IHP LECTURES ON HYPERBOLIC 3-MANIFOLDS

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Abstract. These notes are designed to accompany my mini-course on hyperbolic 3-manifolds at IHP in May 2025. The goal is to develop enough understanding of hyperbolic 3-manifolds to understand the background statements and some consequences of the Tameness Theorem, the Ending Lamination Theorem and the Density Theorem. The notes contain much more material than we will be able to cover in the actual mini-course.

I left the four lane highway took a blacktop seven miles
Down by the old country school I went to as a child
Two miles down a gravel road I could see the proud old home
A tribute to a way of life that's almost come and gone.

The roots of my raising run deep
I come back for the strength that I need
And hope comes no matter how far down I sink
The roots of my raising run deep.
——————Tommy Collins[1]

0. Background material

This section consists of background material on hyperbolic space and Teichmüller space. We will not cover this material in the lectures, but we include the material for any participants who are not familiar with this material or would like a reminder.

0.1. The hyperbolic plane

Recall that the upper half-plane model for the hyperbolic plane is given by

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

with Riemannian metric

$$ds^2_{hyp} = \frac{1}{y^2}\, dxdy.$$

Prosaically, if $\vec{v} \in T_{(x,y)}\mathbb{H}^2$, then its hyperbolic length $||\vec{v}||_{hyp} = \frac{|\vec{v}|}{y}$ where $|\vec{v}|$ is the Eucliden length of $\vec{v}$

One may easily check that the y-axis L is a geodesic in this metric, since if $p : \mathbb{H}^2 \to L$ is Euclidean perpendicular projection, then $||Dp(\vec{v})||_{hyp} \leq ||\vec{v}||_{hyp}$ with equality if and only if $\vec{v}$ is vertical. Moreover, segments of L are the only geodesic joining points on L. One may check that Möbius transformations with real co-efficients act as orientation-preserving isometries of $\mathbb{H}^2$, by

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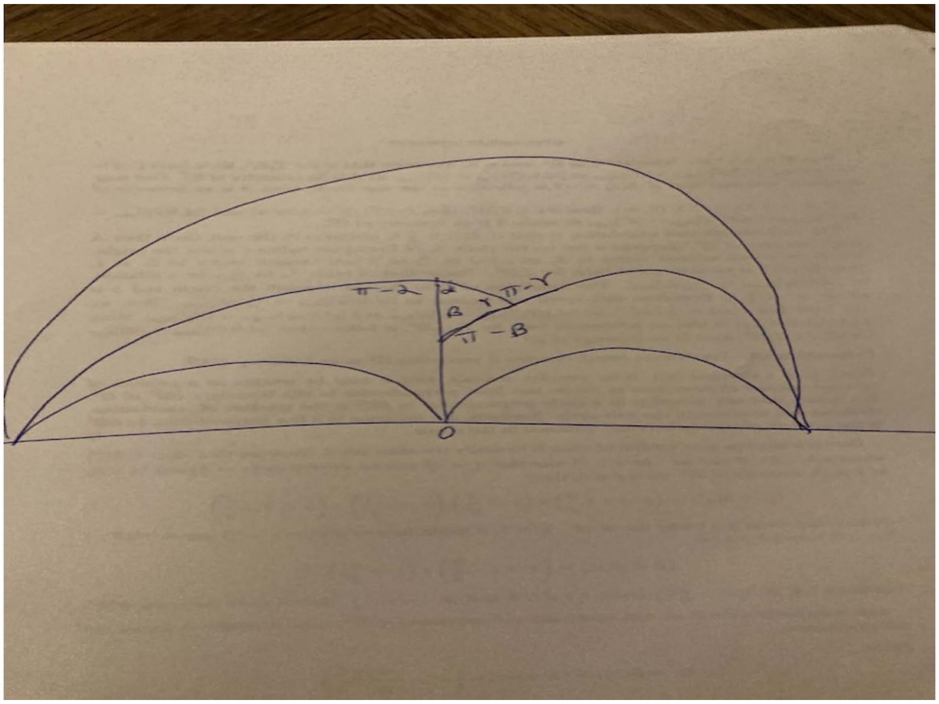
[1]The most famous version of this song is sung by Merle Haggard



a simple calculation. Or you can notice that all such Mobius transformations are generated by maps of the form $z \to a$, $z \to \frac{1}{\bar{z}}$ and $z \to \lambda z$ and checking that each of these maps is an isometry. It follows that all lines and semi-circles perpendicular to the x-axis are geodesics and that these are the only geodesics. Therefore, an orientation-preserving isometry is determined by its action on a single unit tangent vector. Since $\mathsf{PSL}(2,\mathbb{R})$ acts transitively on T^1H^2, we see that

$$\mathrm{Isom}_+(\mathbb{H}^2) = \mathsf{PSL}(2,\mathbb{R}).$$

An ideal triangle is determined by the three geodesics joining any three points in $\partial\mathbb{H}^2$. Since $\mathsf{PSL}(2,\mathbb{R})$ acts transitively on triples of distinct points in $\partial\mathbb{H}^2$, any two ideal triangles are isometric. We say that the angle at an ideal vertex is 0. One may move the end points to 1, -1 and ∞ and compute that the triangle has area π. We say a geodesic triangle is 1/3-ideal if it has two endpoints in $\partial\mathbb{H}^2$ and the other in $\mathbb{H}^2$. We may move the two ideal vertices to -1 and ∞ and arrange that the other vertex lies on the unit circle. If the internal angle at the non-ideal vertex is α, then the vertex must lie at the point $(\cos\alpha, \sin\alpha)$. Hence any two 1/3-ideal triangles with internal angle α are isometric and one can compute that they have area $\pi - \alpha$.

In general if a geodesic triangle T has internal angles α, β and γ, we may assume that one vertex lies at $(0,1)$, one edge emanating from $(0,1)$ travels downward and that the other travels to the right of the y-axis. The following picture then proves that $\mathrm{Area}(T) = \pi - (\alpha+\beta+\gamma)$. (I include this mainly because it is one of my favorite picture proofs.)

More generally, if P is a geodesic n-gon in $\mathbb{H}^2$ with internal angles $\{\alpha_1, \ldots, \alpha_n\}$, then

$$\text{Area}(P) = \pi(n-2) - \sum_{i=1}^{n} \alpha_i.$$

Another prominent model for $\mathbb{H}^2$ is the Poincaré Disk model which is the unit disk D^2 with the metric

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

so if $\vec{v} \in T_{(x,y)}D^2$, then

$$||\vec{v}||_{hyp} = \frac{2|\vec{v}|}{(1 - x^2 - y^2)}.$$

One may check that any Möbius transformation taking the upper half-plane to D^2 is an isometry with respect to the hyperbolic metrrics. For example, one may take $T(z) = \frac{z-i}{z+i}$. It follows that geodesics in this model are lines and semi-circles perpendicular to $\partial D^2 = S^1$. Moreover, the group of orientation-preserving isometries is the group of Möbius transformation which preserve D^2.

The main advantage of this model is the rotational symmetry about the origin. One can compute that if $r \in (0,1)$ and $z \in S^1$, then

$$d_{hyp}(0, rz) = \log \frac{1+r}{1-r} = 2\tanh^{-1}(r).$$

One may then easily compute that the ball of hyperbolic radius R about the origin is a ball of Euclidean radius $\tanh\left(\frac{R}{2}\right)$. Therefore, one may compute that this ball has hyperbolic circumference

$$2\pi \sinh R = 2\pi \frac{\tanh\left(\frac{R}{2}\right)}{1 - \tanh^2\left(\frac{R}{2}\right)}$$

and hyperbolic area

$$2\pi \cosh R - 2\pi = \int_0^R 2\pi \sinh t dt.$$

Since the isometry group of $\mathbb{H}^2$ acts transitively on $\mathbb{H}^2$, every circle of hyperbolic radius R has hyperbolic length $2\pi \sinh R$ and bounds a ball of hyperbolic area $2\pi \cosh R - 2\pi$.

0.2. **Life in the hyperbolic plane**

One tenet of the Thurstonian viewpoint is that one should obtain a visceral feeling for what it is like to love there. Here we explore sports in the hyperbolic plane.

We assume, for simplicity, that the baseball field is a quadrant of a disk with radius 300 feet and that the infield is contained within a disk of radius 100 feet. In Euclidean space, the outfield has area approximately 62,832 square feet. It can be covered by 3 outfielders, so each outfielder covers approximately 20,000 square feet.

In hyperbolic space, the outfield has area $\frac{\pi}{2}\cosh(300) > 10^{100}$. If you assume that each outfielder can still cover 20,000 square feet, you would need more than 10^{94} outfielders to play hyperbolic baseball.

Suppose that you are 300 feet (100 yards) from the pin on the golf course. If you hit the ball exactly 300 feet but one degree off-line, you can use distance along the circle to calculate that you are roughly $\frac{2\pi(300)}{360} = 5.24$ feet from the hole. This estimate is accurate to two significant digits.

In hyperbolic space, the circular estimate would suggest that you are roughly $\frac{2\pi\cosh(300)}{360} > 10^{97}$ feet from the hole, which can't be correct. In fact, you will be over 590 feet from the hole. So hitting it by only one degree off-line is almost as bad as hitting the ball straight backwards.

In Euclidean beachball, a ball of radius one foot which is r feet away takes up roughly $\frac{1}{\pi R}$ of your field of vision (assuming you can see in exactly half the directions). So at 30 feet it takes up roughly 1 percent of your field of vision and at 300 feet it takes up roughly .1 percent of your field of vision.

In hyperbolic beachball, a ball of radius one foot which is r feet away takes up roughly $\frac{1}{\pi \sinh R}$ of your field of vision. So even if you have such good eyesight that you can see things that only take up .01 percent of your field of vision you won't be able to see the beachball if it is more than seven feet from you.

All these calculations are done in American hyperbolic space where the units are feet. In European hyperbolic space the units are meters.

0.3. **Teichmüller space**

A complete orientable Riemannian surface X is said to be **hyperbolic** if it is locally isometric to $\mathbb{H}^2$. In this case, the universal cover $\tilde{X}$ is a simply connected complete Riemannian manifold locally isometric to $\mathbb{H}^2$ and hence can be identified with $\mathbb{H}^2$. Therefore, $X = \mathbb{H}^2/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^2) \cong \mathsf{PSL}(2,\mathbb{R})$. Notice that Γ is only well-defined up to conjugacy, since the identification of $\tilde{X}$ with $\mathbb{H}^2$ is not canonical.

A **marked hyperbolic structure** on a closed oriented surface S is a pair (X, f) where $f : S \to X$ is an orientation-preserving homeomorphism and X is a hyperbolic surface. If $X = \mathbb{H}^2/\Gamma$, then $f_* : \pi_1(S) \to \pi_1(X) \cong \Gamma$ is an isomorphism and hence we obtain a discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$. However, ρ is only well-defined up to conjugation in $\mathsf{PSL}(2,\mathbb{R})$.

One may build a hyperbolic surface of genus two, by starting with a regular hyperbolic octagon, all of whose internal angles are $\frac{\pi}{4}$ and then gluing by the standard gluing pattern. Similarly, one may build a hyperbolic surface of genus g by starting with a regular $(4g-4)$-gon with internal angles $\frac{\pi}{2g}$.

We will choose to formalize Teichmüller space by using representations. Recall that a marked hyperbolic structure on a closed surface S, gives rise to a (conjugacy class of a) discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$. In turn, a discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$ gives rise to a hyperbolic surface $X_\rho = \mathbb{H}^2/\rho(\pi_1(S))$. Since X_ρ is homotopy equivalent to S, it is homeomorphic to S. Moreover, there is homeomorphism $h_\rho : S \to X_\rho$ so that $(h_\rho)_*$ is conjugate to ρ. (Here, we are using a special property of the topology of closed surfaces. The Nielsen-Baer Theorem, see Farb-Margalit [36, Chapter 8], gives that every homotopy equivalence of a closed orientable surface is homotopic to a homeomorphism.) We then let

$$\widetilde{\mathcal{T}}(S) = \{\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R}) \mid \rho \text{ discrete, faithful, and } h_\rho \text{is orientation-preserving}\}$$

and the Teichmüller space of S is the quotient

$$\mathcal{T}(S) = \widetilde{\mathcal{T}}(S)/\mathsf{PSL}(2,\mathbb{R})$$

where $\widetilde{\mathcal{T}}(S)$ inherits a topology as a subset of $\mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$, $\mathsf{PSL}(2,\mathbb{R})$ acts by conjugation and $\mathcal{T}(S)$ inherits the quotient topology.

Alternatively, one may define $\mathcal{T}(S)$ to be the space of marked hyperbolic structure on S up to the equivalence $(X_1, f_1) \sim (X_2, f_2)$ if and only if $f_2 \circ f_1^{-1}$ is homotopic to an isometry. One may think of X as hyperbolic clothing for the naked topological surface S and f as instructions for how to wear the clothing. The equivalence relation allows one to adjust the clothing, but not to wear it backwards or to stick your head through the hole designated for the arm.

It is a classical theorem, going back to the 19th century, that $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$ if $g \geq 2$ is the genus of S. (Notice that $\pi_1(S)$ has a presentation with $2g$ relations and one relation, one would expect that $\mathrm{DF}(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$ has dimension $(2g)3 - 3 = 6g - 3$, so one would predict that Teichmüller space has dimension $6g - 6$.) The *mapping class group* $\mathrm{Mod}(S)$ is the group of (isotopy classes of) self-homeomorphisms of S. Fricke showed that the mapping class group acts properly discontinuously, but not freely, on $\mathcal{T}(S)$ and its quotient is the **moduli space** of unmarked hyperbolic structures on S.

We now give a quick sketch of the Fenchel-Nielsen coordinates on Teichmüller space. Suppose that X is a closed orientable hyperbolic surface of genus $g \geq 2$. Recall that, since X is negatively curved, every homotopically non-trivial closed curve is homotopic to a unique closed geodesic. Morever, if two homotopically non-trivial simple closed curves are disjoint and non-parallel, then their geodesic representatives are also disjoint. Let $\alpha = \{\alpha_1, \ldots, \alpha_{3g-3}\}$ be a maximal collection of disjoint simple closed curves and let α^* be their geodesic representatives on X. The components of $X - \alpha^*$ are a collection of $2g - 2$ hyperbolic pairs of pants with geodesic boundary. (A topological **pair of pants** is a disk with two holes.) Therefore, every closed hyperbolic surface may be built from hyperbolic pairs of pants.

If P is a hyperbolic pair of pants with geodesic boundary and s_1, s_2 and s_3 are the shortest paths joining boundary components (called seams), then $P - \{s_1, s_2, s_3\}$ is a pair of all-right hyperbolic hexagons (i.e. hexagons all of whose interior angles are $\frac{\pi}{2}$). An all-right hexagon is determined by the lengths of any 3 non-consecutive sides. Moreover, any 3 lengths can be achieved. It follows that P is the double of the unique all-right hexagon with alternate sides having lengths agreeing with the lengths of the seams of P. Moreover, we can build a geodesic pair of pants with any collection of boundary lengths and this geodesic pair of pants is entirely determined by its boundary lengths.

So the hyperbolic structure on X is determined, up to isometry, by the lengths of the components of α^* and instructions for gluing the pants together. Since the pants are glued along closed geodesic curves, there is a one-dimensional space of ways to glue them. This suggests more forcefully that the space of hyperbolic structures on X has dimension $6g - 6$.

More formally, we get a map $L : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3}$ where

$$L(X, f) = \Big(\ell_X(f(\alpha_i)^*)\Big)_{i=1}^{3g-3}.$$

At each element of α we can define a twist coordinate in $\mathbb{R}$ which records how the geodesic pairs of pants are glued along $f(\alpha_i)^*$, so we obtain $\Theta : \mathcal{T}(S) \to \mathbb{R}^{3g-3}$. It is natural to think at first that the twist should lie in S^1. One way to see that this is not the case is to observe that because we have marked the surface, we can detect the homotopy class of the shortest curve crossing α. When you make a full positive twist, the shortest such curve changes by a full negative twist (at least if it is unique). One can then see that

$$(L, \Theta) : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \cong \mathbb{R}^{6g-6}$$

is a homeomorphism. For a careful discussion of twist coordinates see, for example, Thurston [76, Section 4.6], Farb-Margalit [36, Section 10.6] or Martelli [51, Chapter 7].

References: Farb and Margalit [36] give a nice treatment of Teichmüller space from a modern geometrical/topological viewpoint. Bers' survey paper [7] is a beautiful treatment of the classical complex analytic approach. Thurston [76, Section 4.6] gives a concise treatment of the Fenchel-Nielson coordinates. Abikoff [1] gives a treatment of the classical theory with an eye towards the modern viewpoint.

0.4. **Hyperbolic 3-space**

The upper half space model for hyperbolic 3-space is given by

$$\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} | t > 0\}$$

with hyperbolic metric given

$$ds^2 = \frac{dx^2 + \cdots + dy^2 + dt^2}{t^2}$$

where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

It is easy to check, just as in $\mathbb{H}^2$, that the t-axis L is a geodesic and that the only geodesics joining points on L are given by segments of L. Möbius transformations (this time with complex co-efficients) extend to orientation-preserving isometries of $\mathbb{H}^3$. One can do this by writing down a painful formula, or by noting that translations in $\mathbb{C}$ extend to horizontal translations in $\mathbb{H}^3$, dilations $z \to \lambda z$ extend to dilations $(z,t) \to (\lambda z, |\lambda| t)$, inversion in the unit circle extends to inversion in the unit sphere and reflection in the y-axis extends to reflection in the $y-t$-plane. One may then check that each of these extensions is an isometry of $\mathbb{H}^3$. It follows that all geodesics in $\mathbb{H}^3$ are semi-circles or lines perpendicular to $\partial\mathbb{H}^3$ and that an isometry is determined by its action on a single orthonormal frame at a point in $\mathbb{H}^3$. Since, the group generated by these inversions, translations, and dilations, acts transitively on the orthonormal frame bundle of $\mathbb{H}^3$, we see that this group is the full isometry group of $\mathbb{H}^3$. In particular, e can identify $\mathrm{Isom}_+(\mathbb{H}^3)$ with $\mathsf{PSL}(2,\mathbb{C})$. Therefore, $\mathbb{H}^3$ has constant sectional curvature, and since it contains a totally geodesic copy of $\mathbb{H}^2$, the constant is -1.

