TOPOLOGICAL RESTRICTIONS ON ANOSOV REPRESENTATIONS

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ABSTRACT. We characterize groups admitting Anosov representations into $\mathsf{SL}(3,\mathbb{R})$, projective Anosov representations into $\mathsf{SL}(4,\mathbb{R})$, and Borel Anosov representations into $\mathsf{SL}(4,\mathbb{R})$. More generally, we obtain bounds on the cohomological dimension of groups admitting P_k-Anosov representations into $\mathsf{SL}(d,\mathbb{R})$ and offer several characterizations of Benoist representations.

1. INTRODUCTION

Anosov representations of hyperbolic groups into higher rank semi-simple Lie groups were introduced by Labourie [40] in his work on Hitchin representations and, after further development by Guichard-Wienhard [28], Guéritaud-Guichard-Kassel-Wienhard [26], Kapovich-Leeb-Porti [36, 37] and others, are widely recognized as the natural higher rank analogue of convex cocompact representations into rank one Lie groups. However, very little is known about which hyperbolic groups admit Anosov representations. Most known Anosov representations either have free groups or surface groups as domain groups or arise by considering a convex cocompact representation ρ into a rank one Lie group H and a "nice" representation $\tau : \mathsf{H} \to \mathsf{G}$ of H into a higher rank Lie group G, and deforming $\tau \circ \rho$. The only examples which are not of this form are due to Benoist [10] and Kapovich [35] and, more recently, to Danciger, Guéritaud, Kassel, Lee and Marquis [20, 22, 41].

In this paper, we initiate a study of the class of torsion-free hyperbolic groups admitting Anosov representations into $\mathsf{SL}(d,\mathbb{R})$. We begin by characterizing groups admitting projective Anosov representations into $\mathsf{SL}(3,\mathbb{R})$ or $\mathsf{SL}(4,\mathbb{R})$. We then obtain, for any d, restrictions on the cohomological dimension of groups admitting Anosov representations into $\mathsf{SL}(d,\mathbb{R})$. We further show that any group admitting a Borel Anosov representation into $\mathsf{SL}(4,\mathbb{R})$ is either a surface group or a free group. Finally, we study and obtain characterizations of Benoist representations, i.e. those projective Anosov representations whose images act properly and cocompactly on strictly convex domains in projective space.

We say that a representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is P_k-Anosov if $k \leq \frac{d}{2}$ and ρ is Anosov with respect to the parabolic group which is the stabilizer of a k-plane in $\mathbb{R}^d$. We will refer to P_1-Anosov representations as *projective Anosov* representations, while representatios which are P_k-Anosov for all k will be called *Borel Anosov*. See Section 2 for detailed definitions.

Theorem 1.1. *If Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(3,\mathbb{R})$ is an Anosov representation, then Γ is either a free group or a surface group.*

Theorem 1.2. *If Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(4,\mathbb{R})$ is a projective Anosov representation, then Γ is isomorphic to a convex cocompact subgroup of $\mathsf{PO}(3,1)$. In particular, Γ is the fundamental group of a compact hyperbolizable 3-manifold.*

Our most general result is that, with four explicit exceptions, if $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-Anosov then Γ has cohomological dimension at most $d - k$. We will see in the proof that each exception is related to one of the four Hopf fibrations.

Theorem 1.3. *Suppose Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-Anosov.*

(1) *If (d, k) is not $(2, 1)$, $(4, 2)$, $(8, 4)$ or $(16, 8)$, then Γ has cohomological dimension at most $d - k$.*

(2) *If (d, k) is $(2, 1)$, $(4, 2)$, $(8, 4)$ or $(16, 8)$, then Γ has cohomological dimension at most $d - k + 1$. Moreover, if Γ has cohomological dimension $d - k + 1$, then $\partial \Gamma$ is homeomorphic to S^{d-k} and, if $k \neq 1$, ρ is not projective Anosov.*

Benoist representations are one of the richest classes of examples of Anosov representations, see for example [7, 11]. We recall that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a *Benoist representation*, if ρ has finite kernel and $\rho(\Gamma)$ preserves and acts properly discontinuously and cocompactly on a strictly convex domain in $\mathbb{P}(\mathbb{R}^d)$. Notice that, almost by definition, $\rho(\Gamma)$ must have virtual cohomological dimension $d - 1$ and recall that Benoist representations are projective Anosov (see [28, Prop 6.1]). Theorem 1.3 implies that they are *only* projective Anosov.

