{-1}(0)$ is an analytic submanifold of R. Moreover, by Theorem 2.7, $-I^\phi(\rho,\eta)$ is the slope of the tangent line to $P_R^{-1}(0)\cap\{(\rho,\eta)\times \hat{D}_\phi\}$ at the point $(\rho,\eta,(h(\rho),0))$, so $I^\phi(\rho,\eta)$ is analytic. Since entropy is analytic, it follows immediately that $J^\phi(\rho,\eta)$ is analytic.

The final claim follows directly from Theorem 2.7. □

Let $\widetilde{\mathcal{P}}^{irr}_\theta(\Gamma,d)$ be the set of irreducible representations in $\widetilde{\mathcal{P}}_\theta(\Gamma,d)$ and let $\mathcal{P}^{irr}_\theta(\Gamma,d) = \widetilde{\mathcal{P}}^{irr}_\theta(\Gamma,d)/\mathsf{PSL}(d,\mathbb{R})$. The argument of [7, Proposition 7.1] shows that the action of $\mathsf{PSL}(d,\mathbb{R})$ on $\widetilde{\mathcal{P}}^{irr}_\theta(\Gamma,d)$ is free, proper and analytic. It follows that if W is an analytic submanifold of $\mathcal{P}^{irr}_\theta(\Gamma,d)$, then its pre-image $\tilde{W}$ is an analytic submanifold of $\widetilde{\mathcal{P}}_\theta(\Gamma,d)$.

In this setting, we get the following result which generalizes Theorem 1.2 from the introduction.

Corollary 3.2. *If W is an analytic submanifold of $\mathcal{P}^{irr}_\theta(\Gamma,d)$ and $\phi\in\Delta_\theta$, then $h^\phi(\rho)$ varies analytically over W and I^ϕ and J^ϕ vary analytically over $W\times W$. Moreover, if $\rho,\eta\in W$, then*

$$J^\phi(\rho,\eta)\ge 1$$

and $J^\phi(\rho,\eta)=1$ if and only if $\ell^\phi_\rho(\gamma) = \frac{h^\phi(\eta)}{h^\phi(\rho)}\ell^\phi_\eta(\gamma)$ for all $\gamma\in\Gamma$.

Given $\phi\in\Delta_\theta$, we define a pressure form on any analytic submanifold W of $\mathcal{P}^{irr}_\theta(\Gamma,d)$ by letting

$$\mathbb{P}^\phi\Big|_{\mathsf{T}_\rho W} = \mathrm{Hess}(J^\phi(\rho,\cdot)).$$

If $\vec{v} = \frac{d}{dt}\big|_{t=0}[\rho_t]$ where $\{\rho_t\}_{t\in(-\epsilon,\epsilon)}$ is a one-parameter analytic family in W, then

$$\mathbb{P}^\phi(\vec{v},\vec{v}) = \frac{d^2}{dt^2}\Big|_{t=0} J^\phi(\rho_0,\rho_t).$$

We note that the exact same definitions apply when Γ is a cocompact lattice, see [9, Sect. 5.5]. We observe the following immediate properties.

Proposition 3.3. *If W is an analytic submanifold of $\mathcal{P}^{irr}_\theta(\Gamma,d)$ and $\phi\in\Delta_\theta$, then $\mathbb{P}^\phi$ is analytic and non-negative, i.e. if $\vec{v}\in\mathsf{T}W$, then $\mathbb{P}^\phi(\vec{v},\vec{v})\ge 0$. Moreover, if M is a subgroup of the mapping class group of Γ and W is M-invariant, then $\mathbb{P}^\phi$ is M-invariant.*

Proof. The pressure form $\mathbb{P}^\phi$ is analytic, since J^ϕ is analytic and is non-negative since J^ϕ achieves its minimum along the diagonal, see Theorem 2.7. If $\psi \in M$, then $\ell^\phi_\gamma(\rho\circ\psi) = \ell^\phi_{\psi^{-1}(\gamma)}(\rho)$, so $R^\phi_T(\rho\circ\psi) = \psi^{-1}\big(R^\phi_T(\rho)\big)$ and $h^\phi(\rho) = h^\phi(\psi\circ\rho)$, so $J^\phi(\rho,\eta) = J^\phi(\rho\circ\psi,\eta\circ\psi)$, which implies that $\mathbb{P}^\phi$ is ψ-invariant. □

The following degeneracy criterion for pressure metrics is standard in the setting of finite Markov shifts, see for example [9, Cor. 2.5], but requires a little more effort in the setting of countable Markov shifts. In our setting, this criterion is established exactly as in Lemma 8.1 in [4].