An element of $\mathsf{PSL}(2,\mathbb{C})$ is said to be **hyperbolic** if it fixes two points in $\partial\mathbb{H}^3$ and no points in $\mathbb{H}^3$. In this case it is conjugate to $H_\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ for some $\lambda > 1$.. If $A = BH_\lambda B^{-1}$ for some $B \in \mathsf{PSL}(2,\mathbb{C})$, then A acts as a translation by $2\log\lambda$ along its **axis** which is the B-image of the t-axis. It has **attracting fixed point** $B(\infty)$ and **repelling fixed point** $B(0)$. Every other point in $\mathbb{H}^3$ is moved a distance greater than $2\log\lambda$, which is called the **translation distance** of A.

An element of $\mathsf{PSL}(2,\mathbb{C})$ is said to be **parabolic** if it fixes exactly one point in $\partial\mathbb{H}^3$ and no points in $\mathbb{H}^3$. In this case it is conjugate to $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. If $A = BPB^{-1}$ for some $B \in \mathsf{PSL}(2,\mathbb{C})$, then A has **parabolic fixed point** $B(\infty)$. We say that A has translation distance 0, since

$$\inf_{x \in \mathbb{H}^3} d(x, A(x)) = 0,$$

even though this infimum is not acheived.

An element of $\mathsf{PSL}(2,\mathbb{C})$ is said to be **elliptic** if it fixes a point in $\mathbb{H}^3$. In this case it is conjugate to $E_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for some $\theta \in (0,2\pi)$.. If $A = BE_\theta B^{-1}$ for some $B \in \mathsf{PSL}(2,\mathbb{C})$, then A is a hyperbolic rotation of angle θ about the geodesic which is the B-image of the t-axis.

The Jordan normal form theorem implies that we have classified the non-trivial elements of $\mathsf{PSL}(2,\mathbb{C})$.

1. Convex cocompact hyperbolic 3-manifolds

1.1. Basic definitions

For simplicity, we will assume throughout these lectures that all groups are **finitely generated**, **non-abelian** and **torsion-free**. Selberg's Lemma guarantees that every finitely generated subgroup of any linear group contains a finite-index torsion-free subgroup, so the assumption that groups are torsion-free is not very restrictive.

A **Kleinian group** Γ is a discrete subgroup of $\mathsf{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}_+(\mathbb{H}^3)$. Since we have assumed that Γ is torsion-free,

$$N_\Gamma = \mathbb{H}^3/\Gamma$$

is a hyperbolic 3-manifold.

The **limit set** $\Lambda(\Gamma)$ of a Kleinian group Γ is the set of accumulation points of an orbit in the boundary of $\mathbb{H}^3$, i.e. if $x_0 \in \mathbb{H}^3$, then

$$\Lambda(\Gamma) = \overline{\Gamma(x_0)} - \Gamma(x_0) \subset \partial\mathbb{H}^3.$$

One may easily check that $\Lambda(\rho)$ does not depend on the choice of basepoint x_0 (If $x_0, y_0 \in \mathbb{H}^3$, $\{\gamma_n\} \subset \Gamma$ and $\gamma_n(x_0) \to z \in \partial\mathbb{H}^3$, then $\gamma_n(y_0) \to z \in \partial\mathbb{H}^3$).

For example, if $\Gamma \subset \mathbb{PSL}(2,\mathbb{R}) \subset \mathsf{PSL}(2,\mathbb{C})$ and $\mathbb{H}^2/\Gamma$ is a closed surface, then $\Lambda(\Gamma) = \mathbb{R} \cup \{0\} \subset \partial\mathbb{H}^3$.

A better example is provided by looking at Schottky groups. If $\{C_1, C_2, \ldots, C_{2n-1}, C_{2n}\}$ is a family of disjoint round circles in $\overline{\mathbb{C}} = \partial\mathbb{H}^3$ bounding disjoint (closed) disks $\{D_1, D_2, \ldots, D_{2n-1}, D_{2n}\}$ (whose closures are disjoint in $\mathbb{C}$), then we may construct a free Kleinian groups by letting γ_i be a Möbius transformation taking D_i to $\mathbb{C} - \mathrm{int}(D_{2i})$ for all i. The classical ping-pong lemma implies that $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ is a free group and that if $F = \mathbb{C} - \bigcup_{i=1}^n D_i$ and if $\gamma \neq id \in \Gamma$, then $\Gamma(F)$ is disjoint from F (and that $\gamma_i(F)$ abuts F along ∂D_{2i}). If you think about this picture a bit you can convince yourself that $\overline{\mathbb{C}} - \Gamma(\overline{F})$ is a Cantor set which is, by construction, Γ-invariant (where $\overline{F}$ is the closure of F). In fact, $\Lambda(\Gamma) = \overline{\mathbb{C}} - \Gamma(\overline{F})$.

Lemma 1.1. *If Γ is a Kleinian group, which is not virtually abelian, then Γ acts on $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial\mathbb{H}^3$ as a (discrete)* **convergence group**. *Specifically, if $\{\gamma_n\}$ is a sequence of distinct elements of Γ, then there exists $w, z \in \Lambda(\Gamma)$ (not necessarily distinct) and a subsequence, still called $\{\gamma_n\}$ so that $\gamma_n(x) \to z$ uniformly on compact subsets of $\overline{\mathbb{H}^3} - \{w\}$.*

Proof. Let $\{\gamma_n\}$ is a sequence of distinct elements of Γ and let $x_0 = (0,0,1) \in \mathbb{H}^3$. We may pass to a subsequence so that $\gamma_n(x_0) \to z$ and $\gamma_n^{-1}(x_0) \to w$. If $x \in \mathbb{H}^3$, then $d(\gamma_n(x_0), \gamma_n(x)) = d(x, x_0)$, so since $\gamma_n(x_0) \to z$, we see that $\gamma_n(x) \to z$.

Now suppose that $x \in \partial\mathbb{H}^3 - \{w\}$. Let R_n be the geodesic ray joining $\gamma_n^{-1}(x_0)$ to x. Then R_n converges to the geodesic L joining w to x. It follows that $d(x_0, R_n)$ is uniformly bounded, so there exists $x_n \in R_n$ so that $d(x_n, x_0)$ is bounded. Now consider any convergent subsequence of $\gamma_n(R_n)$, it will be a geodesic ray emanating from x_0. Moreover, since $d(\gamma_n(x_0), \gamma_n(x_n)$ is bounded and $\gamma_n(x_0) \to z$, the ray R must emanate from z. Therefore, $\gamma_n(x) \to z$.

In order to check that this convergence is uniform on compacta, we just need to check that if $\{x_n\}$ is a sequence in $\overline{\mathbb{H}^3} - \{w\}$ converging to $x \in \overline{\mathbb{H}^3} - \{w\}$, then $\gamma_n(x_n) \to z$. If $x \in \mathbb{H}^3$, then $d(x_n, x)$, and hence $d(x_n, x_0)$ is bounded for all large enough n, so $\gamma_n(x_n) \to z$. If $x \in \partial\mathbb{H}^3$, let R_n be the geodesic ray (or segment) joining $\gamma_n^{-1}(x_0) \to x_n$ (for all large enough n) and proceed exactly as before. $\square$

Exercise: *Prove that* $\Lambda(\Gamma)$ *is the closure of the set of fixed points of elements of* Γ*. Observe that if* $\Lambda(\Gamma)$ *is not virtually abelian, then* $\Lambda(\Gamma)$ *is the smallest non-empty closed subset of* $\partial\mathbb{H}^3$ *which is invariant under* Γ*. Moreover, in this case* $\Lambda(\Gamma)$ *is perfect, hence uncountable.*

The complement $\Omega(\Gamma) = \partial\mathbb{H}^3 \setminus \Lambda(\Gamma)$ of the limit set is called the domain of discontinuity.

Lemma 1.2. *If* Γ *is a Kleinian group, then* Γ *acts properly discontinuously on* $\Omega(\Gamma)$*.*

Proof. Let K be a compact subset of $\Omega(\Gamma)$. Suppose there exists a sequence γ_n of distinct elements of Γ so that $\gamma_n(K) \cap K$ is non-empty. Then, by Lemma 1.1, there exists a subsequence $\{\gamma_n\}$ and $w, z \in \Lambda(\Gamma)$ so that $\gamma_n(x) \to z$ uniformly on compact subsets of $\partial\mathbb{H}^3 - \{w\}$. Since K is a compact subset of $\partial\mathbb{H}^3 - \{w\}$, this is a contradiction. $\square$

We can then consider the **conformal boundary**

$$\partial_c N_\Gamma = \Omega(\Gamma)/\rho(\Gamma)$$

which has the structure of a Riemann surface. Notice that the conformal boundary can be empty, for example if N_Γ is cocompact. We can combine the hyperbolic manifold and its conformal boundary to obtain the **conformal bordification**

$$\widehat{N}_\Gamma = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma = N_\Gamma \cup \partial_c N_\Gamma$$

If Γ is a Fuchsian group uniformizing a closed surface S, then $\partial_c N_\Gamma$ is two copies of the surface S (one the quotient of the upper half-plane and the other is the quotient of the lower half plane) and their is an anti-conformal involution of $\partial_c N_\Gamma$ which switches the two surfaces (which is the just the quotient of the map $z \to \bar{z}$). Moreover, $\widehat{N}_\Gamma \cong S \times [0, 1]$.

If Γ is the Schottky group constructed above, then $\overline{F}$ is a fundamental domain for the action of Γ on $\Omega(\Gamma)$, so one may easily check that $\partial_c N_\Gamma$ is a closed surface of genus n. One can further check that $\widehat{N}$ is a handlebody of genus n.

The **convex hull** of the limit set $CH(\Lambda(\Gamma))$ is defined to be the smallest closed convex subset of $\mathbb{H}^3$ containing all bi-infinite geodesics with end points in $\Lambda(\Lambda(\Gamma))$. Concretely, $CH(\Lambda(\Gamma))$ is the union of all ideal tetrahedra with endpoints in the limit set. (The convex hull clearly contains all such ideal tetrahedra and is clearly closed, it only remains that any geodesic joining points in ideal tetrahedra T_1 and T_2 lies in the union of all ideal tetrahedra whose endpoints lie in the union of the endpoints of T_1 and T_2.) The **convex core** is then given by

$$C(N_\Gamma) = CH(\Lambda(\Gamma))/\Gamma \subset N_\Gamma$$

Notice that since $CH(\Lambda(\Gamma))$ is convex, there exists a well-defined retraction, called the **nearest point retraction**,

$$\tilde{r} : \mathbb{H}^3 \to CH(\Lambda(\Gamma))$$

so that $\tilde{r}(x)$ is the unique point on $CH(\Lambda(\Gamma))$ closest to x. (Notice that if $x \neq y$ and $d(z,x) = d(z,y)$ then there exists a point $u \in \overline{xy}$ so that $d(z,u) < d(z,x)$.) Since $CH(\Lambda(\Gamma))$ is $\rho(\Gamma)$-invariant, $\tilde{r}$ is ρ-equivariant, so descends to a retraction, still called the nearest point retraction,

$$r : N_\Gamma \to C(N_\Gamma).$$

It is often useful to consider the closed neighborhood $C_1(N_\rho)$ of radius one of the convex core. $C_1(N_\rho)$ is the quotient of the closed neighborhood of radius one of the convex hull of the limit set. One key feature here is that $C_1(N)$ is strictly convex (since if $d(x,w) = 1$ and $d(y,z) = 1$ and u lies in the interior of $\overline{xy}$, then $d(u, \overline{wz}) < 1$).

Lemma 1.3. *If Γ is a Kleinian group which is not virtually abelian, then $\partial C_1(N_\rho)$ is a C^1-submanifold.*

Proof. Consider the map $f : \mathbb{H}^3 - CH(\Lambda(\rho)) \to (0,\infty)$, given by $f(x) = d(x, r(x))$. By the regular value theorem, It suffices to show that f is a C^1-submersion. If $y \in \mathbb{H}^3 - CH(\Lambda(\Gamma))$, then let P_y be the totally geodesic plane through x which is perpendicular to $\overline{yr(y)}$, which is a support plane for $CH(\lambda(\rho)$ if $g(x) = d(y,x)$ and $h(y) = d(x, P_y)$, then $h(x) \leq f(x) \leq g(x)$ for all x in the same component of $\mathbb{H}^3 - P_x$ as x. Moreover, $f(y) = g(y) = h(y)$ and f and h are differentiable at y and the have the same derivative at y (which is simply the dot product with the unit tangent vector to $\overline{yr(y)}$ pointing towards y), so f is differentiable at y. Notice that f is a submersion at y, since the restriction of f to $\overrightarrow{r(y)y}$ is a submersion. $\square$

Remark: With a little more care, one can show that $\partial C_1(N_\Gamma)$ is actually $C^{1,1}$.

Suppose that $x, y \in N_\Gamma - C_1(N_\Gamma)$ and $\overrightarrow{r(x)x}$ and $\overline{r(y)y}$ intersect $\partial C_1(N_\Gamma)$ at the same point z, then $r(z) = r(y) = r(x)$, so $\overrightarrow{r(x)x} = \overrightarrow{r(y)y} = \overrightarrow{r(z)z}$. It follows that there is a homeomorphism

$$F : N_\Gamma - C(N_\Gamma) \to \partial C_1(N_\Gamma) \times (0,\infty) \quad \text{given by} \quad F(x) = \left(\overrightarrow{r(x)x} \cap \partial C_1(N_\Gamma), d(x, r(x))\right).$$

(Notice that if $r(x) = r(y)$ it need not be the case that $\overrightarrow{r(x)x} = \overrightarrow{r(y)y}$, which is another main reason we work mostly with $C_1(N_\Gamma)$.)

One may continuously extend $\tilde{r}$ to a map $\partial\tilde{r} : \Omega(\Gamma) \to CH(\Lambda(\Gamma))$ by letting $\tilde{r}(z)$ be the first point of contact of an expanding family of horospheres based at z with $CH(\Lambda(\Gamma))$. Again, $\tilde{r}$ descends to a map

$$\partial r : \partial_c N_\Gamma \to C(N_\Gamma).$$

We can then define a homeomorphism

$$\partial F : \partial_c N_\Gamma \to \partial C_1(N_\Gamma) \quad \text{given by} \quad \partial F(z) = \overrightarrow{r(z)z} \cap \partial C_1(N_\Gamma).$$

One may use F and ∂F to construct a homeomorphism

$$G : \widehat{N}_\rho \to C_1(N_\Gamma)$$

where if $x \in C(N_\Gamma)$ then $G(x) = x$, if $x \in N_\Gamma - C(N_\Gamma)$, then

$$G(x) = F^{-1}\left(\overrightarrow{r(x)x} \cap \partial C_1(N), 1 - \frac{1}{1 + d(r, r(x))}\right)$$

and if $z \in \partial_c N_\Gamma$, then $G(z) = \partial F(z)$. So we see:

Proposition 1.4. *If* Γ *is a Kleinian group, then* $\widehat{N}_\Gamma$ *is homeomorphic to* $C_1(N_\Gamma)$*.*

We say that Γ is *convex cocompact* if $C_1(N_\Gamma)$ is compact. Notice that this is equivalent to requiring that either $\widehat{N}_\Gamma$ or $C(N_\Gamma)$ is compact.

If ρ is convex cocompact, then we identified $N_\rho - C_1(N_\rho)$ with $C_1(N_\rho) \times (1,\infty)$. We may then show that in these coordinates $N_\rho - C_1(N_\rho)$ is bilipschitz to the metric

$$\cosh^2 t ds^2_{\partial C_1(N_\rho)} + dt^2$$

where $ds^2_{\partial C_1(N_\rho)}$ is the intrinsic metric on $\partial C_1(N_\rho)$ and t is the real coordinate. (This fact is implicit in the work of Epstein-Marden [35]. See Section 4.2 of Biringer-Souto [11] for a more precise discussion.)

We observe that convex cocompact Kleinian groups contain no parabolic elements.

Corollary 1.5. *If* $N_\Gamma = \mathbb{H}^3/\Gamma$ *is a convex cocompact hyperbolic 3-manifold, then* Γ *contains no elliptic or parabolic elements, so every non-trivial element of* Γ *is hyperbolic.*

Proof. Let Γ be a convex cocompact Kleinian group and let $\pi : \mathbb{H}^3 \to N_\Gamma$ be the obvious covering map. Γ contains no elliptic elements, since Γ acts freely on $\mathbb{H}^3$. If Γ contains a parabolic element α, then there exists a sequence $\{y_n\} \subset CH(\Lambda(\Gamma))$ so that $d(y_n, \alpha(y_n)) \to 0$. (One obtains such a sequence by approaching the parabolic fixed point of α along a geodesic in $CH(\Lambda(\Gamma))$. This implies that the injectivity radius $\mathrm{inj}_{N_\Gamma}(\pi(y_n))$ of $\pi(y_n)$ in N_Γ converges to 0. This is a contradiction since $C(N)$ is compact, $\pi(y_n) \in C(N)$ and every point in N has positive injectivity radius. $\square$

1.2. A characterization of convex cocompactness in terms of quasi-isometries

It will be useful to give another characterization of convex cocompactness. This definition is well-adapted to show that small deformations of convex cocompact groups remain convex cocompact.