Corollary 1.4. *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a Benoist representation and $\frac{d}{2} \geq k \geq 2$, then ρ is not P_k-Anosov.*

Conversely, we will see that, when $d \geq 4$, Benoist representations are characterized, among Anosov representations, entirely by the cohomological dimension of their domain group. Notice that the Anosov representations of surface groups into $\mathsf{SL}(3, \mathbb{R})$ studied by Barbot [5] are counterexamples to the statement of Theorem 1.5 when $d = 3$.

Theorem 1.5. *If $d \geq 4$, an Anosov representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ of a torsion-free hyperbolic group Γ is a Benoist representation if and only if Γ has cohomological dimension $d - 1$.*

Labourie [40] showed that Hitchin representations are irreducible and Borel Anosov. Andres Sambarino asked whether any torsion-free Borel Anosov subgroup of $\mathsf{SL}(d, \mathbb{R})$ is either free or a surface group. Theorem 1.1 settles this question in the affirmative when $d = 3$. Here, we answer Sambarino's question when $d = 4$. Tsouvalas [49] subsequently answered Sambarino's questions when $d = 4q + 2$ for some $q \in \mathbb{N}$. We know of no counterexamples in any dimension.

Theorem 1.6. *If Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(4, \mathbb{R})$ is Borel Anosov, then Γ is either a surface group or a free group.*

Hitchin representations are the only known Borel Anosov representations of surface groups when d is even. Hitchin representations are irreducible (see [40, Lemma 10.1]), but whenever d is odd one may use Barbot's construction to produce reducible Borel Anosov representations of surface groups into $\mathsf{SL}(d, \mathbb{R})$. In Proposition 7.2, we show that every Borel Anosov representation of a surface group into $\mathsf{SL}(4, \mathbb{R})$ is irreducible. One might hope that all Borel Anosov representations of surface groups into $\mathsf{SL}(4, \mathbb{R})$ are Hitchin. In Proposition 8.1 we show that the restriction of a Borel Anosov representation to an infinite index surface subgroup cannot be Hitchin.

We also extend Theorem 1.5 to replace the assumption that ρ is Anosov with the simpler assumption that ρ admits a non-constant limit map into $\mathbb{P}(\mathbb{R}^d)$.

Theorem 1.7. *Suppose that $d \geq 4$ and Γ is a torsion-free hyperbolic group. A representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a Benoist representation if and only if Γ has cohomological dimension $d - 1$ and there is a ρ-equivariant continuous non-constant map $\xi : \partial \Gamma \to \mathbb{P}(\mathbb{R}^d)$.*

In Section 10 we discuss examples and questions related to our results.

Remark: One can obtain versions of all the results above in the case when Γ has torsion. We recall that a representation is P_k-Anosov if and only if its restriction to a finite index subgroup is P_k-Anosov and that finitely generated linear groups have finite index torsion-free subgroups. It follows that if Γ is not assumed to be torsion-free in the statements of Theorems 1.1 and 1.6, one can still conclude that $\rho(\Gamma)$ has a finite index subgroup which is a free group or a surface group, while in Theorem 1.2 one can conclude that $\rho(\Gamma)$ has a finite index subgroup isomorphic to a convex cocompact subgroup of $\mathsf{PO}(3, 1)$. In Theorem 1.3 one can conclude that $\rho(\Gamma)$ has the same bounds on its virtual cohomological dimension, which one obtains on the cohomological dimension of Γ in the torsion-free setting. If Γ has a finite index torsion-free subgroup, one gets the same bounds on the virtual cohomological dimension of $\rho(\Gamma)$. In Theorems 1.5 and 1.7 one must replace the assumption that Γ has cohomological dimension $d-1$ with the assumption that $\rho(\Gamma)$ has virtual cohomological dimension $d-1$.

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2. BACKGROUND

2.1. **Anosov representations.** We briefly recall the definition of an Anosov representation and its crucial properties. We first set some notation. If Γ is a hyperbolic group, we will fix a finite generating set for Γ and let $|\gamma|$ denote the minimal word length of γ and let $||\gamma||$ denote the minimal word length of an element conjugate to γ (i.e. the minimal translation length of the action of γ on Γ). Let $\partial\Gamma$ denote the Gromov boundary of Γ. If $A \in \mathsf{GL}(d, \mathbb{R})$, we let

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_d(A)$$

denote the ordered moduli of the eigenvalues of A (with multiplicity).