Proposition 3.4. *Suppose that W is an analytic submanifold of $\mathcal{P}^{irr}_\theta(\Gamma,d)$ and $\phi\in\Delta_\theta$. If $\vec v \in \mathsf{T}W$ and $\phi \in \Delta_\theta$, then $\mathbb{P}^\phi(\vec v,\vec v) = 0$ if and only if*

$$\mathrm{D}_{\vec v}\left(h^\phi \ell^\phi_\gamma\right) = 0$$

for all $\gamma\in\Gamma$.

We next observe that $\mathrm{D}_{\vec v}\log \ell^\phi_\gamma$ is independent of γ if $\vec v$ is degenerate and γ is hyperbolic.

Lemma 3.5. *Suppose that W is an analytic submanifold of $\mathcal{P}^{irr}_\theta(\Gamma,d)$ and $\phi\in\Delta_\theta$. If $\vec v\in\mathsf{T}W$ and $\mathbb{P}^\phi(\vec v,\vec v)=0$, then,*

$$\mathrm{D}_{\vec v}\ell^\phi_\gamma = K\ell^\phi_\gamma$$

for all $\gamma\in\Gamma$, where $K = -\frac{\mathrm{D}_{\vec v}h^\phi}{h^\phi(\rho)}$.

Proof. By Proposition 3.4, if $\vec v$ is degenerate, then

$$h^\phi(\rho)\mathrm{D}_{\vec v}\ell^\phi_\gamma = -\big(\mathrm{D}_{\vec v}h^\phi\big)\ell^\phi_\gamma$$

for all hyperbolic $\gamma\in\Gamma$, so $\mathrm{D}_{\vec v}\ell^\phi_\gamma = K\ell^\phi_\gamma$ for all hyperbolic $\gamma\in\Gamma$. If $\gamma\in\Gamma$ is parabolic, then ℓ^ϕ_γ is the zero function, so the condition holds trivially. □

Recall that if M is a real analytic manifold, an analytic function $f : M \to \mathbb{R}$ has *log-type* K at $v\in\mathsf{T}_uM$ if $f(u)\neq 0$ and

$$\mathrm{D}_u\log(|f|)(v) = K\log(|f(u)|).$$

In this language, Lemma 3.5 implies that if Λ^ϕ_γ is the function defined by $\rho\to e^{\ell^\phi_\rho(\gamma)}$ and $\mathbb{P}^\phi(\vec v,\vec v)=0$ then there exists K so that Λ^ϕ_γ has log-type K at $\vec v$ for all $\gamma\in\Gamma$.

4. The spectral radius pressure metric

We are now ready to establish our generalization of the main theorem of [7].

Theorem 1.6. *If W is an analytic submanifold of $\mathcal{P}^{irr}_{1,d-1}(\Gamma,d)$, H is a reductive subgroup of $\mathsf{PSL}(d,\mathbb{R})$ and every representation in W has image in H and is H-generic, then $\mathbb{P}^{\omega_1}|_{\mathsf{T}W}$ is an analytic Riemannian metric on W. Moreover, if W is invariant under a subgroup M of the mapping class group, then $\mathbb{P}^{\omega_1}|_{\mathsf{T}W}$ is M-invariant.*

Since every cusped Hitchin representation is irreducible and $\mathsf{SL}(d,\mathbb{R})$-generic, Theorem 1.3 follows immediately from Theorem 1.6.

Proof of Theorem 1.6. Proposition 3.3 implies that we only need to prove that every non-zero vector $\vec{v} \in \mathsf{T}W$ is non-degenerate. Suppose $\vec{v}_0 \in \mathsf{T}_{\rho_0}W$ is a degenerate vector. Lemma 3.5 implies that there exists K so that $\Lambda^{\omega_1}_\gamma$ has log-type K at $\vec{v}_0$ for all $\gamma \in \Gamma$. Our first step consists of showing that $K = 0$.