Quasi-isometries and quasi-isometric embeddings are natural classes of mappings in the context of geometric group theory. They are generalizations of bilipschitz homeomorphisms and embeddings which ignore the local structure. However, they need not even be continuous. For example, an infinite line is quasi-isometric to both an infinite Euclidean cylinder and to $\mathbb{Z}$ and all compact metric spaces are quasi-isometric. One justification for working in this looser context, is that the natural geometric structure on a group, given by a word metric associated to some (finite) generating set, is only well-defined up to quasi-isometry.

We will always work in the setting of proper length spaces. A metric space is **proper** if all closed metric balls are compact. A proper metric space X is a **length space** if given any $x, y \in X$, then there exists a rectifiable path joining x to y of length $d(x,y)$. If J is an interval in $\mathbb{R}$ and $\alpha : J \to X$ is a path so that $d(\alpha(s), \alpha(t)) = |t-s|$ for all $s, t \in J$, then we say that α is a **geodesic**. Notice that in this case $\alpha([s,t])$ has length $t-s$ if $t > s$. An action of a group Γ on X is **properly discontinuous** if whenever $K \subset X$ is compact, $\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\}$ is finite. (I include this definition since some standard texts in general topology include the non-standard assumption that the group acts freely to the definition of proper discontinuity.)

A map $f : Y \to Z$ between metric spaces is a **quasi-isometric embedding** if there exists $K \geq 1$ and $C \geq 0$ such that

$$\frac{1}{K} d_Y(a,b) - C \leq d_Z(f(a), f(b)) \leq K d_Y(a,b) + C$$

for all $a, b \in Y$. If we want to remember the constants, we say that f is a (K,C)-quasi-isometric embedding. We say that $f : X \to Y$ is a **quasi-isometry** if there exists $K \geq 1$ and $C \geq 0$ so that f is a (K,C)-quasi-isometric embedding and if $y \in Y$, then there exists $x \in X$ so that $d(f(x), y) \leq C$, i.e. f is a quasi-isometric embedding which is coarsely surjective. One may think of quasi-isometric embeddings as bilipschitz embeddings "in the large," where you don't care at all what happens on the "scale" of the additive constant C.

If $f : X \to Y$ is a quasi-isometry, one may define a **quasi-inverse** $g : Y \to X$, i.e. a quasi-isometry so that there exists $\hat{C}$ so that $d_X(x, g(f(x)) \leq \hat{C}$ and $d_Y(y, f(g(y)) \leq \hat{C}$ for all $x \in X$ and $y \in Y$. There is only one sensible way to construct g. Given $y \in Y$, there exists some $x \in X$ so that $d(f(x), y)) \leq C$, and we set $g(y) = x$. If you haven't done so before, I recommend checking the claim that g is a quasi-inverse for yourself. Notice that the quasi-inverse is far from canonical.

If Γ is generated by a finite set S, then we can define an associated **word metric** d_S on S by letting $d_S(\alpha, \beta)$ be the minimum word length of $\alpha^{-1}\beta$ in the generating set S. Notice that the action of Γ on itself by multiplication on the left is an isometric action with respect to d_S. If S and T are two finite generating sets, then the identity map from (Γ, d_S) to (Γ, d_T) is a quasi-isometry.

If $K_{\Gamma,S}$ is the Cayley graph of Γ associated to the finite generating set S, we can metrize K_Γ by giving each edge length 1. In that case, the inclusion map $(\Gamma, d_S) \to K_{\Gamma,S}$ is an isometric embedding. Moreover, one may easily construct a quasi-isometry from $K_{\Gamma,S} \to (\Gamma, d_S)$ by mapping each point to a nearest vertex.

The Milnor-Svarc lemma assures us that if a group acts properly discontinuously and cocompactly on two spaces, then the space is quasi-isometric to the group This allows one to freely study finitely generated groups by studying their actions on spaces. We will later give a proof in our setting.

Lemma 1.6. (Milnor-Svarc Lemma) *If Γ acts properly discontinuously, cocompactly and by isometries on a proper, length space X and $x \in X$ then Γ has a finite generating set S so that the orbit map $\Gamma \to X$ given by $\gamma \mapsto \gamma(x)$, for all $\gamma \in \Gamma$, is a quasi-isometry from (Γ, d_S) to X.*

We now observe that a Kleinian group is convex cocompact if and only if its orbit map is a quasi-isometric embedding.

Proposition 1.7. *A Kleinian group is convex cocompact if and only if its orbit map $\tau_x : \Gamma \to \mathbb{H}^3$ is a quasi-isometric embedding for some (any) $x \in \mathbb{H}^3$*

Proof. The forward direction follows quickly from the Milnor-Svarc lemma. If Γ is convex cocompact, then Γ acts properly discontinuously, cocompactly and by isometries on $CH(\Lambda(\Gamma))$. So, if we pick $x \in CH(\Lambda(\Gamma))$, the Milnor-Svarc lemma implies that the orbit map $\tau_x : \Gamma \to CH(\Lambda(\Gamma))$ given by $\gamma \to \gamma(x)$ is a quasi-isometry with respect to the metric d_S associated to some finite generating set for Γ. Since the inclusion of $\mathcal{CH}(\Lambda(\Gamma))$ into $\mathbb{H}^3$ is an isometric embedding, it follows that $\tau_x : \Gamma \to \mathbb{H}^3$ is a quasi-isometric embedding with respect to the metric d_S.

So, suppose $\tau_x : \Gamma \to \mathbb{H}^3$ is a quasi-isometric embedding into $\mathbb{H}^3$ for some $x \in \mathbb{H}^3$ and the metric d_S associated to some finite generating set for Γ. If $K_{\Gamma,S}$ is the Cayley graph associated to S. we may extend τ_x to a map $\hat{\tau}_x : K_{\Gamma,S} \to \mathbb{H}^3$ by taking each edge of K_Γ to a geodesic (by a map which is proportional to arc length). Since (Γ, d_S) is quasi-isometric to $K_{\Gamma,S}$, $\hat{\tau}_x$ is a (K, C)-quasi-isometric embedding for some $K > 1$ and $C \geq 0$.

We make use of a special case of the Fellow Traveller Property, which we will sketch a proof of later.

Theorem 1.8. (Special case of Fellow Traveller Property*) Given $K \geq 1$ and $C \geq 0$ there exists $R = R(K, C)$ so that if J is a closed interval in $\mathbb{R}$ and $f : J \to \mathbb{H}^3$ is a (K, C)-quasi-isometric embedding, $[a, b] \subset J$ and L is a geodesic in $\mathbb{H}^3$ joining $f(a)$ to $f(b)$, then the Hausdorff distance between L and $f([a, b])$ is at most R.*

Reminder: Suppose that C and D are closed subsets of a metric space Y . We say that the **Hausdorff distance** between C and D is at most R if both

(1) $d(c, D) \leq R$ for all $c \in C$, and
(2) $d(d, C) \leq R$ for all $d \in D$.

Alternatively, one can say that C lies in the (closed) metric neighborhood of radius R of D and D lies in the (closed) metric neighborhood of radius R of C. The Hausdorff distance is symmetric, satisfies the triangle inequality, and equals 0 if and only if $C = D$, but is not truly a distance, since two closed sets can fail to be a finite Hausdorff distance apart.

Recall that $\mathrm{CH}(\Lambda(\Gamma))$ is the union of all ideal tetrahedra in $\mathbb{H}^3$ with endpoints in the limit set. The Fellow Traveller Property implies that there exists $R = R(K, C)$ so that if $[a, b]$ is a geodesic segment in $K_{\Gamma,S}$, then $\hat{\tau}_\rho([a, b])$ is a Hausdorff distance at most R apart from the geodesic $\overline{\hat{\tau}_x(a))\hat{\tau}_x(b)}$ joining $\hat{\tau}_x(a)$ to $\hat{\tau}_x(b)$.

If $z \neq w \in \Lambda(\rho)$, then there exists $\{\gamma_n\}$ and $\{\beta_n\}$ in Γ so that $\tau_x(\gamma_n) \to z$ and $\tau_x(\beta_n) \to w$. Then the geodesic $\overline{\tau_x(\gamma_n)\tau_x(\beta_n)}$ lies in the (closed) neighborhood $\mathcal{N}_R(\hat{\tau}_x(K_{\Gamma,S}))$ of $\hat{\tau}_\rho(K_{\Gamma,S})$ of radius R, for all n. Since $\overline{\tau_x(\gamma_n)\tau_x(\beta_n)} \to \overline{zw}$ we see that $\overline{zw} \subset \mathcal{N}_R(\hat{\tau}_x(C_\Gamma))$.

There exists B so that if T is an ideal tetrahedra in $\mathbb{H}^s$, then every point in T lies within B of an edge of T. Therefore, every point in $\mathcal{CH}(\Lambda(\Gamma))$ lies with $R + B$ of a point in $\hat{\tau}_\rho(K_{\Gamma,S})$. It follows that if D is the diameter of the bouquet of circles $\hat{\tau}_\rho(K_{\Gamma,S})/\Gamma \subset N_\Gamma$, then $C(N_\Gamma)$ has diameter at most $R + B + D$. Since $C(N_\Gamma)$ is a closed subset of the complete hyperbolic 3-manifold N_Γ, this implies that $C(N_\Gamma)$ is compact, and hence that Γ is convex cocompact. $\square$

Exercise: Prove that there exists B so that if T is an ideal tetrahedra in $\mathbb{H}^3$, then every point in T lies within B of an edge of T. (Hint: Triangles have area at most π and T is a union of triangles with every edge contained in a face of T.)

Exercise: If Γ is a finitely generated group and $x, y \in \mathbb{H}^3$, then τ_x is a quasi-isometric embedding if and only if τ_y is a quasi-isometric embedding (although the constants K and C may change). Moreover, if τ_x is a quasi-isometric embedding with respect to a finite generating set S, then it is a quasi-isometric embedding with respect to any finite generating set.

1.3. **Other characterizations**

Beardon and Maskit [6] showed that a Kleinian group is convex cocompact if and only every point in the limit set is a conical limit point. We say that $z \in \Lambda(\rho)$ is a *conical limit point* if whenever $\overrightarrow{xz}$ is a geodesic ray ending at z, there exists R and a sequence $\{\gamma_n\} \subset \pi_1(M)$ so that $\gamma_n(x) \to z$ and $d(\gamma_n(x), \overrightarrow{xz}) \le R$ for all n.

Marden [50] showed that a Kleinian group is convex cocompact if and only if it has a finite-sided convex fundamental domain and contains no parabolic elements.

Exercises: (1) Prove that if Γ is a Kleinian group, which is not virtually abelian, then every point in $\Lambda(\rho)$ is conical if and only if Γ is convex cocompact.

(2) Prove that $z \in \Lambda(\Gamma)$ is conical if and only if there exists a sequence $\{\gamma_n\} \subset \Gamma$ and $a \ne b \in \Lambda(\Gamma)$ so that $\gamma_n(x) \to a$ for all $x \in \overline{\mathbb{H}^3} - \{z\}$ and $\gamma_n(z) \to b$.

Marden [50] showed that a Kleinian group is convex cocompact if and only if it has a finite-sided convex fundamental domain and contains no parabolic elements.

1.4. **A digression on hyperbolic spaces**

We will say that a proper length space X is (Gromov) **δ-hyperbolic** if whenever T is a geodesic triangle in X with sides s_1, s_2 and s_3 and $y \in s_1$, then $d(y, s_2 \cup s_3) \le \delta$. If X is δ-hyperbolic for some δ, we often simply say that it is **Gromov hyperbolic** or simply **hyperbolic**.

The simplest examples of Gromov hyperbolic spaces are trees, which are 0-hyperbolic. The name is motivated, in part, by the observation that $\mathbb{H}^3$ is hyperbolic.

Lemma 1.9. *Hyperbolic space $\mathbb{H}^3$ is $\cosh^{-1}(2)$-hyperbolic for any n.*

Proof. Let T be a geodesic triangle in $\mathbb{H}^3$ with sides s_1, s_2 and s_3. Since any three points in $\mathbb{H}^3$ are contained in a totally geodesic, isometrically embedded copy of $\mathbb{H}^2$, we may assume that $n = 2$.

By the Gauss-Bonnet Theorem, T has area at most π. If $y \in s_1$ and $r = d(y, s_2 \cup s_3)$, then T contains a half-disk D of hyperbolic radius r. Since D has area $\pi \cosh r - \pi$, we see that

$$\pi \cosh r - \pi \le \pi,$$

so $r \le \cosh^{-1}(2) \approx 1.317$. □

We say that a group is **Gromov hyperbolic** if its Cayley graph, with respect to some finite generating set, is a Gromov hyperbolic metric space. It is a consequence of the Fellow Traveler Property that if two spaces X and Y are quasi-isometric, then X is Gromov hyperbolic if and only if Y is Gromov hyperbolic. Since Cayley graphs of a fixed group, with respect to different finite generating sets are quasi-isometric, this notion is well-defined independent of the (finite) choice of generating set. So we obtain the following immediate consequence of Corollary 1.7.

Corollary 1.10. *If Γ is a convex cocompact Kleinian group, then Γ is Gromov hyperbolic.*

Remarks: 1) Actually, $\mathbb{H}^3$ is δ-hyperbolic for $\delta = \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 0.8814$.

2) A stronger notion of negative curvature is given by considering CAT(-1)-spaces. One says that a proper length space is CAT$(-k)$, for some $k \ge 0$, if every geodesic triangle is at least as thin as the triangle with the same lengths in a simply connected, complete Riemannian surface of curvature $-k$. The Comparision Theorem in Riemannian geometry implies that any simply

connected Riemannian manifold with sectional curvature $\leq -k$ is CAT$(-k)$. The above lemma implies that CAT$(-k)$ spaces are $\cosh^{-1}(2)/k^2$-hyperbolic if $k > 0$.

The Fellow Traveller Property we used earlier for hyperbolic spaces, generalizes to the setting of Gromov hyperbolic spaces. Notice that it is far from true in Euclidean geometry.

Theorem 1.11. (Fellow Traveller Property*) Given* (K, C) *and* δ *there exists* R *so that if* X *is* δ*-hyperbolic and* $f : [a, b] \to X$ *is a* (K, C)*-quasi-isometric embedding and* L *is a geodesic joining* $f(a)$ *to* $f(b)$*, then the Hausdorff distance between* L *and* $f([a, b])$ *is at most* R*.*

We sketch the proof in the case when f is a K-Lipschitz, (K, C)-quasi-isometric embedding into $\mathbb{H}^3$. Notice that in this case f is rectifiable and its image has length at most $K|b-a|$. This situation contains all the key ideas of the general proof.

Sketch of Proof: The key observation is that it is "exponentially inefficient" for a path to wander far from the geodesic joining the endpoints. One manifestation of this principle is that if β is a path joining the endpoints of a geodesic of length $2A$ in $\mathbb{H}^3$ and lies entirely outside the ball of radius A about the midpoint x_0, then β has length at least $\pi \sinh A$ (which is the length of the shortest such path in the sphere of radius A about x_0).

We first bound how far any point on L can lie from $f([a, b])$. Choose a point $x_0 \in L$ which lies furthest from $f([a, b])$, i.e.

$$D = d\big(x_0, f([a, b])\big) = \sup\{d\big(x, f([a, b])\big) \mid x \in L\}.$$

Choose a point y on L so that y lies between $f(a)$ and x_0 and $d(y, x_0) = 2D$ (or $y = f(a)$ if $d(f(a), y) \leq 2D$). Choose $s \in [a, b]$ so that $d(f(s), y) \leq D$ (or $s = a$ if $y = f(a)$). Choose a point z on L which lies between x_0 and $f(b)$ and and $d(z, x_0) = 2D$ (or $z = f(b)$ (if $d(f(b), x_0) \leq 2D$.) Choose $t \in [a, b]$ so that $d(f(t), y) \leq D$ (or $t = b$ if $z = f(b)$). We then concatenate a geodesic joining y to $f(s)$, $f([s, t])$ and the geodesic joining $f(t)$ to z to produce a path γ joining y to z. Since $d(f(s), f(t)) \leq 6D$, $|s - t| < 6KD + KC$, and since f is K-lipschitz, $\ell(f([s, t])) \leq 6DK^2 + K^2C$, so

$$\ell(\gamma) \leq 6DK^2 + 2KD + K^2C.$$

Let $\hat{y}$ be the point between x_0 and y so that $d(x_0, \hat{y}) = D$ and let $\hat{z}$ between x_0 and z so that $d(x_0, \hat{z}) = D$, and form a path joining $\hat{y}$ to $\hat{z}$ by appending to γ segments in L joining y to $\hat{y}$ and joining z to $\hat{z}$. Then

$$\ell(\hat{\gamma}) \leq 6DK^2 + K^2C + 4D$$

and $\hat{\gamma}$ lies entirely outside of the ball of radius D about x_0. Therefore,

$$\ell(\hat{\gamma}) \geq \pi \sinh D$$

so

$$D \leq \sinh^{-1}\left(\frac{6DK^2 + KC + 4D}{\pi}\right) = D_0.$$

We now bound the distance from any point on $f([a, b])$ to L. Let $f([s, t])$ be maximal subsegment of $f([a, b])$ which stays outside of an open neighborhood of L of radius D_0. Notice that the subset of L consisting of points within D_0 of $f([a, s])$ is closed and the subset of L consisting of points within D_0 of $f([t, b])$ is closed. On the other hand their union is all of L, by the previous paragraph, so, since L is connected, their intersection is non-empty. So, there exists $r \in [a, s]$, $u \in [t, b]$ and $w \in L$ so that $d(w, f(r)) \leq D_0$ and $d(w, f(u)) \leq D_0$.