We will use a recent theorem of Kassel-Potrie [39] (see also [38, Thm 4.3]) to give a simple definition of P_k-Anosov representations. If Γ is a hyperbolic group and $1 \leq k \leq \frac{d}{2}$ a representation $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ is said to be P_k-*Anosov* if there exist constants $\mu, C > 0$ so that

$$\frac{\lambda_k(\rho(\gamma))}{\lambda_{k+1}(\rho(\gamma))} \geq C e^{\mu||\gamma||}$$

for all $\gamma \in \Gamma$. A representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is said to be *Anosov* if it is P_k-Anosov for some $1 \leq k \leq \frac{d}{2}$ and is said to be *Borel Anosov* if it is P_k-Anosov for every $1 \leq k \leq \frac{d}{2}$. It follows immediately from this definition that Anosov representations have discrete image and finite kernel. (This definition is based on earlier definitions in terms of singular values due to Kapovich-Leeb-Porti [37], Guéritaud-Guichard-Kassel-Wienhard [26] and Bochi-Potrie-Sambarino [14].)

If a representation is P_1-Anosov, we call it *projective Anosov*. Projective Anosov representations are in some sense the most general class of Anosov representations. If $\rho : \Gamma \to \mathsf{G}$ is an Anosov representation into any semisimple Lie group, then there exists an irreducible representation $\tau : \mathsf{G} \to \mathsf{SL}(d, \mathbb{R})$ so that $\tau \circ \rho$ is projective Anosov (see [28, Prop. 4.3]).

If a representation $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ is P_k-Anosov representation, then it admits ρ-equivariant continuous maps

$$\xi_\rho^k : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d) \quad \text{and} \quad \xi_\rho^{d-k} : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$$

into the Grassmanian of k-planes and $(d-k)$-planes in $\mathbb{R}^d$. (If $d = \frac{k}{2}$, then $\xi_\rho^k = \xi_\rho^{d-k}$.) If $\gamma^+ \in \partial\Gamma$ is the attracting fixed point of γ, then $\xi_\rho^k(\gamma^+)$ and $\xi_\rho^{d-k}(\gamma^+)$ are the attracting k-planes and $(d-k)$-planes for $\rho(\gamma)$. Moreover, if $x, y \in \partial\Gamma$ are distinct, then

$$\xi_\rho^k(x) \subset \xi_\rho^{d-k}(x) \quad \text{and} \quad \xi_\rho^k(x) \oplus \xi_\rho^{d-k}(y) = \mathbb{R}^d.$$

If $1 \leq j < k \leq \frac{d}{2}$ and ρ is both P_j-Anosov and P_k-Anosov, then

$$\xi_\rho^j(x) \subset \xi_\rho^k(x) \subset \xi_\rho^{d-k}(x) \subset \xi_\rho^{d-j}(x)$$

for all $x \in \partial\Gamma$. (See Guichard-Wienhard [28, Section 2] for a careful discussion of limit maps of Anosov representations.)

If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a P_k-Anosov representation we may define a sphere bundle

$$E_{\rho,k} = \bigcup_{x \in \partial\Gamma} S(\xi^k(x)) \subset S(\mathbb{R}^d)$$

over $\partial\Gamma$ where if V is subspace of $\mathbb{R}^d$, then $S(V)$ is the unit sphere of V. The bundle map $p_{\rho,k} : E_{\rho,k} \to \partial\Gamma$ is defined so that $p_{\rho,k}(S(\xi^k(x)) = x$. Notice that $p_{\rho,k}$ is well-defined, since, by transversality, $S(\xi^k(x))$ is disjoint from $S(\xi^k(y))$ if $x \neq y$.

Lemma 2.1. *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a P_k-Anosov representation, then $p_{\rho,k} : E_{\rho,k} \to \partial\Gamma$ is a fibre bundle with fibres homeomorphic to S^{k-1}. In particular, if $\partial\Gamma$ has topological dimension m, then $E_{\rho,k}$ has topological dimension $m + k - 1$.*

Proof. If $y \neq x$, let $\pi_{x,y} : \xi^k(y) \to \xi^k(x)$ denote orthogonal projection and notice that there exists an open neighborhood U_x of x in $\partial\Gamma$ such that if $y \in U_x$ then $\pi_{x,y}$ is an isomorphism. Then there is a homeomorphism $\phi_x : p_{\rho,k}^{-1}(U_x) \to U_x \times S(\xi^k(x))$ where

$$\phi_x(z) = \left(p_{\rho,k}(z), \frac{\pi_{x,p(z)}(z)}{||\pi_{x,p(z)}(z)||} \right).$$

Therefore, $p_{\rho,k}$ is a fibre bundle with fibres homeomorphic to S^{k-1}. Since $E_{\rho,k}$ is locally a topological product of $\mathbb{R}^{k-1}$ and a compact Hausdorff space $\partial\Gamma$ of topological dimension m, it has topological dimension $m + k - 1$ (see [30]). $\square$

2.2. **Semisimple representations.** We recall that a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is said to be *semisimple* if the Zariski closure of $\rho(\Gamma)$ in $\mathsf{SL}(d, \mathbb{R})$ is a reductive real algebraic Lie group. Moreover, if ρ is semisimple, there exists a decomposition $\mathbb{R}^d = V_1 \oplus ... \oplus V_k$ into irreducible $\rho(\Gamma)$-modules.