Let $\rho_0(\alpha)$ be an H-generic element. We claim that α is hyperbolic and, consequently that $\rho_0(\alpha)$ is biproximal. If not, α is parabolic, and $\rho_0(\alpha)$ is weakly unipotent, see Theorem 2.4 part (2). In particular, $\rho_0(\alpha)$ is not diagonalizable over $\mathbb{C}$, so its centralizer cannot be a maximal torus, which contradicts the assumption that $\rho_0(\alpha)$ is H-generic.

Let β be an element of Γ, so that α and β generate a free convex cocompact subgroup Γ_0 of Γ. Lemma 2.3 then implies that the restriction $\rho_0|_{\Gamma_0}$ is $P_{\{1,d-1\}}$-Anosov in the traditional sense. One may then choose an open neighborhood W_0 of ρ_0 in W so that if $\rho \in W_0$, then $\rho|_{\Gamma_0}$ is $P_{\{1,d-1\}}$-Anosov and $\rho(\alpha)$ is H-generic. We then can consider the analytic family $\{\rho|_{\Gamma_0}\}_{\rho \in W_0}$ and apply [7, Lemma 9.8] to conclude that $K = 0$.

Next, we show that if $\Lambda^{\omega_1}_\gamma$ is log-type zero at $\vec{v}_0$ for all $\gamma \in \Gamma$, then $\vec{v}_0 = 0$.

Recall that Labourie defines a continuous cross-ratio $\mathbf{b}$ on pairs of mutually transverse hyperplanes (P_1, P_2) and lines (L_1, L_2) by setting

$$\mathbf{b}(P_1, P_2, L_1, L_2) = \frac{\phi_1(\vec{v}_1)\phi_2(\vec{v}_2)}{\phi_1(\vec{v}_2)\phi_2(\vec{v}_1)}$$

where ϕ_1 and ϕ_2 are linear functionals with kernel P_1 and P_2 and $\vec{v}_1$ and $\vec{v}_2$ are non-zero vectors in L_1 and L_2. One may then define a continuous cross-ratio $\mathbf{b}_\rho : \Lambda(\Gamma)^{(4)} \to \mathbb{R}$, where $\Lambda(\Gamma)^{(4)}$ is the set of pairwise distinct quadruples in $\Lambda(\Gamma)$, by setting

$$\mathbf{b}_\rho(x, y, z, w) = \mathbf{b}(\xi^{d-1}_\rho(x), \xi^{d-1}_\rho(y), \xi^1_\rho(z), \xi^1_\rho(w)).$$

Note that $\mathbf{b}_\rho$ is well-defined because ρ is $P_{\{1,d-1\}}$-Anosov. Suppose that (α, β) is a pair of hyperbolic elements of Γ generating a rank two Schottky subgroup of Γ, then $(\rho(\alpha), \rho(\beta))$ generate a projective Anosov Schottky group, so [7, Prop. 10.4] gives that

$$\mathbf{b}_\rho(\alpha^-, \beta^-, \beta^+, \alpha^+) = \lim_{n\to\infty} \frac{\Lambda^{\omega_1}_{\alpha^n\beta}(\rho)}{\Lambda^{\omega_1}_{\alpha^n}(\rho)}.$$

It follows that $\mathbf{b}_\rho(\alpha^-, \beta^-, \beta^+, \alpha^+)$ is of log-type zero (since ratios of log-type zero functions are log-type zero, as are limits of log-type zero functions). Since such quadruples are dense in $\Lambda(\Gamma)^{(4)}$, it follows that $\mathbf{b}_\rho(x, y, z, w)$ is log-type zero for all quadruples in $\Lambda(\Gamma)^{(4)}$.