Since $d(f(r), f(u)) \leq 2D_0$, we see that $|r-u| \leq 2KD_0 + KC$ and, since f is K-lipschitz,

$$\ell(f([u,r])) \leq 2K^2 D_0 + K^2 C$$

so if $q \in [s,t] \subset [r,u]$, then

$$d(f(q), L) \leq D_0 + K^2 D_0 + \frac{K^2 C}{2} = R.$$

Therefore, the Hausdorff distance between $f([a,b])$ and L is at most R. □

The following special case of the Milnor-Svarc lemma assures us that it suffices for our purposes to consider the case f is a K-Lipschitz, (K,C)-quasi-isometric embedding into $\mathbb{H}^3$.

Lemma 1.12. (Specialized Milnor-Svarc Lemma) *If Γ is a convex cocompact Kleinian group and $x \in CH(\Lambda(\Gamma))$, then there exists a finite generating set S for Γ, $K \geq 1$ and $C \geq 0$ so that $\hat{\tau}_x : K_{\Gamma,S} \to CH(\Lambda(\Gamma))$ is K-bilipschitz and a (K,C)-quasi-isometry.*

Proof. Let R be the diameter of $C(N_\Gamma)$ and let $S = \{\gamma \in \Gamma \mid \gamma(D(x,3R)) \cap D(x,3R) \neq \emptyset\}$, where $D(x,3R)$ is the closed ball of radius $3R$ about x. Since Γ is discrete, S is finite.

Let $\gamma \in \Gamma$ and let L be a geodesic segment in X joining x_0 to $\gamma(x_0)$. Divide L up into

$$n = \left\lfloor \frac{d(x_0, \gamma(x_0))}{R} \right\rfloor + 1$$

segments of equal length, with endpoints $\{x_0, x_1, \ldots, x_n\}$. Notice that each segment has length less than R. Since $C(N_\Gamma)$ has diameter R and $x_0 \in CH(\Lambda(\rho))$, there exists, for each i, $\gamma_i \in \Gamma$ so that $d(x_i, \gamma_i(x)) \leq R$ where we may choose $\gamma_0 = id$ and $\gamma_n = \gamma$. Then, since $d(\gamma_i(x), \gamma_{i+1})(x)) \leq 3R$ (by the triangle inequality), $\gamma_i^{-1}\gamma_{i+1} \in S$. Therefore, S is a finite generating set for Γ and

$$d_S(id, \gamma) \leq n = \left(\left\lfloor \frac{d(x, \gamma(x))}{R} \right\rfloor + 1 \right) \leq \frac{1}{R} d(x, \gamma(x)) + 1$$

so

$$R d_S(id, \gamma) - R \leq d(x, \gamma(x))$$

and, since τ_x is Γ-equivariant,

$$R d_S(\alpha, \beta) - R \leq d(\tau_x(\alpha), \tau_x(\beta)$$

for all $\alpha, \beta \in \Gamma$.

If $a, b \in K_{\Gamma,S}$, then there exists $\alpha, \beta \in \Gamma$, so that $d_{K_{\Gamma,S}}(a, \alpha) \leq \frac{1}{2}$ and $d_{K_{\Gamma,S}}(b, beta) \leq \frac{1}{2}$. Then

$$d(\hat{\tau}_x(a), \hat{\tau}_x(b)) \geq d(\tau_x(\alpha), \tau_x(\beta)) - R \geq R d_S(\alpha, \beta) - 2R \geq R d_{K_{\Gamma,S}} - 3R.$$

On the other hand, since τ_x is $3R$-Lipschitz on each edge.

$$d(\hat{\tau}_x(a), \hat{\tau}_x(b)) \leq 3R d_{K_{\Gamma,S}}(a, b).$$

Finally, notice that, every point in $CH(\Lambda(\Gamma)$ lies within R of $\tau_x(\Gamma)$. Therefore, $\hat{\tau}$ is $3R$-Lipschitz and is a $\left(\max\{\frac{1}{R}, 3R\}, 3R\right)$-quasi-isometry. □

2. Deformation spaces of Kleinian groups

2.1. Basic Definitions

Throughout the lecture course, M will denote a compact, oriented, irreducible 3-manifold, possibly with boundary, with infinite fundamental group. A 3-manifold M is said to be irreducible if every embedded 2-sphere in M bounds a 3-ball in M. These assumptions guarantee that the universal cover of M is contractible. We will often be considering the simple case where $M = S \times [0,1]$ and S is a closed surface. Moreover, S will always denote a closed oriented surface in these notes.

If $\rho : \pi_1(M) \to \mathsf{PSL}(2, C)$ is a discrete, faithful representation, then $\rho(\pi_1(M))$ then we obtain a hyperbolic 3-manifold

$$N_\rho = \mathbb{H}^3/\rho(\pi_1(S)).$$

Since M and N_ρ both have contractible universal cover and ρ gives an identification of the fundamental groups of M and N_ρ, we obtain a homotopy equivalence

$$h_\rho : M \to N_\rho$$

so that $(h_\rho)_* : \pi_1(M) \to \pi_1(N_\rho) = \rho(\pi_1(M))$ is conjugate to ρ. We think of h_ρ as a marking of a hyperbolic 3-manifold (much as in the setting of Teichmüller space) and think of the pair (M_ρ, h_ρ) as a marked hyperbolic 3-manifold.

Let $\widetilde{AH}(M) \subset \mathrm{Hom}(\pi_1(M), \mathsf{PSL}(2,\mathbb{C})$ denote the space of discrete faithful representations of $\pi_1(M)$ into $\mathsf{PSL}(2,\mathbb{C})$ and let $AH(M)$ be its image in the quotient character variety

$$X(M) = \mathrm{Hom}(\pi_1(M), \mathsf{PSL}(2,\mathbb{C})//\mathsf{PSL}(2,\mathbb{C}).$$

(The double back-slash indicates that we are taking the geometric invariant theory quotient which gives $X(M)$ the structure of the variety. We will not need to worry about this construction but there is an open neighborhood of the set of discrete faithful representation on which the quotient is simply the usual quotient and the image of the neighborhood lies in the smooth part of the character variety.) In analogy with Teichmüller space, we may think of this as the space of marked hyperbolic 3-manifolds homotopy equivalent to M (up to orientation-preserving isometry).

It is a classical consequence of the Margulis lemma that $AH(M)$ is a closed subset of $X(M)$. When $AH(M)$ is not a single point, it will not be an open subset of $X(M)$.

Theorem 2.1. *If M is a compact 3-manifold whose fundamental group is not virtually abelian, then $AH(M)$ is a closed (possibly empty) subset of $X(M)$.*

Exercise: Show that if H_2 is a handlebody of genus two, then $AH(H_2)$ is not open in $X(M)$. Hint: It suffices to consider representations into $\mathsf{PSL}(2,\mathbb{R})$.

A discrete, faithful representation $\rho : \pi_1(M) \to \mathsf{PSL}(2,\mathbb{C})$ is said to be **convex cocompact** if $\rho(\pi_1(M)$ is convex cocompact. Let $CC(M)$ be the set of convex cocompact representations in $AH(M)$. Moreover, a discrete, faithful representation $\rho : \pi_1(M) \to \mathsf{PSL}(2,\mathbb{C})$ is said to be a **convex cocompact uniformization of** M if $h_\rho : M \to N_\rho$ is homotopic to an orientation-preserving homeomorphism $j_\rho : M \to C_1(N_\rho)$. Let $CC_0(M)$ be the set of convex cocompact uniformizations of M in $AH(M)$.

Thurston completely characterized which manifolds with boundary have convex cocompact uniformizations.

Thurston's Hyperbolization Theorem (special case): *If M is an irreducible, compact 3-manifold with non-empty boundary and $\pi_1(M)$ is infinite and does not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, then M admits a convex cocompact uniformization.*

The best source for Thurston's original proof of his hyperbolization theorem is the article of Morgan [61], although portions of Thurston's proof appear in his papers [74, 78]. Otal [65] and Kapovich [43] discuss other proofs of his theorem.

2.2. **Stability**

It is a crucial property of convex cocompact representations, known as stability, that $\widetilde{CC}(M)$ is open in $\mathrm{Hom}(\Gamma, \mathsf{PSL}(2, \mathbb{C}))$. This was first established by Marden [50]. Informally, if you wiggle a convex cocompact representation a little bit it remains convex cocompact.

One can easily see how this phenomena works for Schottky groups. If $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ is the Schottky group constructed in Section 1, we may view it as the image of a convex cocompact representation of $F_n = \langle a_1, \ldots, a_n \rangle$, $\rho : F_n \to \mathsf{PSL}(2, C)$ where $\rho(a_i) = \gamma_i$. If $\sigma \in \mathrm{Hom}(F_n, PSL(2, C)$ is close enough to ρ, then

$$\{C_1, \ldots, C_n, \rho(a_1)(C_1), \ldots, \rho(a_n)(C_n)\}$$

is a collection of disjoint circle. So, the image of σ is a Schottky group and σ is convex cocompact. It follows that ρ lies in the interior of $CC(H_n)$.

Theorem 2.2. *If $\rho : \pi_1(M) \to \mathsf{PSL}(2, \mathbb{C})$ is convex cocompact, then there exists a neighborhood U of ρ in $\mathrm{Hom}(\pi_1(M), \mathsf{PSL}(2, \mathbb{C}))$ such that if $\sigma \in U$, then σ is convex cocompact.*

Theorem 2.2 was first establshed by Marden [50, Theorem 10.1]. Thurston [75, Proposition 8.3.3] observed that it followed from argument of the form due to Weyl, see also Canary-Epstein-Green [23, Section I.2.5].

Proof. The proof relies on the following local-to-global principle. We will sketch a proof in our setting in the next subsection.

Theorem 2.3. (Local to Global Principle) *Given $K \geq 1$, $C \geq 0$, there exists $\hat{K}$, $\hat{C}$ and A so that if J is an interval in $\mathbb{R}$ and $h : J \to \mathbb{H}^3$ is a (K, C)-quasi-isometric embedding restricted to every connected subsegment of J with length $\leq A$, then h is a $(\hat{K}, \hat{C})$-quasi-isometric embedding.*

Let K_M denote the Cayley graph of $\pi_1(M)$ with respect to some generating set S and choose $x_0 \in \mathbb{H}^3$. There exists (K, C) so that the orbit map $\tau_\rho : K_M \to \mathbb{H}^3$ (with basepoint x_0) is a (K, C)-quasi-isometric embedding. The local-to-global principle, see Theorem 2.5, implies that there exists $A \geq 1$, $\hat{K}$, and $\hat{C}$ so that if $f : J \to \mathbb{H}^3$ (where J is an interval in $\mathbb{R}$) is a $(K + C + 1, 2K + 3C + 5)$-quasi-isometry on all segments of length at most A, then f is a $(\hat{K}, \hat{C})$-quasi-isometry.

Let U be an open neighborhood of ρ in $\mathrm{Hom}(\pi_1(M), \mathsf{PSL}(2,))$ so that if $\sigma \in U$, $\gamma \in \pi_1(M)$ and $d_S(1, \gamma) \leq A$, then $d(\rho(\gamma(x_0)), \sigma(\gamma)(x_0)) < 1$. (We may do so since there are only finitely many elements of γ within A of id.)

If $\sigma \in U$, let τ_σ be the orbit map of σ with basepoint x_0. We see that if $d_S(1,\gamma) \leq A$, then

$$\frac{1}{K} d_S(id,\gamma) - C - 1 \leq d(\tau_\sigma(id), \tau_\sigma(\gamma)) \leq K d_S(id,\gamma) + C + 1.$$

Since τ_σ is σ-equivariant, we see that if $\alpha, \beta \in \Gamma$ and $d_S(\alpha,\beta) \leq A$, then

$$\frac{1}{K} d_S(\alpha,\beta) - C - 1 \leq d(\tau_\sigma(\alpha), \tau_\sigma(\beta)) \leq K d_S(\alpha,\beta) + C + 1.$$

We now check that τ_σ is a quasi-isometric embedding on all geodesic segments in K_M of length at most A. Since τ_σ is $(K+C+1)$-bilipschitz on each edge with a vertex at the origin and it is σ-equivariant, we conclude that τ_σ is $(K+C+1)$-bilipschitz on each edge of K_M and hence globally bilipschitz.

Let $\overline{ab}$ be a segment in K_M of length at most A joining a to b. Since τ_σ is $(K+C+1)$-bilipschitz,

$$d(\tau_\sigma(a), \tau_\sigma(b)) \leq (K+C+1) d_{K_M}(a,b).$$

We must work a little harder to get the lower bound. Let α be the vertex on $[a,b]$ closest to a and let β be the vertex on $[a,b]$ closest to b. Then $d(a,\alpha) < 1$, $d(b,\beta) < 1$ and $d(\alpha,\beta) \leq A$. So

$$\begin{aligned}
d(\tau_x(a), \tau_x(b)) &\geq d(\tau_\sigma(\alpha), \tau_\sigma(\beta)) - 2(K+C+1) \\
&\geq \frac{1}{K} d_S(\alpha,\beta) - C - 1 - 2(K+C+1) \\
&\geq \frac{1}{K}(d_{K_M}(a,b) - 2) - 2K - 3C - 3 \\
&\geq \frac{1}{K} d_{K_M}(a,b) - 2K - 3C - 5
\end{aligned}$$

Therefore, the extended orbit map $\hat{\tau}_\sigma$ is a $(K+C+1, 2K+3C+5)$-quasi-isometry on all geodesic segments in K_M of length at most A. Since $\hat{\tau}_\sigma$ is σ-equivariant, $\hat{\tau}_\sigma$ is a $(K+2, C+2)$-quasi-isometric embedding on all geodesic segments in C_M of length at most A. Therefore, $\hat{\tau}_\sigma$ is a $(\hat{K}, \hat{C})$-quasi-isometric embedding on all geodesic segments in K_M, which implies that $\hat{\tau}_\sigma$ is a $(\hat{K}, \hat{C})$-quasi-isometric embedding. Therefore, σ is convex cocompact. □

Since the set $CC(M)$ is invariant under conjugation, we immediately see that both $\widetilde{CC}(M)$ and its quotient $CC(M)$ are open.

Corollary 2.4. *If M is a compact, irreducible 3-manifold with non-empty boundary, then $CC(M)$ is open in $X(M)$.*

Remark: If M is a closed hyperbolic 3-manifold which admits a hyperbolic structure, then Mostow's Rigidity Theorem [62] implies that $CC(M)$ consists of two points (one for each orientation on M),

2.3. A proof of the Local-to-global principle

In a hyperbolic space, the local-to-global principle takes the following form:

Theorem 2.5. (Local to Global Principle) *Given $K \geq 1$, $C \geq 0$ and $\delta \geq 0$, there exists $\hat{K}$, $\hat{C}$ and A so that if J is an interval in $\mathbb{R}$, X is δ-hyperbolic and $h : J \to X$ is a (K, C)-quasi-isometric embedding restricted to every connected subsegment of J with length $\leq A$, then h is a $(\hat{K}, \hat{C})$-quasi-isometric embedding.*

We will sketch a proof of Theorem 2.5 in the case that $X = \mathbb{H}^3$ and $J = \mathbb{R}$ (The assumption that $J = \mathbb{R}$ is simply for convenience, while the restriction to $X = \mathbb{H}^3$ significantly simplifies the proof). See Coornaert-Delzant-Papadopoulos [33, Thm. 3.1.4] for a complete proof.

Sketch of proof: We will make use of an elementary lemma in hyperbolic geometry.

Lemma 2.6. *Given $S > 0$, $T > 0$, there exists $B = B(S, T) > 0$ so that if P and Q are totally geodesic hyperplanes in $\mathbb{H}^3$, $p \in P$, $q \in Q$ and $x \in \mathbb{H}^3$, and $\overline{px}$ and $\overline{qx}$ are geodesic segments perpendicular to P and Q respectively, so that $d(p, x) \geq B$, $d(q, x) \geq B$, and $d(x, \overline{pq}) \leq S$, then $d(P, Q) \geq T$.*

(The idea of the proof of the lemma is that if B is large enough, then $\overline{px}$ and $\overline{pq}$ are nearly tangent, so $\overline{pq}$ is nearly orthogonal to P. Similarly, $\overline{pq}$ is nearly orthogonal to Q. So we choose B large enough that the angles between $\overline{pq}$ and both P and Q is at least .75. Notice that $\overline{pq}$ has length at least $2B - 2S$. Let $\mathcal{C} = \{(P, Q, L)\}$ be the set of triples where P and Q are geodesic hyperplanes which are joined by a geodesic segment L which makes angle at least .75 with each of P and Q and $d(P, Q) \leq T$. If we also assume $\overline{pq}$ passes through a fixed point, then $\mathcal{C}$ is a compact set of configurations. Therefore, there is an upper bound R on the length of L. So, if we also choose B large enough that $2B - 2S > R$, then our assumptions guarantee that $d(P, Q) \geq T$.)

Given $K \geq 1$ and $C \geq 0$, let $R = R(K, C, \cosh^{-1}(2))$ be the constant provided by the Fellow Traveller property and let $B = B(2R, 2R)$ be the constant provided by Lemma 2.6. Choose $A \geq 4K(B + C + R)$.