If ρ is a semisimple projective Anosov representation, then the restriction of ρ to the subspace spanned by the image of its limit map is irreducible.

Proposition 2.2. *If Γ is a non-elementary hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a semisimple projective Anosov representation, then the action of $\rho(\Gamma)$ on the subspace $W \subset \mathbb{R}^d$ spanned by $\xi_\rho^1(\partial\Gamma)$ is irreducible. Moreover, if $\rho_W : \Gamma \to \mathsf{SL}^\pm(W)$ is given by $\rho_W(\gamma) = |\det(\rho(\gamma)|_W)|^{-1/\dim(W)}\rho(\gamma)|_W$, then ρ_W is projective Anosov.*

Proof. Let $\mathbb{R}^d = V_1 \oplus ... \oplus V_k$ be the decomposition into irreducible $\rho(\Gamma)$-modules. Let γ be an infinite order element of Γ. Then, since ρ is projective Anosov, $\rho(\gamma)$ is proximal and $\xi_\rho^1(\gamma^+)$ is the attracting eigenline of $\rho(\gamma)$. Since $\xi_\rho^1(\gamma^+)$ is an attracting eigenline, it must be contained in one of the V_i, so we may assume $\xi_\rho^1(\gamma^+) \subset V_1$. Since V_1 is $\rho(\Gamma)$-invariant we have that $\xi_\rho^1(\alpha(\gamma^+)) = \rho(\alpha)(\xi_\rho^1(\gamma^+)) \subset V_1$ for all $\alpha \in \Gamma$. Moreover, since the orbit of γ^+ is dense in $\partial\Gamma$,

we conclude that that $\langle \xi(\partial_\infty \Gamma) \rangle$ is a $\rho(\Gamma)$-invariant vector subspace of V_1. The restriction of ρ to V_1 is irreducible, so it follows that $\langle \xi_\rho^1(\partial \Gamma) \rangle = V_1$.

Notice that ρ_W is projective Anosov, since ρ is projective Anosov and

$$\lambda_1(\rho(\gamma)|_W) = |\det(\rho(\gamma)|_W)|^{-1/\dim(W)} \lambda_1(\rho(\gamma)) \text{ and } \lambda_2(\rho(\gamma)|_W) \leq |\det(\rho(\gamma)|_W)|^{-1/\dim(W)} \lambda_2(\rho(\gamma))$$

for all $\gamma \in \Gamma$. $\qquad\square$

Benoist [6] used work of Abels-Margulis-Soifer [1] to establish the following useful relationship between eigenvalues and singular values for semisimple representation (see [26, Thm 4.12] for a proof).

Theorem 2.3. *If Γ is a finitely generated group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is semisimple, then there exists a finite subset A of Γ and $M > 0$ so that if $\gamma \in \Gamma$, then there exists $\alpha \in A$ so that*

$$|\log \lambda_i(\rho(\gamma\alpha)) - \log \sigma_i(\rho(\gamma))| \leq M$$

for all i, where $\sigma_i(\rho(\gamma))$ is the ith singular value of $\rho(\gamma)$.

Gueritaud, Guichard, Kassel and Wienhard [26, Section 2.5.4] observe that given any representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$, one may define a *semisimplification* $\rho^{ss} : \Gamma \to \mathsf{SL}(d, \mathbb{R})$. They further show that the Jordan projections agree, so ρ and ρ^{ss} share the same Anosov qualities.

Lemma 2.4. ([26, Prop 2.39, Lemma 2.40]) *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation of a hyperbolic group and ρ^{ss} is a semisimplification of ρ, then $\lambda_i(\rho^{ss}(\gamma)) = \lambda_i(\rho(\gamma))$ for all i and all $\gamma \in \Gamma$. In particular, ρ is P_k-Anosov if and only if ρ^{ss} is P_k-Anosov.*

Proposition 2.2 and Lemma 2.4 have the following useful corollary.