Given a projective frame $F = (L_1, \ldots, L_{d+1})$ for $\mathbb{P}(\mathbb{R}^d)$ and a projective frame $F^* = (P_1, \ldots, P_{d+1})$ for $\mathbb{P}((\mathbb{R}^d)^*)$, one can define a smooth injective immersion

$$\mu_{F,F^*} : \mathsf{PSL}(d, \mathbb{R}) \to W(d)$$

where $W(d)$ is the quotient of the space of $(d+1) \times (d+1)$ matrices via a multiplicative action of $(\mathbb{R} - \{0\})^{2(d+1)}$ whose action on the coefficients of the matrix is given by $(a_1, \cdots, a_{d+1}, b_1, \cdots, b_{d+1})(M_{ij}) = a_i b_j M_{ij}$ (see [7, Section 10.2]). (Recall that a projection frame for $\mathbb{P}(\mathbb{R}^d)$ is a collection of $d+1$ lines so that no d lines are contained in any hyperplane.) Specifically one chooses non-zero vectors $\vec{v}_i \in L_i$ and covectors $\phi_i \in P_i$ so that $\sum \vec{v}_i = 0$ and $\sum \phi_i = 0$ and defines

$$\mu_{F,F^*}(A) = [\phi_i(A(\vec{v}_j))].$$

This smooth injective immersion and the cross ratio $\mathbf{b}_\rho$ are related by the following crucial property, whose proof proceeds exactly as in [7, Lemma 10.7].

Lemma 4.1. *Suppose* $\{x_1,\ldots,x_{d+1}\}$ *and* $\{y_1,\ldots,y_{d+1}\}$ *are two collections of pairwise distinct points in* $\Lambda(\Gamma)$, *and that* $F=\{\xi^1_\rho(x_1),\ldots,\xi^1_\rho(x_{d+1})\}$ *and* $F^*=\{\xi^{d-1}_\rho(y_1),\ldots,\xi^{d-1}_\rho(y_{d+1})\}$ *are projective frames. Then*

$$\mu_{F,F^*}(\rho(\alpha))=[\mathbf{b}_\rho(y_i,z,\alpha(x_j),w)]$$

for arbitrary $z,w\in\Lambda(\Gamma)$ *and for all* $\alpha\in\Gamma$.

The remainder of the proof then simply mimics the proof of [7, Lemma 10.8].

Let $\{\rho_t\}$ be a path in W so that $\frac{d}{dt}\rho_t|_{t=0}=\vec{v}_0$. Since ρ is irreducible, see [7, Lemma 2.17], we may choose $\{x_1,\ldots,x_{d+1}\}$ and $\{y_1,\ldots,y_{d+1}\}$ so that their images $F=\{\xi^1_\rho(x_1),\ldots,\xi^1_\rho(x_{d+1})\}$ and $F^*=\{\xi^{d-1}_\rho(y_1),\ldots,\xi^{d-1}_\rho(y_{d+1})\}$ are projective frames. Let $F_t=\{\xi^1_{\rho_t}(x_1),\ldots,\xi^1_{\rho_t}(x_{d+1})\}$ and $F^*_t=\{\xi^{d-1}_{\rho_t}(y_1),\ldots,\xi^{d-1}_{\rho_t}(y_{d+1})\}$. We may assume, by restricting the path, that F_t and F^*_t are projective frames, and, by conjugating, that $F_t=F_0$ for all t.

Then $\mu_{F_t,F_t^*}(\rho_t(\alpha))=[\mathbf{b}_{\rho_t}(y_i,z,\alpha(x_j),w)]$, for all $\alpha\in\Gamma$ and all t. So, since our cross-ratios have log type zero, we see that

$$\frac{d}{dt}\Big|_{t=0}\mu_{F_t,F_t^*}(\rho_t(\alpha))=0$$

for all $\alpha\in\Gamma$.

By construction, if $B\in\mathsf{PSL}(d,\mathbb{R})$ and F and F^* are any projective frames, then $\mu_{F,B^*F^*}(A)=\mu_{F,F^*}(B^{-1}A)$. So, if we choose $C_t\in\mathsf{PSL}(d,\mathbb{R})$ so that $(C_t^{-1})^*F_t^*=F_0^*$, then

$$0=\frac{d}{dt}\Big|_{t=0}\mu_{F_t,F_t^*}(\rho_t(\alpha))=\frac{d}{dt}\Big|_{t=0}\mu_{F_0,F_0^*}(C_t\rho_t(\alpha))=D\mu_{F_0,F_0^*}\left(\frac{d}{dt}\Big|_{t=0}C_t\rho_t(\alpha)\right)$$