For all $i \in \mathbb{Z}$, let $t_i = \frac{iA}{2}$ and $y_i = h(t_i)$. Let $G_i = \overline{y_i y_{i+1}}$ be the geodesic segment with vertices y_i and y_{i+1} and midpoint m_i. Notice that $d(y_i, y_{i+1}) \geq \frac{A}{2K} - C$, so

$$d(m_i, y_{i+1}) \geq \frac{A}{4K} - \frac{C}{2} \geq B.$$

Similarly, $d(m_{i+1}, y_i) \geq L$.

By the Fellow Traveller Property, there exists $s_i \in [t_i, t_{i+1}]$ such that $d(f(s_i), m_i) \leq R$. The Fellow traveller property, then implies that $d\left(y_i, \overline{h(s_i)h(s_{i+1})}\right) \leq R$. Choose $z_i \in \overline{h(s_i), h(s_{i+1})}$, so that $d(z_i, y_i) \leq R$. Since $\overline{m_i, m_{i+1}}$ are $\overline{h(s_i)h(s_{i+1})}$ are geodesics whose endpoints are a distance at most R apart, the convexity of the distance function implies that the Hausdorff distance between $\overline{m_i, m_{i+1}}$ are $\overline{h(s_i)h(s_{i+1})}$ is at most R. Therefore, $d(z_i, \overline{m_i m_{i+1}}) \leq R$, so $d(y_i, \overline{m_i m_{i+1}}) \leq 2R$. Lemma 2.6 then implies that

$$d(P_i, P_{i+1}) \geq 2R \qquad \text{for all } \; i \in \mathbb{Z}.$$

We next claim that P_{i-1} and P_{i+1} lie on opposite sides of P_i. If not, then y_{i-1} and y_{i+1} lie on the same side of P_i, so the geodesic segment $\overline{y_{i-1} y_{i+1}}$ lies on the opposite side of P_i from y_i,

but
$$d(y_i, P_i) = d(y_i, m_i) \geq \frac{A}{4K} - C > 2R, \qquad \text{so} \qquad d\big(f(s_i), \overline{y_{i-1}, y_{i+1}}\big) > R$$
which would contradict the Fellow Traveller Property. It follows that P_{i-1} lies on the opposite side of P_i as P_{i+1}. Therefore, since $d(P_i, P_{i+1}) \geq 2R$ for all i and are ordered monotonically, we see that
$$d(y_m, y_n) \geq (|m-n|-1)2R \qquad \text{for all} \;\; m, n \in \mathbb{Z},$$
If $a, b \in \mathbb{R}$, choose $m, n \in \mathbb{Z}$ so that $a \in [t_{m-1}, t_m]$ and $b \in [t_n, t_{n+1}]$. Then
$$|a - t_m| < \frac{A}{2}, \quad |b - t_n| < \frac{A}{2}, \quad d(f(a), f(t_m)) \leq \frac{KA}{2} + C \quad d(f(b), f(t_n)) \leq \frac{KA}{2} + C$$
so
$$\begin{aligned} d(f(a), f(b)) &\geq 2R(|m-n|-1) - KA - 2C \\ &= \frac{4R}{A}|t_m - t_n| - 2R - KA - 2C \\ &\geq \frac{4R}{A}|b-a| - 6R - KA - 2C \end{aligned}$$

Since, $|t_i - t_{i+1}| = \frac{A}{2}$, $d(f(t_i), f(t_{i+1})) \leq \frac{KA}{2} + C$, so we see that
$$d(f(t_m), f(t_n)) \leq |m-n| \left(\frac{KA}{2} + C\right).$$
Therefore,
$$\begin{aligned} d(f(a), f(b)) &\leq |m-n| \left(\frac{KA}{2} + C\right) + KA + 2C \\ &= \frac{2}{A}|t_m - t_n| \left(\frac{KA}{2} + C\right) + KA + 2C \\ &\leq |b-a| \left(K + \frac{2C}{A}\right) + A\left(K + \frac{2C}{A}\right) + KA + 2C \end{aligned}$$
We conclude that f is a $(\hat{K}, \hat{C})$-quasi-isometry where
$$\hat{K} = \max\left\{\frac{A}{R}, K + \frac{2C}{A}\right\} \qquad \text{and} \qquad \hat{C} = A\left(K + \frac{2C}{A}\right) + 6R + KA + 2C.$$
$\square$

3. Bers' Simultaneous uniformization

If S is a closed oriented surface, we call a convex cocompact uniformization of $S \times [0,1]$ (and its image) **quasifuchsian**. We also use the simplifying notation
$$QF(S) = CC_0(S \times [0,1]).$$
Since $S \times [0,1]$ admits an orientation-reversing homeomorphism homotopic to the identity, we may assume that h_ρ is orientation-preserving. Therefore, we get a map
$$B : QF(S) \to \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \quad \text{given by} \quad B(\rho) = (\partial_c N_\rho, h_\rho)$$

where $\bar{S}$ is S with the opposite orientation. The main goal of this section is to show that B is a homeomorphism.

Remark: If M is a compact, irreducible 3-manifold and $h : S \times [0,1] \to M$ is a homotopy equivalence, then h is homotopic to an orientation-preserving homeomorphism (see Section 10 of Hempel [39]). It follows that

$$CC(S \times [0,1]) = CC_0(S \times [0,1]).$$

3.1. Quasiconformal maps and Beltrami differentials

We begin with a brief survey of the theory of quasiconformal maps (without proof). Roughly, quasiconformal maps are orientation preserving homeomorphisms which distort the conformal structure by a bounded amount. One may view them as the conformal analogue of bilipschitz maps, which distort the metric structure a bounded amount. Good references for the theory of quasiconformal maps are the books of Lehto-Virtanen [48] and Lehto [47]. This section is plagiarized from the research monograph of Canary-McCullough [30].

Given a function $f\colon D \to \overline{\mathbb{C}}$ defined on a domain D in $\overline{\mathbb{C}}$, we may write it as $f(x,y) = u(x,y) + iv(x,y)$. We say f is *ACL* (absolutely continuous on lines) if given any rectangle $R = [a,b] \times [c,d]$ in D both u and v are absolutely continuous restricted to almost every vertical and almost every horizontal line segment in R. If f is ACL then the partial derivatives of u and v exist almost everywhere and we define $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$. Then, we let $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$. (Recall that the Cauchy-Riemann equations assert that if f is analytic then $f_{\bar{z}} = 0$ for all $z \in D$.) We define the *Beltrami differential* of f to be $\mu_f = \dfrac{f_{\bar{z}}}{f_z}$ Notice that if f is differentiable at a point z and $Jf(z)$ is its Jacobian, then the image of the unit circle (in the tangent space $T_z(D)$) under $Jf(z)$ is an ellipse, the ratio of the lengths of the axes is given by $K(z) = \dfrac{1+|\mu_f(z)|}{1-|\mu_f(z)|}$ and the angle that the preimage of the (longer) axis makes with the x-axis is $\frac{1}{2}\arg(\mu_f(z))$.

One says that an orientation-preserving homeomorphism $f\colon D \to D'$ is K-*quasiconformal* if f is ACL and $|\mu_f| \leq \frac{K-1}{K+1}$ almost everywhere. This says that, typically, very small circles are taken to curves very much like ellipses with eccentricity at most K. One way of formalizing this is by defining

$$H(z) = \limsup_{r\to 0} \frac{\max_\theta |f(z+re^{i\theta}) - f(z)|}{\min_\theta |f(z+re^{i\theta}) - f(z)|} .$$

An orientation-preserving homeomorphism $f\colon D \to \mathbb{C} \cup \{\infty\}$ is K-quasiconformal if and only if H is bounded on $D - \{\infty, f^{-1}(\infty)\}$ and $H(z) \leq K$ almost everywhere in D (see pages 177 and 178 in Lehto [47]). If one uses the spherical metric on $\overline{\mathbb{C}}$, then one need not exclude ∞ and $f^{-1}(\infty)$ from consideration.

One may check that the composition of a K_1-quasiconformal map and a K_2-quasiconformal map is a K_1K_2-quasiconformal map. Another useful fact is:

Proposition 3.1. (Lehto-Virtanen [48, Thm. 1.5.1]) *A quasiconformal map is conformal if and only if it is* 1*-quasiconformal.*

The most fundamental result concerning quasiconformal maps is the Measurable Riemann Mapping Theorem (see Ahlfors-Bers [5] or Lehto [47]) which asserts that the Beltrami differential determines the quasiconformal map (up to normalization) and that every Beltrami differential (of norm less than 1) determines a quasiconformal map.

Measurable Riemann Mapping Theorem: *Suppose that $\mu \in L_\infty(\overline{\mathbb{C}}, \mathbb{C})$ and $||\mu||_\infty < 1$. Then there exists a unique quasiconformal map $\phi_\mu \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ whose Beltrami differential is μ and such that ϕ_μ fixes 0, 1, and ∞. Moreover, ϕ_μ depends analytically on μ.*

Notice that one may combine the Measurable Riemann Mapping Theorem and the traditional Riemann Mapping Theorem to observe that the same result holds for the upper half-plane $\mathbb{H}^2$. This version of the result is used in traditional Teichmüller theory and also plays a role in our proof of the Quasiconformal Parameterization Theorem.

Measurable Riemann Mapping Theorem (Disk version): *Suppose that $\mu \in L_\infty(\mathbb{H}^2, \mathbb{C})$ and $||\mu_\infty|| < 1$. Then there exists a unique quasiconformal map $\phi_\mu \colon \mathbb{H}^2 \to \mathbb{H}^2$ whose Beltrami differential is μ and such that ϕ_μ fixes i, $2i$, and $3i$. Moreover, ϕ_μ depends analytically on μ.*

An alternative characterization of quasiconformal mappings of $\overline{\mathbb{C}}$ is obtained by considering bilipschitz homeomorphisms of $\mathbb{H}^3$. The Fellow Traveller property may be used to show that any bilipschitz homeomorphism of $\mathbb{H}^3$ to itself extends continuously to a homeomorphism of $\partial\mathbb{H}^3$ to itself. One must work a little harder (although not too much harder) to show that this extension is quasiconformal. (I like how this is written up in Thurston's notes [75], but one may find this argument many places.) It is a deeper fact that any quasiconformal automorphism of $\partial\mathbb{H}^3$ extends to a bilipschitz map of $\mathbb{H}^3$. (One place to read an exposition is in Matsuzaki-Taniguchi [55, Thm 5.3.1].)

Proposition 3.2. *Let $\phi \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an orientation-preserving homeomorphism. Then ϕ is quasiconformal if and only if it extends to a homeomorphism $\Phi \colon \mathbb{H}^3 \cup \overline{\mathbb{C}} \to \mathbb{H}^3 \cup \overline{\mathbb{C}}$ whose restriction to $\mathbb{H}^3$ is bilipschitz (with respect to the hyperbolic metric).*

Notice that one can show that a bilipschitz map extends to a well-defined map of the boundary using only the Fellow Traveller Property. One must work a little harder (although not too much harder) to show that this extension is quasiconformal. The fact that quasiconformal maps extend to bilipschitz maps is a deeper fact.

It will also be useful to know that quasiconformal homeomorphisms take sets of measure zero to sets of measure zero (see [48]).

Theorem 3.3. *If $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a quasiconformal map and $E \subset \mathbb{C}$ has measure zero, then $f(E)$ has measure zero.*

3.2. **Quasifconformal conjugacy**

We now show that any two quasifuchsian representations are quasiconformally conjugate.

Theorem 3.4. *If $\rho_1, \rho_2 \in QF(S)$, then there exists a quasiconformal map $\phi : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$, so that $\rho_2(\gamma) = \phi \circ \rho_2(\gamma) \circ \phi^{-1}$ for all $\gamma \in \Gamma$.*

Proof. Let $h_{\rho_i} : S \times [0, 1] \to C_1(N_{\rho_i})$ be orientation-preserving homeomorphism in the homotopy class of ρ_i. Since $C_1(N_{\rho_i})$ is a compact C^1-manifold, we may assume that h_{ρ_i} is bilipschitz. So

$$h = h_{\rho_2} \circ h_{\rho_1}^{-1} : C_1(N_{\rho_1}) \to C_1(N_{\rho_2})$$

is a bilipschitz homeomorphism. We may then extend h radially in the coordinates to obtain an orientation-preserving bilipschitz homeomorphism

$$H : N_{\rho_1} \to N_{\rho_2}$$

where if $x \in N_{\rho_1} - C_1(N_{\rho_1})$ has coordinates (y,t), then $H(x)$ has coordinates $(h(y),t)$.

Then H lifts to an orientation-preserving bilipschitz homeomorphism $\tilde{H} : \mathbb{H}^3 \to \mathbb{H}^3$ which conjugates the action of $\rho_1(\pi_1(S))$ to the action of $\rho_2(\pi_1(S)$, so, perhaps after precomposition by an element of $\rho_1(\pi_1(S))$,

$$\tilde{H} \circ \rho_1(\gamma) = \rho_2(\gamma) \circ \tilde{H}$$

for all $\gamma \in \pi_1(S)$. Then $\tilde{H}$ extends to a quasiconformal homeomorphism $\phi : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ so that

$$\phi \circ \rho_1(\gamma) = \rho_2(\gamma) \circ \phi$$

for all $\gamma \in \pi_1(S)$. □

If ρ_1 is Fuchsian, then its limit set has measure zero, which implies that $\Lambda(\rho_2) = \phi(\Lambda(\rho_1))$ also has measure zero.

Corollary 3.5. *If $\rho \in QF(S)$, then $\Lambda(\rho)$ has measure zero.*

3.3. **Simultaneous uniformization**

Bers [8] showed that B gives a complete parametrization of $QF(S)$.

Theorem 3.6. *The map*

$$B : QF(S) \to \mathcal{T}(S) \times \mathcal{T}(\bar{S})$$

is a homeomorphism.

Proof. We first prove that B is injective. If $B(\rho_1) = B(\rho_2)$, then there exists a conformal map $\psi : \partial_c N_{\rho_1} \to \partial_c N_{\rho_2}$ in the homotopy class of $h_{\rho_2} \circ h_{\rho_1}^{-1}$. This map lifts to a conformal homeomorphism $\tilde{\psi} : \Omega(\rho_1) \to \Omega(\rho_2)$ so that $\rho_2(\gamma) = \tilde{\phi} \circ \rho_1(\gamma) \circ \tilde{\phi}^{-1}$ for all $\gamma \in \pi_1(S)$.

We may use the radial coordinates on $\widehat{N}_{\rho_i} - C_1(N_{\rho_i})$ to extend ψ to a differentiable bilipschitz homeomorphism Ψ from $N_{\rho_1} \setminus \mathrm{int} C_1(N_{\rho_1})$ to $N_{\rho_2} \setminus \mathrm{int} C_1(N_{\rho_2})$. Since $C_1(N_{\rho_1})$ and $C_1(N_{\rho_2})$ are homeomorphic, we may then extend Ψ to a differentiable bilipschitz diffeomorphism from N_{ρ_1} to N_{ρ_2}. The map Ψ lifts to a bilipschitz map from $\mathbb{H}^3$ to $\mathbb{H}^3$ which admits a continuous extension to a quasiconformal homeomorphism C of $\partial\mathbb{H}^3$. By construction, C agrees with $\tilde{\psi}$ on $\Omega(\rho_1)$. (Bers uses a more analytic argument to construct the extension C.)

Since $\Lambda(\rho_1)$ has measure zero and C is conformal on $\Omega(\rho_1)$, we conclude that C is conformal and that $\rho_2(\gamma) = C \circ \rho_1(\gamma) \circ C^{-1}$ for all $\gamma \in \pi_1(S)$. Therefore $\rho_1 = \rho_2 \in QF(S)$.

The fact that B is surjective is an application of the Measurable Riemann Mapping Theorem. Suppose that $(Y, \bar{Z}) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$. Let ρ_0 be a Fuchsian group so that $B(\rho_0) = (X, \bar{X})$. Let $\psi : X \cup \bar{X} \to Y \cup \bar{Z}$ be an orientation-preserving diffeomorphism (in the correct homotopy class). We may lift ψ to a map $\tilde{\psi} : \Omega(\rho_0) \to \mathbb{H}^2 \cup \mathbb{H}^2$ and compute its Beltrami differential $\mu_{\tilde{\psi}}$. We may then extend $\mu_{\tilde{\psi}}$ to a Beltrami differential on $\partial\mathbb{H}^3$ by setting it equal to 0 on $\Lambda(\rho_0)$. The Measurable Riemann Mapping Theorem implies that there exists a quasiconformal map $\phi : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ with Beltrami differential μ.

By construction, $\psi \circ \rho_0(g)\psi^{-1}$ is conformal, so $\psi \circ \rho_0(g)$ and ψ have the same Beltrami differential for all $g \in \pi_1(S)$. Thus, $\phi \circ \rho_0(g)$ and ϕ have the same Beltrami differential on

$\Omega(\rho_0)$, and hence on $\hat{C}$, for all $g \in \pi_1(S)$. The uniqueness portion of the Measurable Riemann Mapping Theorem then implies that $\phi \circ \rho_0(g)$ and ϕ differ by postcomposition by a Möbius transformation. Therefore, $\phi \circ \rho_0(g) \circ \phi^{-1} i$ is conformal, for all $g \in \pi_1(S)$, so we obtain a faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{C})$ defined by $\rho(\gamma) = \phi \circ \rho_0(\gamma) \circ \phi^{-1}$ for all $\gamma \in \Gamma$. ρ is discrete since $\rho(\pi_1(S))$ acts properly discontinuously on $\phi(\Omega(\rho_0))$.