Corollary 2.5. *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a reducible projective Anosov representation and ρ^{ss} is its semisimplification, then the action of $\rho^{ss}(\Gamma)$ on the proper subspace W of $\mathbb{R}^d$ spanned by $\xi_{\rho^{ss}}^1(\partial \Gamma)$ is irreducible. Moreover, ρ_W^{ss} is projective Anosov.*

2.3. **Convex cocompactness.** Danciger-Guéritaud-Kassel [20, 21] and Zimmer [52] have recently shown that many projective Anosov representations can be understood as convex cocompact actions on properly convex domains in projective space. We recall that a domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is said to be *properly convex* if it is a bounded subset of an affine chart $A = \mathbb{P}(\mathbb{R}^d - V)$ where V is a $(d - 1)$-plane in $\mathbb{R}^d$ and Ω is convex in A. The domain Ω is *strictly convex* if it is a bounded, strictly convex subset of some affine chart. We say that $\rho(\Gamma)$ is a *convex cocompact* subgroup of $\mathrm{Aut}(\Omega)$ if $\rho(\Gamma)$ preserves Ω and there is a closed convex $\rho(\Gamma)$-invariant subset C of Ω so that $C/\rho(\Gamma)$ is compact. See [21] or [52] for details. (We note that in [21], they require convex cocompact groups to act cocompactly on the convex hull of the full orbital limit set in Ω and refer to groups which merely act cocompactly on some convex subset of Ω as naively convex cocompact. In the setting of projective Anosov groups the two definitions are equivalent.)

Theorem 2.6. (Zimmer [52, Thm 1.25]) *Suppose that Γ is a torsion-free hyperbolic group with connected boundary $\partial \Gamma$ which is not a surface group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is an irreducible projective Anosov representation. Then there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ so that $\rho(\Gamma)$ is a convex cocompact subgroup of $\mathrm{Aut}(\Omega)$.*

We recall that a hyperbolic group has connected boundary if and only if it is one-ended.

Danciger, Gueritaud and Kassel [21] describe the maximal domain that a convex cocompact group acts on.

Proposition 2.7. (Danciger-Gueritaud-Kassel [21, Proposition 8.1], Zimmer [52, Theorem 1.27]) *Suppose* $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ *is an irreducible projective Anosov representation and* $\rho(\Gamma)$ *preserves a properly convex open subset of* $\mathbb{P}(\mathbb{R}^d)$. *Then* $\rho(\Gamma)$ *acts convex cocompactly on*

$$\Omega_{max} = \mathbb{P}(\mathbb{R}^d) - \bigcup_{x \in \partial\Gamma} \mathbb{P}(\xi_\rho^{d-1}(x))$$

and $\xi_\rho^1(\partial\Gamma) \subset \partial\Omega_{max}$.

We will also need the following result which is implicit in Zimmer's work [52].

Proposition 2.8. *Suppose that* Γ *is a hyperbolic group with connected boundary* $\partial\Gamma$, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ *is a representation, and* $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ *is a* ρ-*equivariant continuous map. If* $\xi(\partial\Gamma)$ *spans* $\mathbb{R}^d$ *and lies inside an affine chart, then there exists a properly convex domain* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *so that* $\rho(\Gamma)$ *preserves* Ω.

Proof. By assumption $S = \xi(\partial\Gamma)$ is a connected, compact subset of an affine chart $A \subset \mathbb{P}(\mathbb{R}^d)$. We may assume that

$$A = \big\{ [x_1 : ... : x_d] \ \big| \ x_1 \neq 0 \big\}$$

so every point in A has a unique representative of the form $[1 : u]$ where $u \in \{0\} \times \mathbb{R}^{d-1}$. Then

$$CH(S) = \left\{ [1 : u] \ \middle| \ u = \sum_{i=1}^{d} t_i u_i, \ \sum_{i=1}^{d} t_i = 1, \ t_i \geq 0 \text{ and } [1 : u_i] \in S \text{ for all } i \right\}$$

is the convex hull of S in the affine chart A. Since S spans $\mathbb{R}^d$, $CH(S)$ has non-empty interior.