for all $\alpha\in\Gamma$. Since μ_{F_0,F_0^*} is an immersion, this implies that

$$\frac{d}{dt}\Big|_{t=0}C_t\rho_t(\alpha)=0$$

for all $\alpha\in\Gamma$, so

$$C_0\circ\frac{d}{dt}\Big|_{t=0}\rho_t(\alpha)+\left(\frac{d}{dt}\Big|_{t=0}C_t\right)\circ\rho_t(\alpha)=0$$

for all $\alpha\in\Gamma$. By considering the case where $\alpha=id$, we see that $\dot{C}_0=\left(\frac{d}{dt}\Big|_{t=0}C_t\right)=0$. Since $C_0=I$, we see that $\frac{d}{dt}\Big|_{t=0}\rho_t(\alpha)=0$ for all $\alpha\in\Gamma$, which implies that $\vec{v}_0=0$. □

5. The Hilbert length pressure metric

Theorem 1.6 has the following immediate corollary:

Corollary 5.1. *If* S *is a simple subgroup of* $\mathsf{PSL}(d,\mathbb{R})$ *and* W *is a submanifold of* $\mathcal{H}_d(\Gamma)$ *consisting of representations whose Zariski closure is* S, *then* $\mathbb{P}^{\omega_H}$ *is non-degenerate on* $\mathsf{T}W$.

Proof. Consider the Adjoint representation $Ad : \mathsf{S} \to \mathsf{SL}(V)$ where V is the Lie algebra of S. Then $\mathsf{H} = Ad(\mathsf{S})$ is an irreducible reductive subgroup of $\mathsf{SL}(V)$. Moreover, if $\rho \in W$, then $Ad \circ \rho$ is irreducible and H-generic. Theorem 1.6 implies that the pressure form $\mathbb{P}^{\omega_1}$ is non-degenerate on $\mathsf{T}Ad(W)$.

Note that Ad is an immersion as the adjoint representation $ad\colon V \to \mathfrak{sl}(V)$ is injective. Recall that $\omega_1(Ad \circ \rho(\gamma)) = \omega_H(\rho(\gamma))$. Therefore, $\mathbb{P}^{\omega_H}|_{\mathsf{T}W}$ is the pull-back of $\mathbb{P}^{\omega_1}|_{\mathsf{T}Ad(W)}$ and hence non-degenerate. □

The proof of [7, Lemma 13.1] immediately generalizes to give the following lemma. (See Appendix 7 for a proof.)

Lemma 5.2. *Let W_0 be a smooth manifold and let $W_n \subset W_{n-1} \subset \cdots W_1 \subset W_0$ be a nested collection of submanifolds of W_0 so that W_i has positive codimension in W_{i-1} for all i. Suppose that g is a smooth non-negative symmetric 2-tensor g such that*

- *g is positive definite on $\mathsf{T}_x W_{i-1}$ if $x \in W_{i-1} \setminus W_i$,*
- *the restriction of g to $\mathsf{T}_x W_n$ is positive definite if $x \in W_n$.*

Then the path pseudo metric defined by g is a metric.

One thus obtains a pressure path metric on $\mathcal{H}_d(\Gamma)$ associated to the Hilbert length functional, by applying Sambarino's analysis of the possible Zariski closures of Hitchin representations (see Theorem 2.9).

Theorem 1.5. *Suppose that $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is torsion-free and geometrically finite. If $\vec{v} \in \mathsf{T}\mathcal{H}_d(\Gamma)$ is non-zero, then $\mathbb{P}^{\omega_H}(\vec{v},\vec{v}) = 0$ if and only if $\vec{v}$ is anti-self-dual. Moreover, $\mathbb{P}^{\omega_H}$ gives rise to a mapping class group invariant path metric on $\mathcal{H}_d(\Gamma)$ which is an analytic Riemannian metric off of the self-dual locus.*