Then $\widehat{N}_\rho$ is an irreducible, orientable manifold homotopy equivalent to S. It has two boundary components which are homotopic, hence homologous, which implies that $\widehat{N}_\rho$ is compact, and hence that ρ is convex cocompact. Since ϕ and $\tilde{\psi}$ have the same Beltrami differential on $\Omega(\rho_0)$, we see that $\phi \circ \tilde{\psi}^{-1}$ is a conformal homeomorphism from $\mathbb{H}^2 \cup \mathbb{H}^2$ to $\Omega(\rho)$ which descends to a conformal homeomorphism from $Y \cup \bar{Z}$ to $\partial_c N_\rho$. Therefore, $B(\rho) = (Y, \bar{Z})$. (If one prefers a more geometric argument, one may use work of Douady and Earle [34] to extend the quotient of ϕ to a homeomorphism of $\widehat{N}_{\rho_0}$ to $\widehat{N}_\rho$ which is bilipschitz on N_{ρ_0}.) Therefore, $\rho \in QF(S)$ and $B(\rho) = (Y, \bar{Z})$.) We have now shown that B is surjective which completes the proof. $\square$

3.4. **The general case**

The same outline of proof will always produce a natural parametrization of $CC_0(M)$ in general.

We say that a compact, irreducible 3-manifold M has **compressible boundary** if there is a closed curve α in ∂M which is not homotopically trivial in ∂M but is homotopically trivial in M. Otherwise, we say that M has **incompressible boundary**. Equivalently, M has incompressible boundary if and only if whenever S is a boundary component of ∂M, the inclusion map of S into M induces an injection from $\pi_1(S)$ into $\pi_1(M)$.

If M and M' have incompressible boundary, then Waldhausen's theorem implies that any two homotopic orientation-preserving homeomorphisms between M and M' are isotopic. Thus, when M has incompressible boundary, the generalization of Bers' Simultaneous Uniformization theorem has the following simple form.

Theorem 3.7. (Bers) *If M is compact, irreducible 3-maniifold with non-empty, incompressible boundary and $CC_0(M)$ is non-empty, then*

$$CC_0(M) \cong \mathcal{T}(\partial M).$$

Here, if $\{S_1, \ldots, S_n\}$ are the components of ∂M, then

$$\mathcal{T}(\partial M) = \prod_{i=1}^{n} \mathcal{T}(S_i).$$

The Loop Theorem says that M has compressible boundary if and only if there is a homotopically non-trivial simple closed curve in ∂M which bounds an embedded disk in M. A disk of this form is called a **compressing disk**. Thus, M has compressible boundary if and only if $\pi_1(M)$ splits, non-trivially, as a free product of subgroups.

If M has compressible boundary, then Dehn twists about compressing disks are orientation-preserving homeomorphisms homotopic to the identity. We let $\mathrm{Mod}_0(M)$ denote the group of isotopy classes of homeomorphism of M which are homotopic to the identity. This group is always generated by Dehn twists about compressing disks and is typically infinitely generated. Notice that if $\rho \in CC_0(M)$, then j_ρ is only well-defined modulo $\mathrm{Mod}_0(M)$. Moreover, $Mod_0(M)$

is naturally identified with a subgroup of the mapping class group of ∂M and hence acts on $\mathcal{T}(\partial M)$.

If M has compressible boundary, the generalization of Bers' Simultaneous Uniformization theorem has the following less simple form.

Theorem 3.8. (Bers) *If M is compact, irreducible 3-maniifold with non-empty, compressible boundary and $CC_0(M)$ is non-empty, then*

$$CC_0(M) \cong \mathcal{T}(\partial M)/\mathrm{Mod}_0(M).$$

The eagle-eyed reader will have noticed that one portion of the above sketch of proof admits no obvious generalization. We used the fact that the limit set of a Fuchsian uniformization of a closed surface is round circle in $\partial\mathbb{H}^3$, to conclude that every quasifuchsian group has a limit set of measure zero. We used this in turn to conclude that any quasiconformal conjugacy which is conformal on the domain of discontinuity is a globally conformal conjugacy. We may replace this step with a result of Ahlfors.

Ahlfors' Measure Theorem: *If Γ is a convex cocompact Kleinian group and $\Lambda(\Gamma) \neq \partial\mathbb{H}^3$, then its limit set $\Lambda(\Gamma)$ has measure zero. Moreover, if $\Lambda(\Gamma) = \partial\mathbb{H}^3$, then Γ acts ergodically on $\partial\mathbb{H}^3$, i.e. if $A \subset \partial\mathbb{H}^3$ is a Γ-invariant measurable subset of $\partial\mathbb{H}^3$, then A either has full measure or zero measure.*

We sketch Ahlfors' beautiful proof [4].

Sketch of proof: If $\Lambda(\Gamma) \neq \partial\mathbb{H}^3$, we define a Γ-invariant function $\tilde{h} : \mathbb{H}^3 \to [0,1]$ by letting $h(x)$ denote the measure, in $T^1_x\mathbb{H}^3$ of the set of unit vectors tangent to geodesic rays emanating from x which end at points in $\Lambda(\Gamma)$. (Here, we scale the Riemannian metric on $T^1_x\mathbb{H}^3$ so that it is a probability measure.) If we work in the Poincaré ball model for $\mathbb{H}^3$, one can choose $A_x \in \mathrm{Isom}_+(\mathbb{H}^3)$ so that $A_x(x) = \vec{0}$ and let $\tilde{h}(x) = \frac{1}{4\pi}m(A_x(\Lambda(\Gamma))$ where dm is the usual Lebesgue measure on S^2. Recalling that $A'_x(z) = \frac{1-|x|^2}{|x-z|^2}$ for all $z \in S^2$, we obtain the formula

$$\tilde{h}(x) = \frac{1}{4\pi}\int_{S^2}\left(\frac{1-|x|^2}{|x-z|^2}\right)^2 dm.$$

One may check that $\tilde{h}$ is harmonic, i.e. that $\mathrm{div}(\mathrm{grad}\ \tilde{h})(x) = 0$ for all $x \in \mathbb{H}^3$.

Notice that if $x \in \mathbb{H}^3 - CH(\Lambda(\Gamma)$, then there exists a totally geodesic plane P_x through x which bounds an open half-space H_x which is disjoint from $CH(\Lambda(\Gamma)$. It follows that $\Lambda(\Gamma)$ is disjoint from the interior of the disk $D_x = \overline{H}_x \cap \partial\mathbb{H}^3$. Since half the geodesic rays emanating from x lie in D_x, it follows that $\tilde{h}(x) \leq \frac{1}{2}$ if $x \in \mathbb{H}^3 - CH(\Lambda(\Gamma)$.

Since h is Γ-invariant, by construction, it descends to a harmonic function

$$h : N_\Gamma \to [0,1]$$

such that $h(x) \leq \frac{1}{2}$ if $x \in N_\Gamma - C(N_\Gamma)$. So, $h(x) \leq \frac{1}{2}$ if $x \in \partial C(N)$. Since $C(N)$ is compact, the maximum principle for harmonic functions implies that h achieves its maximal value on $C(N)$ at a point on $\partial C(N)$. Therefore, $h(x) \leq \frac{1}{2}$ for all $x \in \mathbb{H}^3$.

If $\Lambda(\Gamma)$ does not have measure zero, then it has a point of density z. However, if x_n is a sequence of points approaching z along a geodesic, we see that $\lim h(x_n) = 1$, so we have a contradiction. Recall that z is a point of density for the limit set if

$$\lim_{\epsilon\to 0}\frac{m(B_\epsilon(z)\cap\Lambda(\Gamma)}{m(B_\epsilon(z))}=1.$$

If $\Lambda(\Gamma)=\partial\mathbb{H}^3$, then $N=C(N)$, so, since $C(N)$ is compact, N is a closed manifold. if $A\subset\partial\mathbb{H}^3$ is a Γ-invariant measurable subset of $\partial\mathbb{H}^3$, we can define a Γ-invariant harmonic function $\tilde{h}:\mathbb{H}^3\to[0,1]$ by letting $h(x)$ denote the measure, in $T^1_x\mathbb{H}^3$ of the set of unit vectors tangent to geodesic rays emanating from x which end at points in A. Since N is closed, all harmonic functions are constant (if not consider the volume-preserving flow $\{\phi_t\}$ generated by grad h and notice that $\phi_t(N)$ is a proper subset of N for all $t>0$ which is impossible). If A is neither full measure nor zero measure, then A has a point of density so $h=1$, while $\partial\mathbb{H}^3-A$ also has a point of density, so $h=0$, and we have arrived at a contradiction. ☐

An excellent, analytically oriented, survey of the quasiconformal deformation theory of Kleinian groups is given in a paper of Bers [10]. A full treatment from a more topological viewpoint is given in Canary-McCullough [30].

4. Geometrically tame hyperbolic 3-manifolds

A hyperbolic manifold N_Γ is said to have **no cusps** if Γ contains no parabolic elements. Equivalently, we can say that every non-trivial element of Γ has non-zero translation length and gives rise to a closed geodesic in N_Γ. We will assume from now on, for simplicity, that N_Γ has no cusps. This will only be a technical issue in the results we will discuss, but in their proofs one must consider manifolds with cusps (so we are hiding important details in some sense). We will continue to assume that Γ is finitely generated and torsion-free.

There exist many important hyperbolic 3-manifolds without cusps and we will discuss their topology and geometry in this section.

4.1. The first explicit examples

The most easily understood, and constructed, hyperbolic 3-manifolds which are not convex cocompact arise as covers of closed hyperbolic 3-manifolds which fiber over the circle. Thurston showed that if $\phi:S\to S$ is a pseudo-Anosov homeomorphim then its mapping torus

$$M_\phi=S\times[0,1]/(x,0)\sim(\phi(x),1)$$

is homeomorphic to a hyperbolic 3-manifold N_ϕ. Since N_ϕ is closed, it is itself a convex cocompact hyperbolic 3-manifold. (We recall that ϕ is pseudo-Anosov if whenever C is a simple closed curve on S, then $\phi^n(C)$ is not homotopic to C for any $n>0$.)

If $N_\phi=\mathbb{H}^3/\Gamma_\phi$, then $\Lambda(\Gamma_\phi)=\partial\mathbb{H}^3$ (since if $\Lambda(\Gamma)\neq\partial\mathbb{H}^3$, one may easily check, using the fact that $C(N_\Gamma)\neq N_\Gamma$ and our analysis of the complement of the convex core, that N_Γ has infinite volume).

Let $\Gamma^0_\phi\subset\Gamma_\phi$ denote the normal subgroup of Γ_ϕ associated to the subgroup $\pi_1(S\times\{\frac{1}{2}\}\subset\pi_1(M_\phi)$. Since Γ^0_ϕ is normal in Γ_ϕ, $\Lambda(\Gamma^0_\phi)$ is a closed Γ_ϕ invariant subset of $\partial\mathbb{H}^3$, Therefore,

$$\Lambda(\Gamma_\phi)\subset\Lambda(\Gamma^0_\phi)$$

so $\Lambda(\Gamma^0_\phi)=\partial\mathbb{H}^3$.

Let $N^0_\phi = \mathbb{H}^3/\Gamma^0_\phi$ be the regular infinite cyclic cover of N_ϕ. Since $\Lambda(\Gamma^0_\phi) = \partial\mathbb{H}^3$, we see that $C(N^0_\phi) = N^0_\phi$, so N^0_ϕ is not convex cocompact (since N^0_ϕ has infinite volume). One may construct N^0_ϕ explicitly by cutting N_ϕ along an embedded surface in the homotopy class of the fiber to obtain a "lego block" L which is homeomorphic to $S \times [0,1]$. One then constructs N^0_ϕ from infinitely many lego blocks by gluing the top boundary of the n^{th} lego block to the bottom boundary of the $(n+1)^{st}$ lego block (by an isometry in the homotopy class of ϕ). The group $\langle C \rangle$ of covering transformations of the cover $N^0_\phi \to N_\phi$ then acts by translation (twisted by ϕ) in the collection of lego blocks.

This picture gives us a coarse understanding of the geometry of N^0_ϕ. One initially surprising fact is that the volume of r-balls in N^0_ϕ grows linearly in r. There is, possibly apocryphal, story that when Thurston was first asked which 3-manifolds could be hyperbolic, he guessed that 3-manifolds fibering over the circle could not be, based on this consequence.

We will focus on two features of this example. First, if we take a simple closed curve α on S and consider its associated geodesic α^* of α, we see that $\phi^n(\alpha)^* = C^n(\alpha^*)$ for all $n \in \mathbb{Z}$. This implies that the sequence of geodesics $\{C^n(\alpha^*)\}_n \in \mathbb{N}$ is a sequence of geodesics exiting the "'upward-pointing end" of N^0_ϕ each of which is homotopic to a simple closed curve on S. Similarly, $\{C^{-n}(\alpha^*)\}_n \in \mathbb{N}$ is a sequence of geodesics exiting the "downward-pointing end" of N^0_ϕ each of which is homotopic to a simple closed curve on S.

Second, let Y be a minimal surface in N_ϕ in the homotopy class of the fiber. One key fact is that in its intrinsic metric, Y has curvature bounded above by -1. Then, the pre-image of Y is an infinite family $\{Y_n\}_{n\in\mathbb{Z}}$ of minimal surfaces in N^0_ϕ so that $Y_n = C^n(Y_0)$ for all $n \in \mathbb{Z}$. So $\{Y_n\}_{n\in\mathbb{N}}$ is an infinite family of surface whose intrinsic curvature is bounded above by -1, each of which is in the homotopy class of $S \times \{0\}$, which exits the upward-pointing end of N^0_ϕ.

Historical Remarks: The majority of Thurston's proof of his hyperbolization theorem for closed 3-manifolds which fibre over the circle is contained in his preprint [77]. A complete proof, using the technology of $\mathbb{R}$-trees in place of Thurston's use of pleated surfaces, is given by Otal [66]. The first explicit example was discovered by Jørgensen [42] and is the cover of a finite volume hyperbolic 3-manifold fibering over the circle associated to the fibre subgroup (which was a punctured torus in his case). Greenberg [37] was the first to abstractly show the existence of hyperbolic 3-manifolds with finitely generated fundamental group which are not geometrically finite.

4.2. **Simplicial hyperbolic surfaces**

Instead of working with minimal surfaces, we will work with simplicial hyperbolic surfaces. The intrinsic metric on these manifolds will be hyperbolic except at a finite number of cone points where the total angle around the cone points is greater than 2π. We interpret these cone points as points of concentrated (or infinite) negative curvature. In particular, the intrinsic metric on a simplicial hyperbolic surface is locally CAT(-1), i.e. every geodesic triangle in the universal cover is at least as thin as a triangle in $\mathbb{H}^2$.

We will say that a finite graph T on a surface S is a triangulation if every component of $S - T$ is bounded by 3 edges (I.e. is the interior of a triangle). One key observation is that every essential simple closed curve C on S can be extended to a triangulation of S with a single vertex. One simply chooses a vertex v on C and adds disjoint edges which begin and end at v (no two of which are homotopic rel boundary) until one achieves a triangulation.

A simplicial hyperbolic surface $f : S \to N$ is a map so that there exists a triangulation T of S, so that the image of each face of T is a totally geodesic (filled) triangle in N and the total angle around each vertex is at least 2π. One obtains a pull-back metric on S, which we call τ_f, which is locally isometric to $\mathbb{H}^2$ at every point except (possibly) the vertices. If $v \in T^(0)$ is a vertex, then let $a(v)$ be the total angle of the induced metric about v (i.e. the sums of the internal angles of triangles with a vertex at v.) Since the area of a triangle is its angle defect (i.e. the difference between π and the sum of the internal angles), we see that

$$Area(S, \tau_f) = -2\xi(S) - \sum_{v \in T^{(0)}} a(v) - 2\pi \leq -2\pi\xi(S)$$

where $\xi(S)$ is the Euler characteristic of S.

Suppose that $g : S \to N$ is a π_1-injective map and N has no cusps. If α is a simple closed curve on S, we complete it to a triangulation T with only one vertex v on α. One may then homotope g to a map taking v to a point on $g(\alpha)^*$ and pull the remaining edges and faces tight so that their images are totally geodesic to obtain a map $f : S \to N$. One may check that f is actually a simplicial hyperbolic surface, by noticing that there is an edge through $f(v)$ which is a closed geodesic through $f(v)$, so the total angle occuring on either side of this edge is at least π, so that $a(v) \geq 2\pi$ as required. It may be easiest to understand the picture by looking at the lifted map $\tilde{f} : \tilde{S} \to \mathbb{H}^3$ from the universal cover to $\mathbb{H}^3$. In this picture, one may consider the edges arranged around a lift $\tilde{v}$ of v, then there are two edges, both of which are "lifts" of $g(\alpha)^*$, which meet at $\tilde{v}$ and are antipodal, so any path of edges joining one edge to another transverse a total angle at least π Another way to see this is to look at the set of unit tangent vectors to $f(S)$ at $f(v)$. The set of tangent vectors is a path in $T^1_{f(v)}N$ and $a(v)$ is the length of this path. Since the path contains antipodal points, it has length at least 2π. We will say that f is a simplicial hyperbolic surface which **realizes α with one vertex**.

If g is not π_1-injective, then the above construction works as long as $g(\alpha)$ is homotopically non-trivial and we can choose T so that every edge of T determines a homotopically non-trivial loop in N.