It only remains to show that $CH(S)$ is Γ-invariant. Suppose that $\gamma \in \rho(\Gamma)$. If $[1 : u] \in S$, then $\gamma([1 : u]) = [\gamma(e_1) + \gamma(u)] \in S \subset A$, so

$$\langle \gamma(e_1) + \gamma(u), e_1 \rangle \neq 0.$$

Since S is connected, $\langle \gamma(e_1) + \gamma(u), e_1 \rangle$ always has the same sign if $[1 : u] \in S$. If

$$x = [1 : u] = \left[\sum_{i=1}^{d} t_i : \sum_{i=1}^{d} t_i u_i \right] \in CH(S)$$

let

$$B_i(x) = \langle \gamma(e_1) + \gamma(u_i), e_1 \rangle \quad \text{and} \quad B(x) = \sum_{i=1}^{d} t_i B_i(x),$$

so

$$\begin{aligned}
\gamma(x) &= \left[B(x) : \sum_{i=1}^{d} t_i \Big(\gamma(u_i) + \gamma(e_1) - B_i(x)e_1 \Big) \right] \\
&= \left[1 : \sum_{i=1}^{d} \frac{t_i B_i(x)}{B(x)} \left(\frac{1}{B_i(x)} \big(\gamma(u_i) + \gamma(e_1) - B_i(x)e_1 \big) \right) \right] \\
&= \left[\sum_{i=1}^{d} s_i : \sum_{i=1}^{d} s_i \frac{\gamma(e_i) + \gamma(u_i) - B_i(x)e_1}{B_i(x)} \right] = \left[\sum_{i=1}^{d} s_i \gamma(1 : u_i) \right]
\end{aligned}$$

where

$$s_i = \frac{t_i B_i(x)}{B(x)}.$$

Notice that $s_i \geq 0$ for all i, since $t_i \geq 0$ and $B_i(x)$ and $B(x)$ have the same sign. Therefore, $\gamma(x) \in CH(S)$, so $CH(S)$ is Γ-invariant as required. $\square$

We combine Proposition 2.8 and Proposition 2.7 to show that an irreducible projective Anosov surface group whose limit set lies in an affine chart preserves a properly convex domain of the form given by Proposition 2.7.

Corollary 2.9. *Suppose that Γ is a surface group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation. If $\xi_\rho^1(\partial\Gamma)$ spans $\mathbb{R}^d$ and lies inside an affine chart for $\mathbb{R}^d$, then $\rho(\Gamma)$ acts convex cocompactly on the properly convex domain*

$$\Omega = \mathbb{P}(\mathbb{R}^d) - \bigcup_{x \in \partial\Gamma} \mathbb{P}(\xi_\rho^{d-1}(x))$$

and $\xi_\rho^1(\partial\Gamma) \subset \partial\Omega$.

3. Anosov representations into $\mathsf{SL}(3, \mathbb{R})$

We first characterize Anosov representations into $\mathsf{SL}(3, \mathbb{R})$.

Theorem 1.1: *If Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(3, \mathbb{R})$ is an Anosov representation, then Γ is either a free group or a surface group.*

Proof. Notice that an Anosov representation into $\mathsf{SL}(3, \mathbb{R})$ is, by definition, projective Anosov. If Γ is not free or a surface group, let $\Gamma = \Gamma_1 * \cdots * \Gamma_r * F_s$ be the free product decomposition of Γ where each Γ_i is one-ended, $r \geq 1$ and $s \geqslant 0$. Recall that Γ_1 is quasiconvex in Γ (see [15, Prop. 1.2]), so $\rho|_{\Gamma_1}$ is projective Anosov [18, Lemma 2.3].

If $\rho|_{\Gamma_1}$ is reducible, then Corollary 2.5 implies that there exists a projective Anosov representation of Γ_1 into $\mathsf{SL}^\pm(W)$ where W is a proper subspace of $\mathbb{R}^3$. However, every torsion-free discrete subgroup of $\mathsf{SL}^\pm(W)$ is either a free group or a surface group if W is one or two dimensional.

If $\rho|_{\Gamma_1}$ is irreducible and Γ_1 is not a surface group, then Theorem 2.6 implies that $\rho(\Gamma)$ acts convex cocompactly on a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^3)$, but then Γ is isomorphic to the fundamental group of the surface $\Omega/\rho(\Gamma)$ which is a contradiction. We conclude that Γ_1 is a surface group.