Proof. Let $W_0 = \mathcal{H}_d(\Gamma)$. If $n \geq 4$ is even, let $W_1 \subset \mathcal{H}_d(\Gamma)$ be the submanifold of representations whose Zariski closure is conjugate into $\mathsf{PSp}(n,\mathbb{R})$ and let W_2 be the Fuchsian locus (i.e. representations contained in an irreducible image of $\mathsf{PSL}(2,\mathbb{R})$.) If $n \geq 5$ is odd, and $n \neq 7$, let $W_1 \subset \mathcal{H}_d(\Gamma)$ be the submanifold of representations whose Zariski closure is conjugate into $\mathsf{PSO}(n, n-1)$. If $n = 7$, let $W_1 \subset \mathcal{H}_d(\Gamma)$ be the submanifold of representations whose Zariski closure is conjugate into $\mathsf{PSO}(4,3)$, let $W_2 \subset \mathcal{H}_d(\Gamma)$ be the submanifold of W_1 consisting representations whose Zariski closure is conjugate into G_2 and let W_3 be the Fuchsian locus. If $n = 3$, let W_1 be the Fuchsian locus. (Theorem 1.1 implies that W_0 is a manifold and that W_i is always a submanifold of W_{i-1}).

In all cases, Corollary 5.1 implies that $\mathbb{P}^{\omega_H}$ is non-degenerate on $\mathsf{T}_x W_{i-1}$ if $x \in W_{i-1} \setminus W_i$. Therefore, Lemma 5.2 implies that $\mathbb{P}^{\omega_H}$ gives rise to a path metric on $\mathcal{H}_d(\Gamma)$ which is Riemannian off of W_1. □

6. The (first) simple root pressure metric

6.1. **Trace functions.** Recall that an element in $\mathcal{H}_d(\Gamma)$ is a conjugacy class of Hitchin representations of Γ into $\mathsf{PSL}(d,\mathbb{R})$. It will be convenient to identify $\mathcal{H}_d(\Gamma)$ with a subset $\widehat{\mathcal{H}}_d(\Gamma)$ of the character variety

$$X_d(\Gamma) = \mathrm{Hom}(\Gamma, \mathsf{SL}(d,\mathbb{C}))//\mathsf{SL}(d,\mathbb{C}).$$

The character variety $X_d(\Gamma)$ is the biggest Hausdorff quotient of $\mathrm{Hom}(\Gamma, \mathsf{SL}(d,\mathbb{C}))$ by the $\mathsf{SL}(d,\mathbb{C})$-action by conjugation which coincides with the GIT quotient of $\mathrm{Hom}(\Gamma, \mathsf{SL}(d,\mathbb{C}))$ by this same action (See [34]).

If Γ is cocompact, then Hitchin [19] showed that there is a component, $\widehat{\mathcal{H}}_d(\Gamma)$, of $X_d(\Gamma)$ and an analytic diffeomorphism $F : \mathcal{H}_d(\Gamma) \to \widehat{\mathcal{H}}_d(\Gamma)$, so that $F([\rho])$ is the conjugacy class of a lift to $\mathsf{SL}(d,\mathbb{R})$ of ρ.

Let $\widetilde{\mathcal{H}}_d(\Gamma)$ denote the set of all Hitchin representations of Γ in $\mathsf{PSL}(d,\mathbb{R})$. If Γ is not cocompact, then, since Γ is a free group and $\widetilde{\mathcal{H}}_d(\Gamma)$ is an analytic manifold, it is easy to define an analytic map

$$F : \widetilde{\mathcal{H}}_d(\Gamma) \to \mathrm{Hom}(\Gamma, \mathsf{SL}(d,\mathbb{C}))$$

so that $F(\rho)$ is a lift of ρ. Since Hitchin representations are strongly irreducible, see Theorem 2.4, Schur's Lemma implies that $F(\rho)$ is conjugate to $F(\eta)$ in $\mathsf{SL}(d,\mathbb{C})$ if and only ρ and η are conjugate in $\mathsf{PSL}(d,\mathbb{R})$. Then, again since Hitchin representations are strongly irreducible, it follows that F descends to an analytic embedding $\hat{F} : \mathcal{H}_d(\Gamma) \to X_d(\Gamma)$ whose image lies in the smooth part of $X_d(\Gamma)$. We then let $\widehat{\mathcal{H}}_d(\Gamma) = \hat{F}(\mathcal{H}_d(\Gamma))$. Notice that if d is odd, then $\widehat{\mathcal{H}}_d(\Gamma) = \mathcal{H}_d(\Gamma)$.