4.3. **Simply degenerate ends**

Peter Scott [69] proved that every irreducible 3-manifold N with finitely generated fundamental group contains a compact submanifold C so that the inclusion of C into M is a homotopy equivalence. We call such a submanifold a **compact core** for N. If N is homeomorphic to the interior of a handlebody, it is easy to construct compact cores which are not isotopic, since one may form a compact core by thickening any bouquet of circles in N whose inclusion into N is a homotopy equivalence. McCullough, Miller and Swarup [56] proved that this core is unique in the sense that if C_1 and C_2 are two compact cores for N, then there is a homeomorphism from C_1 to C_2 which is homotopic, within N, to the inclusion of C_1 into N. I believe it is the case that the compact core is unique up to isotopy when the compact core has incompressible boundary (i.e. when $\pi_1(N)$ is freely indecomposable).[2]

[2]This will definitely be the case when N is topologically tame, i.e homeomorphic to the interior of a compact 3-manifold, and its compact core has incompressible boundary

If N_Γ is a hyperbolic 3-manifold, let C_Γ denote a compact core for N_Γ. We will call each component of $N_\Gamma - C_\Gamma$ an **end** of N_Γ. If E is an end of N_Γ, then any open subset of U so that $E - U$ is bounded, will be called a **nieghborhood of the end** E. (See Section 1.2 in Bonahon [13] for a discussion of why this is equivalent to the more abstract definition of ends.)

Inspired by the examples above, we say that an end of a hyperbolic 3-manifold is **simply degenerate** if it has a neighborhood U which is homeomorphic to $S \times (0, \infty)$ (where S is homeomorphic to the boundary of E) and there exists a sequence $\{f_n : S \to U\}$ of simplicial hyperbolic surfaces so that $\{f_n(S)\}$ **exits the end** E (i.e. for any compact subset K of the closure of E, there exists N so that if $n \geq N$, then $f_n(S) \cap K = \emptyset$) and, for all n, f_n is homotopic, within U to the inclusion map $\iota : S \to S \times \{1\}$ given by $\iota(x) = (x, 1)$.

Furthermore, we say that an end of N_Γ is convex cocompact if it has a neighborhood disjoint from $C(N_\Gamma)$. Notice, that N_Γ is convex cocompact if and only if every end of N_Γ is convex cocompact.

We say that N_Γ is **geometrically tame** if every end of N_Γ is either simply degenerate or convex cocompact. Notice that if N_Γ is geometrically tame, then every end has a neighborhood of the form $S \times (0, \infty)$. It follows, that if N_Γ is geometrically tame, then it is also **topologically tame**, i.e. homeomorphic to the interior of a compact 3-manifold.

We note that these definitions can be exended to the setting of hyperbolic 3-manifolds with cusps, but we will forego the technicalities involved.

Historical Remarks: (1) Thurston's original definition of a simply degenerate end was only given in the setting where the compact core has incompressible boundary. In this case, the assumption that each simplicial hyperbolic surface f_n is homotopic to the inclusion map ι within U is unnecessary (in fact, follows from results in 3-manifolds). His simpler definition was that there exist a sequence of simple closed curves α_n on ∂E whose geodesic representatives $\{\alpha_n^*\}$ exit E. Given such a sequence of curves one may construct simplicial hyperbolic surfaces f_n which realize α_n with one vertex, and one may easily check that the resulting hyperbolic surfaces must also exit E. However, in the compressible case this additional assumption is necessary to ensure that the regions homologically bounded by $f_n(S)$ and ∂E exhaust E. This new definition first appears in [25].

(2) One need not use simplicial hyperbolic surfaces in the definition of simply degenerate ends. Most of Thurston's work uses pleated surfaces instead of simplicia hyperbolic surfaces. Minsky [57] showed that one can use harmonic maps in place of simplicial hyperbolic surfaces. In recent work, authors often simply use 1-Lipschitz maps of hyperbolic surfaces (which include all of these examples). However, one cannot use minimal surfaces, since they are much more sparse than these other examples.

4.4. **The Tameness Theorem**

Marden [50] presciently conjectured that every hyperbolic 3-manifold with finitely generated fundamenal group is topologically tame.

The first major breakthrough was Thurston's proof [75] that if M has incompressible boundary and $\{\rho_n\} \subset CC(M)$ converges to $\rho \in AH(M)$ and $\rho(\pi_1(M))$ contains no parabolic elements, then N_ρ is geometrically tame, and hence topologically tame. Bonahon [13] proved much more generally that if N_Γ has freely indecomposable fundamental group then N_Γ is geometrically tame. Canary [25] used Bonahon's work to show that if N_Γ is topologically tame, then N_Γ is geometrically tame.

Agol [2] and Calegari-Gabai [22] independently completed the picture by proving the full tameness conjecture. Soma [70] later gave a simplified proof combinng aspects of both of their proofs.

Tameness Theorem: (Agol, Calegari-Gabai) *If N is a hyperbolic 3-manifold with finitely generated fundamental group, then N is geometrically tame, and hence topologically tame.*

A sketch of the proof is beyond the reach of a short mini-course, so we will instead discuss a few applications of their result. A more complete discussion of the history of Marden's Tameness Conjecture and its applications is given in the survey article [28]. (In fact, we have self-plagiarized a portion of this article in the next few sections.)

4.5. Ahlfors' Measure Conjecture

One of the major classical conjectures in the study of hyperbolic 3-manifolds is Ahlfors' Measure Conjecture, which was motivated in part by his result for convex cocompact hyperbolic 3-manifolds which we discussed as Ahlfors' Measure Theorem above.

Ahlfors' Measure Conjecture: *If $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold with finitely generated fundamental group, then either $\Lambda(\Gamma)$ has measure zero or $\Lambda(\Gamma) = \partial\mathbb{H}^3$ and Γ acts ergodically on $\partial\mathbb{H}^3$.*

The proof of Ahlfors' measure theorem suggests that we study the behavior of hyperbolic 3-manifolds. Thurston established the following minimum principle for positive superharmonic functions on geometrically tame hyperbolic 3-manifolds with finitely generated fundamental group and Canary observed that with the correct definition of geometric tameness Thurston's proof generalized to the setting of all geometrically tame hyperbolic 3-manifolds.

Theorem 4.1. *(Thurston [75], Canary[25]) Let N be a geometrically tame hyperbolic 3-manifold. If $h : N \to (0,\infty)$ is a positive superharmonic function, i.e.* $\mathrm{div}(\mathrm{grad}\ h) \geq 0$*, then*

$$\inf_{C(N)} h = \inf_{\partial C(N)} h.$$

In particular, if $C(N) = N$, then h is constant.

Idea of proof: If h is non-constant, consider the flow $\{\phi_t\}$ generated by $-\mathrm{grad}\ h$, i.e. the flow in the direction of maximal decrease. The fact that h is superharmonic guarantees that this flow is volume non-decreasing.

The fact that h is positive guarantees that the flow moves more and more slowly as one progresses. More concretely, if $x \in N$, T is a measurable subset of $[0,\infty)$, $A = \{\phi_t(x) : t \in T\}$ is the associated subset of the forward flow line and $\ell(A)$ is the length of A in N, then the Cauchy-Schwartz inequality implies that

$$\ell(A)^2 = \int_A \left(\frac{1}{\sqrt{|\mathrm{grad}\ h|}}\sqrt{|\mathrm{grad}\ h|}\right)^2\ ds \leq \left(\int_A \frac{1}{|\mathrm{grad}\ h|}ds\right)\ \left(\int_A |\mathrm{grad}\ h|\ ds\right) \leq \ell(T)h(x)$$

where ds is the measure of arc length on the flow line. Notice that this implies that $\{\phi_t\}$ exist for all positive time.

Neighborhoods of radius one of (the images of) our simplicial hyperbolic surfaces have bounded volume, so act as narrows for the flow. More concretely,

Lemma 4.2. *Given $A > 0$, there exists $C > 0$ so that if $f : S \to N$ is a simplicial hyperbolic surface and whenever α is a homotopically non-trivial curve on S so that $f(\alpha)$ is homotopically trivial in N, then $f(\alpha)$ has length at least A, then*

$$\mathrm{vol}\big(\mathcal{N}_1(f(S))\big) \leq C|\chi(S)|$$

where vol *denotes volume, $\mathcal{N}_1(f(S))$ is the metric neighborhood of radius one of $f(S)$ and $\xi(S)$ is the Euler characteristic of S.*

If $C(N) \neq N$, we use these facts to show that almost every flow line starting in $C(N)$ must intersect the boundary $\partial C(N)$.

Let x be a point in the interior of $C(M)$ and choose a small ball B about x in $C(N)$ so that $\phi_1(B)$ is disjoint from B. Supposes that there is a measurable subset B_0 of B of positive measure so that if $y \in B_0$, then $\phi_t(y)$ lies in the interior of $C(N)$ for all $t \geq 0$. Since $\phi_n(B) \cap B = \emptyset$ for all $n \geq 0$ and ϕ_t is volume non-decreasing, the flow line emanating from y must leave every compact subset of $C(N)$. So there exists a boundary component of the compact core of N and a family of simplicial hyperbolic surfaces $f_n : S \to C(N)$ so that the flow line through y passes through $\mathcal{N}_1(f_n(S))$. We further assume that the surfaces $\{\mathcal{N}_1(f_n(S))\}$ are mutually disjoint. For any N, let

$$T_y^N = \{t \geq 0 : \phi_t(y) \in \mathcal{N}_1(f_n(S)) \text{ for some } n \leq N\} \text{ and let } A_y^N = \{\phi_t(y) : t \in T^N(y)\}.$$

We know that $\ell(A_y^N) \geq N$ and hence that $\ell(T_Y^N) \geq N^2$. One may then show that

$$\sum_{n\in N} \mathrm{vol}\left(\bigcup_{n\in\mathbb{N}} \phi_n(B_0) \cap \bigcup_{i=1}^{N} \mathcal{N}_1(f_n(S))\right)$$

grows quadratically in N, which contradicts Lemma 4.2. This completes the proof when $C(N) \neq N$.

When $C(N) = N$, the same argument shows that h must be constant, since the flow cannot exist. $\square$

Combining the Tameness Theorem, the minimum principle Theorem 4.1 and Ahlfors' proof of his Measure Theorem, one immediately obtains a proof of Ahlfors' Measure Conjecture.

Corollary 4.3. *If $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold with finitely generated fundamental group, then either $\Lambda(\Gamma)$ has measure zero or $\Lambda(\Gamma) = \partial\mathbb{H}^3$. Moreover, if $\Lambda(\Gamma) = \partial\mathbb{H}^3$ then Γ acts ergodically on $\partial\mathbb{H}^3$, i.e. if $A \subset \partial\mathbb{H}^3$ is measurable and Γ-invariant, then A has either measure zero or full measure.*

Another immediate consequence of this minimum principle is a characterization of which hyperbolic 3-manifolds admit non-constant positive superharmonic functions.

Corollary 4.4. *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. The manifold N is strongly parabolic (i.e. admits no non-constant positive superharmonic functions) if and only if $\Lambda(\Gamma) = \partial\mathbb{H}^3$.*

Sullivan [71] showed that the geodesic flow of N is ergodic if and only if it admits a (positive) Green's function, so one can also completely characterize when the geodesic flow of N is ergodic.

Corollary 4.5. *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. The geodesic flow of N is ergodic if and only if $\Lambda(\Gamma) = \partial\mathbb{H}^3$.*

4.6. Limit sets of Kleinian groups

Another collection of geometric applications of topological tameness involve the Hausdorff dimension of the limit set and the bottom of the spectrum of the Laplacian. Patterson [67] and Sullivan [72] showed that there are deep relationships between these two quantities. In particular, they showed that if $N = \mathbb{H}^3/\Gamma$ is geometrically finite, then

$$\lambda_0(N) = D(\Lambda(\Gamma))(2 - D(\Lambda(\Gamma))$$

unless $D(\Lambda(\Gamma) < 1$ in which case $\lambda_0(N) = 1$. Here, $D(\Lambda(\Gamma))$ denotes the Hausdorff dimension of the limit set and $\lambda_0(N) = \inf \operatorname{spec}(-\Delta)$ is the bottom of the spectrum of the Laplacian $\Delta = \operatorname{div}(\operatorname{grad})$ Moreover, if $D(\Lambda(\Gamma)) < 1$, then Γ is a free group (see Braam [15], Canary-Taylor [31] and Sullivan [71]).

Sullivan [72] and Tukia [79] showed that if $N = \mathbb{H}^3/\Gamma$ is convex cocompact and has infinite volume, then $\lambda_0(N) > 0$ and $D(\Lambda(\Gamma)) < 2$. Canary [24] proved that if N is topologically tame and geometrically infinite, then $\lambda_0(N) = 0$. (One does this by simply using the simplicial hyperbolic surfaces exiting the end to show that the Cheeger constant of a geometrically infinite manifold is 0.)

Theorem 4.6. (Sullivan [72], Tukia[79], Canary[24]) *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. Then $\lambda_0(N) = 0$ if and only if either N has finite volume or is geometrically infinite.*

Bishop and Jones [12] showed that geometrically infinite hyperbolic 3-manifolds have limit sets of Hausdorff dimension 2 without making use of tameness. Combining all the results we have mentioned one gets the following result.

Corollary 4.7. *Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group. Then,*

$$\lambda_0(N) = D(\Lambda(\Gamma))(2 - D(\Lambda(\Gamma))$$

unless $D(\Lambda(\Gamma) < 1$ in which case $\lambda_0(N) = 1$ and Γ is a free group.

4.7. A covering theorem for hyperbolic 3-manifolds

The restrictive structure of simply degenerate ends places serious restrictions on how it can cover another manifold. Thurston established the following result when the covering manifold has freely indecomposable fundamental group and Canary generalized the argument to the general setting.

Covering Theorem: (Thurston[75], Canary [27]) *Let $\hat{N}$ be a geometrically tame hyperbolic 3-manifold which covers another hyperbolic 3-manifold N by a local isometry $p : \hat{N} \to N$. If $\hat{E}$ is a simply degenerate end of $\hat{N}$ then either*

a) $\hat{E}$ has a neighborhood $\hat{U}$ such that p is finite-to-one on $\hat{U}$, or

b) N has finite volume and has a finite cover N' which fibers over the circle such that if N_S denotes the cover of N' associated to the fiber subgroup then $\hat{N}$ is finitely covered by N_S. Moreover, if $\hat{N} \neq N_S$, then $\hat{N}$ is homeomorphic to the interior of a twisted I-bundle which is doubly covered by N_S.

Idea of proof: Let $\hat{U}$ be a neighborhood of $\hat{E}$ homeomorphic to $S \times (0, \infty)$ and let $\{f_n : S \times \hat{U}\}$ be a sequence of simplicial hyperbolic surfaces exiting E (which are homotopic to the inclusion map ι within $\hat{U}$). There exists $L > 0$ so that for all n there exists an essential simple closed curve α_n on S so that $\ell(f(\alpha_n)) \leq L$. (By the usual "blowing a up a balloon" argument, we may take L to be any number so that

$$\text{area} B_{\mathbb{H}^2}(L, i) = 2\pi \cosh L - 2\pi > 2\pi|\xi(S)| \geq \text{area}(S, \tau_f).)$$

It is then not difficult to show that $f(\alpha_n)$ is homotopically non-trivial and $f(\alpha_n)^*$ is homotopic to $f(\alpha_n)$ (within U) for all large enough n and $\{f(\alpha_n)^*\}$ exits E. We may then replace f_n by a simplicial hyperbolic surface $g_n : S \to \hat{U}$, for all large n, which realizes α_n with one vertex so that $\{g_n : S \times \hat{U}\}$ is a sequence of simplicial hyperbolic surfaces exiting E (which are homotopic to the inclusion map ι within $\hat{U}$).

Hatcher [38] showed that any two triangulations of S with one vertex can be joined by a path of triangulations each of which is obtained from the previous one by an "elementary move." Using Hatcher's result, if $n \neq m$, we can either

(1) Produce a continuous family $\{h_t : S \to \hat{N}\}_{t \in [0,1]}$ of simplicial hyperbolic surfaces (with at most two vertices) joining g_m to g_n (i.e. $h_0 = g_m$ and $h_1 = g_n$), or

(2) Produce continuous families $\{h_t : S \to \overline{\hat{U}}\}_{t \in [0,1]}$ and $\{j_t : S \to \hat{N}\}_{t \in [0,1]}$ simplicial hyperbolic surfaces (with at most two vertices) so that $h_0 = g_m$ and $h_1(S_n) \cap \partial\overline{\hat{U}}$ is non-empty and $j_0 = g_n$ and $j_1(S_n) \cap \partial\overline{\hat{U}}$

If S is π_1-injective in N, then case (1) always occurs. In either situation, this allows us to find a smaller neighborhood U of N, so that if $x \in U$, there exists a simplicial hyperbolic surface $h_x : S \to \hat{U}$ (which is isotopic to ι within $\hat{U}$). One corollary of this observation is:

Corollary 4.8. *If N is a hyperbolic 3-manifold with finitely generated fundamental group, then there exists L so that $C(N)$ does not contain an embedded hyperbolic ball of radius L.*

Suppose that p is infinite-to-one on U. So there exists a point $x \in N$ so that $p^{-1}(x) \cap U$ is infinite. Index $p^{-1}(x) \cap U = \{x_i\}_{i \in N}$ so that x_i exits the end E. For all i, let $f_i : S \to \hat{U}$ be a simplicial hyperbolic surface so that $x_i \in f_i(S)$ (homotopic to ι within $\hat{U}$). We may pass to a subsequence so that $p(f_i(S))$ converges. Choose i and j large enough so that there is a short homotopy in N between $p(f_i(S))$ and $p(f_j(S))$ and $f_i(S)$ and $f_j(S)$ are far apart in $\hat{U}$. We concatenate the p-image of a homotopy between $f_i(S)$ and $f_j(S)$ within $\hat{U}$ and the short homotopy between $f_i(S)$ and $f_j(S)$ to construct a map $g : M \to N$ of a closed 3-manifold M which fibers over the circle into N.