Suppose that Γ_1 has infinite index in Γ (i.e. suppose that there is more than one factor). Then $\partial\Gamma_1$ is a proper subset of $\partial\Gamma$. Let $z \in \partial\Gamma - \partial\Gamma_1$. Notice that $\xi_\rho(\partial\Gamma_1)$ is a compact subset of the affine chart $\mathbb{P}(\mathbb{R}^3 - \xi_\rho^2(z))$. Since $\xi_\rho^1(\partial\Gamma_1)$ is an embedded circle in A, $\xi_\rho^1(\partial\Gamma_1)$ must span $\mathbb{R}^3$ (otherwise, $\xi_\rho^1(\partial\Gamma)$ would be contained in the intersection of a projective line with A). Corollary 2.9 then implies that $\rho(\Gamma_1)$ acts cocompactly on

$$\Omega = \mathbb{P}(\mathbb{R}^3) - \bigcup_{x \in \partial\Gamma_1} \mathbb{P}(\xi_\rho^2(x))$$

and that $\partial\Omega = \xi_\rho^1(\partial\Gamma_1)$. In particular, either $\xi_\rho^1(z)$ is contained in $\xi_\rho^2(x)$ for some $x \in \partial\Gamma_1$, which violates transversality, or $\xi_\rho^1(z)$ is contained in Ω which would imply that $\xi_\rho^2(z)$ must intersect $\partial\Omega = \xi_\rho^1(\partial\Gamma_1)$, which agains violates transversality. This final contradiction completes the proof. $\square$

4. Projective Anosov representations into $\mathsf{SL}(4, \mathbb{R})$

The inclusion of any convex cocompact subgroup of $\mathsf{SO}_0(3, 1)$ into $\mathsf{SL}(4, \mathbb{R})$ is a projective Anosov representation (see [28, Sec. 6.1]). We use work of Danciger-Guérituad-Kassel [21] and Zimmer [52] and the Geometrization Theorem to show that these are the only groups admitting projective Anosov representations into $\mathsf{SL}(4, \mathbb{R})$.

Theorem 1.2: *If Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(4, \mathbb{R})$ is a projective Anosov representation, then Γ is isomorphic to a convex cocompact subgroup of $\mathsf{PO}(3, 1)$.*

Proof of Theorem 1.2: We decompose $\Gamma = \Gamma_1 * \cdots * \Gamma_p * F_r$ where each Γ_i is one-ended. Notice that F_r is the fundamental group of a handlebody of genus r, which is a compact irreducible 3-manifold with non-empty boundary. If Γ_i is a surface group, then it is the fundamental group of an interval bundle M_i, so M_i is irreducible and has non-empty boundary.

Now suppose that Γ_i is not a surface group. Since Γ_i is a quasiconvex subgroup of Γ (see [15, Prop. 1.2]), $\rho_i = \rho|_{\Gamma_i}$ is also projective Anosov (see [18, Lemma 2.3]). If ρ_i is reducible, then Corollary 2.5 gives a proper subspace W of $\mathbb{R}^4$ and an irreducible projective Anosov representation

$$\hat{\rho}_i = (\rho_i)^{ss}_W : \Gamma_i \to \mathsf{SL}^{\pm}(W).$$

If W is one-dimensional, then Γ_i is not one-ended. If W is two-dimensional, then $\hat{\rho}_i$ is a Fuchsian representation, so Γ_i is either a free group or a surface group, both of which have been disallowed. If W is 3-dimensional, then Theorem 1.1 again implies that Γ_i is either a free group or a surface group, which is impossible. Therefore, ρ_i is irreducible.

Since ρ_i is irreducible and Γ_i is one-ended and not a surface group, Theorem 2.7 implies that $\rho(\Gamma_i)$ acts convex cocompactly on a properly convex domain

$$\Omega_i = \mathbb{P} \left(\mathbb{R}^4 - \bigcup_{x \in \partial \Gamma_i} \xi^3_\rho(x) \right) \subset \mathbb{P}(\mathbb{R}^4).$$

In particular, Γ_i is the fundamental group of an irreducible 3-manifold $N_i = \Omega_i / \Gamma_i$.

If N_i is a closed 3-manifold, then ρ_i is a Benoist representation. If $\Gamma \neq \Gamma_i$, then there exists $z \in \partial\Gamma - \partial\Gamma_i$. By transversality, $\xi^1_\rho(z)$ cannot lie in $\mathbb{P}(\mathbb{R}^4) - \Omega_i$. On the other hand, if $z \in \Omega_i$, then $\xi^3_\rho(z)$ intersects $\partial\Omega_i = \xi^1_\rho(\partial\Gamma_i)$, which also violates transversality. Therefore, if N_i is closed, $\Gamma = \Gamma_i$, ρ is a Benoist representation and the Geometrization Theorem [44] implies that Γ is isomorphic to a convex cocompact subgroup of $\mathsf{PO}(3, 1)$).

If N_i is not a closed 3-manifold, then the Scott core theorem [47] implies that N_i contains a compact, irreducible submanifold M_i with non-empty boundary and fundamental group Γ_i.