If $\gamma \in \Gamma$, there is a complex analytic trace function $\mathrm{Tr}_\gamma : X_d(\Gamma) \to \mathbb{C}$ so that $\mathrm{Tr}_\gamma([\rho])$ is the trace of $\rho(\gamma)$. It is well-known that derivatives of trace functions generate the cotangent space at any smooth point, see for example Lubotzky-Magid [29]. The following consequence will be used to verify the non-degeneracy of the first simple root pressure form.

Lemma 6.1. *If $[\rho] \in \widehat{\mathcal{H}}_d(\Gamma)$, then $\{\mathrm{D}_{\vec{v}} \mathrm{Tr}_\gamma \mid \gamma \in \Gamma\}$ spans the cotangent space $\mathsf{T}^*_{[\rho]}\widehat{\mathcal{H}}_d(\Gamma)$.*

Even though Tr_γ is not well-defined on $\mathcal{H}_d(\Gamma)$, we will abuse notation by saying that $\mathrm{D}_{\vec{v}}\mathrm{Tr}_\beta = 0$ for some $\vec{v} \in \mathsf{T}\mathcal{H}_d(S)$ if $\mathrm{D}_{DF(\vec{v})}\mathrm{Tr}_\beta = 0$.

6.2. **Nondegeneracy.** Bridgeman, Canary, Labourie and Sambarino [8] prove that if Γ is a closed surface group, then $\mathbb{P}^{\alpha_1}$ is non-degenerate on $\mathcal{H}_d(\Gamma)$. A key tool in their work is the fact, due to Potrie-Sambarino [37], that the topological entropy $h^{\alpha_1}(\rho) = 1$ if $\rho \in \mathcal{H}_d(\Gamma)$. Canary, Zhang and Zimmer [12] generalized Potrie and Sambarino's result to the setting of torsion-free lattices which are not cocompact.

Theorem 6.2 (Potrie-Sambarino [37] and Canary-Zhang-Zimmer [12])**.** *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is a torsion-free lattice, and $\rho \in \mathcal{H}_d(\Gamma)$, then $h^{\alpha_1}(\rho) = 1$.*

With this result in hand, we are ready to establish the non-degeneracy of the first simple root pressure metric.

Theorem 1.4. *If $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is a torsion-free lattice, then the pressure form $\mathbb{P}^{\alpha_1}$ is non-degenerate, so it gives rise to a mapping class group invariant, analytic Riemannian metric on $\mathcal{H}_d(\Gamma)$.*

Proposition 3.3 and Lemma 6.1 together imply that Theorem 1.4 follows from the following proposition.

Proposition 6.3. *If $\vec{v} \in T_{[\eta]}\mathcal{H}_d(\Gamma)$ and $\mathbb{P}^{\alpha_1}(\vec{v},\vec{v}) = 0$, then $\mathrm{D}_{\vec{v}} \mathrm{Tr}_\beta = 0$ for all $\beta \in \Gamma$.*

Here, we will only sketch the proof, since the proof proceeds exactly as in the proof of [8, Prop. 7.4].

Proof. We again abuse notation by identifying $[\rho]$ with $F([\rho]) \in \widehat{\mathcal{H}}_d(\Gamma)$. Since $h^{\alpha_1}(\rho) = 1$ for all $\rho \in \mathcal{H}_d(\Gamma)$, Proposition 3.4 implies that $\mathrm{D}_{\vec{v}}\ell^{\alpha_1}_\beta = 0$ for all $\beta \in \Gamma$.

If $\alpha \in \Gamma$ is parabolic, then Tr_α is constant on $\widehat{\mathcal{H}}_d(\Gamma)$, so $\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\alpha = 0$.