If S is π_1-injective, then we may show that g is π_i-injective. (Notice that the "lift"of the fibre of M to $\hat{N}$ is an arc in $\hat{U}$ joining a point in $f_i(S)$ to $f_j(S)$, so is not a closed loop.) Let N' be the cover of N associated to $g_*(\pi_1(M))$ and consider the lift $\tilde{g} : M \to N'$. Since $\tilde{g}$ is a homotopy equivalence, N' must be closed (for homological reasons). Waldhausen's Theorem (see [39, Thm.13.9] then implies that $\tilde{g}$ is homotopic to a homeomorphism. Moreover, since N' is closed, it must be a finite cover of N.

If S is not π_1-injective, let L_i be the length (in τ_{f_i} of the shortest compressible curve on $f_i(S)$ (i.e. an essential curve in S which is mapped to a homotopically trivial curve in $\hat{N}$). One may show that $L_i \to \infty$. This then contradicts the fact that the $p(f_i(S))$ can be chosen to accumulate in N. $\square$

A subgroup $\hat{\Gamma}$ of Γ is said to be a *virtual fiber subgroup* if there exist finite index subgroups Γ_0 of Γ and $\hat{\Gamma}_0$ of $\hat{\Gamma}$ such that $N_0 = \mathbb{H}^3/\Gamma_0$ fibers over the circle and $\hat{\Gamma}_0$ corresponds to the fiber subgroup. Corollary 4.9 is the key tool in many of the group-theoretic applications of Marden's Tameness Conjecture.

Corollary 4.9. *If $N = \mathbb{H}^3/\Gamma$ is a closed hyperbolic 3-manifold and $\hat{\Gamma}$ is a finitely generated subgroup of Γ, then $\hat{\Gamma}$ is either convex cocompact or a virtual fiber subgroup.*

Thurston gave a beautiful proof, using that a cover of an infinite-volume convex cocompact hyperbolic 3-manifold is convex cocompact if it has finitely generated fundamental group. (See [27] for the general statement and proof.)

Theorem 4.10. *If $N = \mathbb{H}^3/\Gamma$ is an infinite volume convex cocompact hyperbolic 3-manifold and Γ_0 is a finitely generated (non-abelian) subgroup of Γ, then Γ_0 is convex compact.*

Proof. Ahlfors' Finiteness Theorem [3] implies that if Δ is a finitely, generated, torsion-free, non-abelian Klelnian group which does not contain any parabolic elements, then $\partial_c N_\Delta$ is a finite collection (possibly empty) of closed surfaces. It follows from our earlier discussion that $\partial C_1(N_\Delta)$ is also a finite collection of closed surfaces, and hence that $\partial C(N_\Delta)$ is compact.

Since N_Γ is convex cocompact and infinite volume, $\partial C(N_\Gamma)$ is non-empty. Since $C(N_\Gamma)$ is compact, there exists D so that if $x \in C(N_\Gamma)$, then $d(x, \partial C(N_\Gamma)) \leq D$. Lifting to the cover we see that if $y \in CH(\Lambda(\Gamma))$, then $d(y, \partial CH(\Lambda(\Gamma)) \leq D$.

Since $\Gamma_0 \subset \Gamma$, $\Lambda(\Gamma_0) \subset \Lambda(\Gamma)$, so $CH(\Lambda(\Gamma_0)) \subset CH(\Lambda(\Gamma))$. If $y \in CH(\Lambda(\Gamma_0)$, then $y \in CH(\Lambda(\Gamma)$, so $d(y, \partial CH(\Lambda(\Gamma)) \leq D$. But, again since $CH(\Lambda(\Gamma_0)) \subset CH(\Lambda(\Gamma))$, we see that $d(y, \partial CH(\Lambda(\Gamma_0)) \leq D$. So, if $x \in C(N_{\Gamma_0})$, then $d(x, \partial C(N_{\Gamma_0}) \leq D$. Since $\partial C(N_{\Gamma_0})$ is compact (by the above discussion), $C(N_{\Gamma_0})$ is a closed bounded subset of N_{Γ_0}, hence compact. Therefore, Γ_0 is convex cocompact. $\square$

More generally, one may use the covering theorem to completely describe exactly which covers of a hyperbolic 3-manifold with finitely generated fundamental group are convex cocompact (see [27]).

Further discussion of the group theoretic consequences of the Covering Theorem are given in [28].

5. The classification of hyperbolic 3-manifolds

One may view the conformal boundary of a convex cocompact hyperbolic as capturing the asymptotic geometry of the 3-manifolds, so from this viewpoint Bers' parametrization theorems assert that a convex cococompact hyperbolic 3-manifold is determined by its asymptotic geometry.

One may associate a geodesic lamination to a simply degenerate end, which we call its ending lamination, which captures the asymptotic geometry. Thurston conjectured that a hyperbolic 3-manifold is completely determined by its conformal boundary and the ending laminations of its simply degenerate ends. The proof of the Ending Lamination Theorem verified his conjecture. We will attempt to give a complete version of the statement in the case when the manifold has no cusps. We will then discuss some of its consequence, including the proof of the Bers-Sullivan-Thurston Density Conjecture. If you haven't seen this material before, I encourage you to focus only on the case where the compact core has incompressible boundary where the statements are much simpler.

5.1. **Geodesic laminations**

We begin with a brief discussion of the theory of geodesic laminations.

A *geodesic lamination* on a hyperbolic surface X is a closed set which is a disjoint union of simple complete geodesics, i.e. geodesics in the disjoint union are either simple closed geodesics or bi-infinite simple geodesics. The simplest examples are disjoint unions of simple closed geodesics. In fact every geodesic lamination is a (Gromov-Hausdorff) limit of a sequence of a sequence of finite-leaved geodesic laminations.

We now explain how to obtain a maximal finite-leaved geodesic lamination whose complement is a finite collection of ideal triangles. We begin with the case of a pair of pants with geodesic boundary. Consider the three common perpendicular geodesic segments which join pairs of sides. These geodesic segments decompose the pair of pants into two all-right angled hexagons. One can spin the vertices along the geodesic, reducing the angle and lengthening each segment. If one takes a limit as one spins a larger and larger amount, each geodesic segment will converge to a bi-infinite geodesic each of whose ends spirals about one of the closed geodesics. The complement of the three leaves is a union of two ideal triangles.

Given a geodesic pants decomposition of a hyperbolic surface, one may perform the same operation on each pair of pants to obtain a finite leaved lamination which contains the original pants decomposition such that each component of its complement is an ideal triangle. More generally, given a collection C of disjoint simple closed geodesics on the surface, one may place one vertex on each component of C and complete C to a triangulation T. If we spin T about C we obtain a finite-leaved geodesic lamination whose closed geodesics are exactly C and so that all other geodesics are bi-infinite geodesics each of whose ends spiral about a component of C. All maximal finite-leaved laminations may be obtained in this manner.

It is much harder to draw a picture of a "typical" geodesic lamination has uncountably many leaves and the intersection with a short goedesic segment transverse to the lamination is typically a Cantor set.

If $X = \mathbb{H}^2/\Gamma_X$, then a geodesic lamination λ lifts to a Γ_X-invariant geodesic lamination $\tilde{\lambda}$. The collection C_λ of pairs of points in $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ which arise as endpoints of geodesics in $\tilde{\lambda}$ is a Γ_X-invariant subset which determines $\tilde{\lambda}$ and hence λ. If $h : X \to Y$ is a homeomorphism and $Y = \mathbb{H}^2/\Gamma_Y$, then h lifts to a homeomorphism $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ conjugating the action of Γ_X to Γ_Y and $\tilde{h}$ extends to a homeomorphism $\partial h : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ conjugating the action of Γ_X to Γ_Y. One way then transport λ on X to a geodesic lamination on Y, we simply look at the geodesic lamination determined by the pairs of points given by $(\partial h \times \partial h)(C_\lambda)$. The set $(\partial h \times \partial h)(C_\lambda)$ gives rise to a Γ_Y-invariant geodesic lamination on $\mathbb{H}^2$, which descends to a lamination on Y. Therefore, it makes sense to talk about the space $GL(S)$ of geodesic laminations on S without choosing a specific hyperbolic structure on S.

More complete discussions of geodesic laminations are contained in the notes of Canary-Epstein-Green [23] , the book of Casson-Bleiler [32] or Section 8.3 of Martelli [51].

5.2. **Ending laminations**

Suppose that E is a simply degenerate end and let U be a neighborhood of E homeomorphic to $S \times (0, \infty)$ and let $\{f_n : S \times U\}$ be a sequence of simplicial hyperbolic surfaces exiting E (which are homotopic to the inclusion map ι within U). There exists $L > 0$ so that for all n there exists an essential simple closed curve α_n on S so that $\ell(f(\alpha_n)) \leq L$. By the usual "blowing a up a balloon" argument, we may take L to be any number so that

$$\text{area} B_{\mathbb{H}^2}\left(\frac{L}{2}, i\right) = 2\pi \cosh \frac{L}{2} - 2\pi > 2\pi|\xi(S)| \geq \text{area}(S, \tau_f).$$

It is then not difficult to show that $f_n(\alpha_n)$ is homotopically non-trivial and $f_n(\alpha_n)^*$ is homotopic to $f_n(\alpha_n)$ (within U) for all large enough n and $\{f_n(\alpha_n)^*\}$ exits E. Let λ_∞ be a Gromov-Hausdorff limit of (some subsequence of) α_n. Then the **ending lamination** of the end E is obtained from λ_∞ by removing any isolated leaves. (A leaf of a geodesic lamination is **isolated** it if is not a limit of a sequence of other leaves. For example, every leaf in a finite-leaved lamination is isolated.)

If one is familiar with the language of measured laminations one may make a more natural version of this definition. One places a hyperbolic structure X on S, assumes that each α_n is geodesic in X and then considers the sequence of unit length measured laminations $\{\mu_n = \frac{\alpha_n}{\ell_X(\alpha_n)}\}$. Then λ is the support of any measured lamination which arises as the limit of a subsequence of $\{\mu_n\}$.

Thurston [75] showed that if the compact core of N has incompressible boundary then the ending lamination of any simply degenerate end is well-defined. He also showed that the end lamination is filling (intersects every simple closed curve) and minimal (contains no proper sublamination). His proof relied on the following crucial estimate, whose proof and statement were corrected by Bonahon [13].

Lemma 5.1. *Suppose that N is a geometrically tame hyperbolic 3-manifold with a compact core C with incompressible boundary. If E is a simply degenerate end of N bounded by a component S of ∂C, there exists a constant K so that if α_1 and α_2 are essential closed curves on S (not necessarily simple) so that $\alpha_1^*, \alpha_2^* \subset E$, $d(\alpha_i^*, S) \geq D$ for $i = 1, 2$, and each α_i^* is either the core curve of a Margulis tube or disjoint from all Margulis tubes in N, then*

$$i(\alpha_1, \alpha_2) \leq K e^D \ell_N(\alpha_1^*) \ell_N(\alpha_2^*) + 2$$

where $i(\alpha_1, \alpha_2)$ is the geometric intersection number of α_1 and α_2 on S.

It E is a simply degenerate end of a hyperbolic 3-manifold whose compact core has compressible boundary and $S = \partial E$, Canary [25] showed that the ending lamination is well-defined up to the action of $\text{Mod}(C, S)$ which is the subgroup of $\text{Mod}(S)$ generated by Dehn twists about essential simple closed curves in S which bound compressing disks in C. This is consistent with the fact that if E is a convex cocompact end of a hyperbolic 3-manifold whose compact core has compressible boundary, then the conformal structure on the component of the conformal boundary associated to E is only well-defined up to the action of $\text{Mod}(S, C)$. Moreover, in this case the ending lamination is always the support of a projective measured lamination lying in the Masur domain, which is a domain of discontinuity for the action of $\text{Mod}(C, S)$ on the space $PL(S)$ of projective classes of measured laminations. (See [52] for a discussion of the Masur domain.)

5.3. **The Ending Lamination Theorem**

We now describe the ending invariants of a hyperbolic 3-manifold with finitely generated fundamental group and no cusps. Let C be a compact core for N (which is well-defined up to homeomorphism homotopic to the identity). The conformal boundary is identified with a collection of components of ∂C, so gives rise to a conformal structure on these components. The remaining components of ∂C bound simply degenerate ends of N, so each such component inherits an ending lamination. The **ending invariants** of N consist of the compact submanifold C together with either a conformal structure or a filling, minimal lamination on each component of ∂C. If N is a hyperbolic 3-manifold (with

Thurston conjectured that this information determined N up to isometry. Brock, Canary and Minsky [59, 18] proved this conjecture for topologically tame hyperbolic 3-manifolds as the culmination of a long-term program developed by Minsky. The proof depends crucially on work of Masur and Minsky [53, 54] on the curve complex of a surface. Given the resolution of Marden's Tameness Conjecture we have the following:

Ending Lamination Theorem: *A hyperbolic 3-manifold with finitely generated fundamental group is determined up to isometry by its ending invariants.*

One can completely describe the set of ending invariants which arise as ending invariants of hyperbolic 3-manifolds with finitely generated fundamental group, see Theorem 1.3 in Namazi-Souto, so the Ending Lamination Theorem may be viewed as a classification theorem. However t it does not provide a parametrization of $AH(M)$ since the ending invariants do not vary continuously. In fact, Bromberg [21] and Magid [49] showed that $AH(S \times [0,1])$ is never locally connected. A survey of the deformation theory of Kleinian groups, circa 2008, is given in [29].

Historical Remarks: Minsky [58] earlier established the Ending Lamination Theorem for hyperbolic 3-manifolds with a (positive) lower bound on their injectivity radius and freely indecomposable fundamental group. Alternate approaches to the proof of the Ending Lamination Theorem have been given by Bowditch [14] and Rees [68].

5.4. **The Bers-Sullivan-Thurston Density Conjecture and other consequences**

Sulivan [73] proved that $CC(M)$ is the interior of $AH(M)$.

Theorem 5.2. *Let M be a compact, oriented, irreducible 3-manifold with non-empty boundary such that $\pi_1(M)$ is infinite, non-abelian and does not contain a free abelian subgroup of rank 2. Then $CC(M)$ is the interior of $AH(M)$.*

Idea of Proof: Let ρ be an interior point of $AH(M)$. One may use the λ-lemma to show that there exists a neighborhood U of ρ so that every representation in U is quasiconformally conjugate to ρ/ Sullivan proved that the Beltrami differential of every quasiconformal deformation of a finitely generated Kleinian group is supported on the limit set. Therefore, U is identified with an open set in $\mathcal{T}(\partial_c N_\rho)$. Since the dimension of the component of $X(M)$ containing ρ agrees with the dimension of $\mathcal{T}(\partial M)$, it follows that $\mathcal{T}(\partial_c N_\rho)$ has the same dimension as $\mathcal{T}(\partial M)$, so $\chi(\partial M) = \chi(\partial_c N_\rho)$. Topological considerations then imply that $\partial_c N_\rho$ compactifies N_ρ, so ρ is convex cocompact. $\square$

Bers, Sullivan and Thurston conjectured that every representation in $AH(M)$, with our restrictions on M, is a limit of convex cocompact representations. Given Sullivan's theorem we can rephrase this conjecture by saying that $AH(M)$ is the closure of its interior. This rephrasing is valid for all manifolds M whose interior admits a hyperbolic structure.

Density Theorem: *If M is a compact irreducible 3-manifold and $AH(M)$ is non-empty, then $AH(M)$ is the closure of its interior* $\mathrm{int}(AH(M))$.

The proof of the Density theorem has a rather convoluted history. The first major breakthrough was due to Bromberg [20] who established a version of this conjecture for $\rho \in AH(S \times [0,1])$ where N_ρ has no cusp and $\partial_c N_\rho$ has exactly one component. His argument was generalized by Brock-Bomberg [17] to prove the conjecture for $\rho \in AH(M)$ where M has incompressible boundary and N_ρ has no cusps. Their approach makes use of the deformation theory of cone-manifolds developed by Hodgson-Kerckhoff [40, 41] and Bromberg [19]. It only makes use of the proof of the Ending Lamination Conjecture in the case where there is a lower bound on injectivity radius, which was established earlier by Minsky [58].

If M has incompressible boundary, one may derive the Density Theorem from the Tameness Theorem, the Ending Lamination Theorem, and convergence results of Thurston [77, 78] (see the discussion in [18].) Basically, the idea here is to consider the end invariants of a given 3-manifold, use the convergence results to construct a hyperbolic 3-manifold with the given end invariants which arises as a limit of geometrically finite hyperbolic 3-manifolds. One then applies the Ending Lamination Theorem to show that the manifold you constructed agrees with the original hyperbolic 3-manifold.

If M has compressible boundary, one replaces Thurston's convergence results with convergence results of Kleineidam-Souto [44], Lecuire [46] and Kim-Lecuire-Ohshika [45]. However, especially in the case that M is homotopy equivalent to a compression body (e.g. a handlebody), significant technical difficulties arise in showing that the limits have the correct ending invariants. Namazi-Souto [63] and Ohshika [64] overcome these obstacles to complete the proof of the Density Theorem in all cases.

Another consequence of the proof of the Ending Lamination Theorem is the following common generalization of Mostow [62] and Sullivan's [71] rigidity theorems.

Corollary 5.3. *Let G be a finitely generated, torsion-free group which is not virtually abelian. If two discrete faithful representations $\rho_1 : G \to \mathsf{PSL}(2,\mathbb{C})$ and $\rho_2 : G \to \mathsf{PSL}(2,\mathbb{C})$ are conjugate by an orientation-preserving homeomorphism ϕ of $\partial\mathbb{H}^3$, then they are quasiconformally conjugate. Moreover, if ϕ is conformal on $\Omega(\rho_1(G))$, then ϕ is conformal.*

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