If β is hyperbolic, we may choose $\alpha \in \Gamma$, so that α is hyperbolic and α and β have non-intersecting axes. We may pass to powers α^n and β^n which generate a Schottky subgroup of Γ. We are then exactly in the setting of the proof of [8, Prop. 7.4] which shows that $\mathrm{D}_{\vec{v}}\lambda_i(\rho(\beta^n)) = \mathrm{D}_{\vec{v}}\lambda_i(\rho(\beta))^n = 0$ for all i. Therefore, $\mathrm{D}_{\vec{v}}\lambda_i(\rho(\beta)) = 0$ for all i, so $\mathrm{D}_{\vec{v}}\,\mathrm{Tr}_\beta = 0$. □

7. Appendix

We prove:

Lemma 5.2. *Let W_0 be a smooth manifold and let $W_n \subset W_{n-1} \subset \cdots W_1 \subset W_0$ be a nested collections of submanifolds of W_0 so that W_i has non-zero codimension in W_{i-1} for all i. Set $W_{n+1} = \emptyset$. Suppose that g is a smooth non-negative symmetric 2-tensor on W_0 such that for every $i = 0, \ldots, n$, the restriction of g to $\mathsf{T}_x W_i$ is positive definite if $x \in W_i \setminus W_{i+1}$. Then, the path pseudo-metric defined by g is a metric.*

Proof. We proceed iteratively to establish the following claim:

Claim. *If $x \in W_i \setminus W_{i+1}$, then x has a neighborhood U whose closure $\overline{U}$ lies in $W_0 \setminus W_{i+1}$, so that if $u \in \overline{U} \setminus \{x\}$, then $d(x,u) > 0$.*

Once we have proved this claim for all i, we will have completed the proof of the lemma.

If $x \in W_0 \setminus W_1$, then if U is any neighborhood of x whose closure $\overline{U}$ is disjoint from W_1, then g is Riemannian on $\overline{U}$. Therefore, our claim is true for $i = 0$.

Next, we suppose that the claim is true for all $i = j < k$, and prove the claim for $i = k$. This establishes the claim for all i.

Let $n_i = \dim W_i$. If $x \in W_k \setminus W_{k+1}$, we may identify some neighborhood U of x with the Euclidean unit ball in $\mathbb{R}^{n_0}$ (centered at $\vec{0}$) so that x is identified with $\vec{0}$. We may assume that the closure $\overline{U}$ of U is compact and disjoint from W_{k+1} and that if $j \le k$, then $W_j \cap \overline{U}$ is identified with the intersection of the closure $D(\vec{0},1)$ of $B(\vec{0},1)$ with $\mathbb{R}^{n_j} \times \{\vec{0}\}^{n_0 - n_j}$. We will work in coordinates for the rest of this proof. We identify $TD(\vec{0},1)$ with $D(\vec{0},1) \times \mathbb{R}^{n_0}$.

Since the restriction of g to $T(W_k \setminus W_{k+1})$ is Riemannian, there exists $r, s > 0$ so that if $\vec{v}$ is a (Euclidean) unit vector in $(W_k \cap D(\vec{0},1)) \times \mathbb{R}^{n_k} \times \{\vec{0}\}^{n_0 - n_k}$, then $s^2 \ge g(\vec{v},\vec{v}) \ge 4r^2$.

Since g is continuous, it follows that, after possibly restricting to a smaller neighborhood of x, we can assume that if $\vec{v}$ is a unit vector in $D(\vec{0},1) \times \mathbb{R}^{n_k} \times \{\vec{0}\}^{n_0 - n_k}$, then $4s^2 \ge g(\vec{v},\vec{v}) \ge r^2$. It follows that the (Euclidean) projection map from $\pi_k \colon D(\vec{0},1) \to W_k$ is K-Lipschitz where $K = \frac{2s}{r}$. Therefore, since the restriction of g to $T(W_k \setminus W_{k+1})$ is Riemannian, it follows that if $u \in U$ and $\pi_k(u) \ne \vec{0}$, then $d(u,x) > 0$. On the other hand, if $\pi_k(u) = \vec{0}$ and $u \ne x$, then $u \in W_0 \setminus W_k$, so, by our iterative assumption, there exists a neighborhood V of u whose closure lies in $W_0 \setminus W_k$, so that if $v \in \overline{V} \setminus \{u\}$, then $d(v,u) > 0$. It follows that there exists $c > 0$ so that if $v \in \partial V$, then $d(u,v) \ge c$. Since $x \notin V$, this implies that $d(x, u) \geq d(\partial V, u) \geq c > 0$. This completes the proof of the claim and hence the lemma. □

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