Therefore, if ρ is not a Benoist representation, Γ is the fundamental group of the boundary connected sum M of the M_i and a handlebody of genus r. Since M is irreducible, $\pi_1(M)$ is word hyperbolic, torsion-free and infinite, and M has non-empty boundary, it follows from Thurston's original Geometrization Theorem (see Morgan [43]) that the interior of M admits a convex cocompact hyperbolic structure, so Γ is isomorphic to a convex cocompact subgroup of $\mathsf{PO}(3, 1)$. $\square$

5. Cohomological resrictions on P_k-Anosov representations

In this section, we place cohomological restrictions on P_k-Anosov subgroups of $\mathsf{SL}(d, \mathbb{R})$. Benoist [9, Prop 1.3] previously proved that a discrete hyperbolic subgroup of $\mathsf{SL}(d, \mathbb{R})$ consisting entirely of positively semi-proximal elements has virtual cohomological dimension at most $d - 1$, with equality if and only if the inclusion map is a Benoist representation. Guichard and Wienhard [28, Prop. 8.3] obtained bounds on cohomological dimension for various classes of Anosov representations into specified Lie subgroups of $\mathsf{SL}(d, \mathbb{R})$.

Theorem 1.3: *Suppose Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-Anosov.*

   (1) *If (d, k) is not $(2, 1)$, $(4, 2)$, $(8, 4)$ or $(16, 8)$, then Γ has cohomological dimension at most $d - k$.*

(2) *If (d, k) is $(2, 1)$, $(4, 2)$, $(8, 4)$ or $(16, 8)$, then Γ has virtual cohomological dimension at most $d - k + 1$. Moreover, if Γ has cohomological dimension $d - k + 1$ and (d, k) is $(4, 2)$, $(8, 4)$ or $(16, 8)$, then $\partial\Gamma$ is homeomorphic to S^{d-k} and ρ is not projective Anosov.*

Proof of Theorem 1.3: If $d = 2$, then $k = 1$ and $\rho : \Gamma \to \mathsf{SL}(2, \mathbb{R})$ is Fuchsian, so either Γ has cohomological dimension $d - k = 1$ (if Γ is free) or 2 (if Γ is a surface group), which corresponds to the first exceptional case in item (2). Theorem 1.1 handles the case where $d = 3$.

Now suppose that $\frac{d}{2} \geq k \geq 1$ and $d > 3$. Let m be the topological dimension of $\partial\Gamma$. Fix $x_0 \in \partial\Gamma$ and a $(d - k + 1)$-plane V in $\mathbb{R}^d$ which contains $\xi^{d-k}(x_0)$. We define a map

$$F : \partial\Gamma - \{x_0\} \to \mathbb{P}(V - \xi^k(x_0))$$

by letting $F(y)$ be the line which is the intersection of $\xi^k(y)$ with V. (Transversality implies that the intersection of $\xi^k(y)$ and $\xi^{d-k}(x_0)$ is trivial if $y \neq x_0$, so the intersection of $\xi^k(y)$ with V must be a line.)

One sees that F is injective, since if $x \neq y \in \partial\Gamma$, then $\xi^k(x)$ and $\xi^k(y)$ have trivial intersection (by transversality). Moreover, F is proper, since if $\{y_n\}$ is a sequence in $\partial\Gamma - \{x_0\}$ converging to x_0, then, by continuity of limit maps, $\{\xi_\rho^k(y_n)\}$ is converging to $\xi_\rho^k(x_0)$, so $\{F(y_n)\}$ leaves every compact subset of $\mathbb{P}(V - \xi^k(x_0))$. Therefore, F is an embedding. Since $\partial\Gamma - \{x_0\}$ embeds in a $(d - k)$-manifold, $\partial\Gamma$ has topological dimension at most $d - k$ (see [31, Thm III.1]).

Now suppose that $\partial\Gamma$ has topological dimension exactly $d - k$. Then, $F(\partial\Gamma)$ contains an open subset of $\mathbb{P}(V)$ (see [31, Thm. IV.3/Cor. 1]). So, since $\partial\Gamma$ has a manifold point, $\partial\Gamma$ is homeomorphic to S^{d-k}, by Kapovich-Benakli [33, Th. 4.4]. Let $p : E \to \partial\Gamma$ be the fibre bundle provided by Lemma 2.1 where

$$E = \bigcup_{x \in \partial\Gamma} S(\xi^k(x)) \subset S(\mathbb{R}^d).$$

Then E has topological dimension $(d - k) + k - 1 = d - 1$. Since $\partial\Gamma$ is homeomorphic to S^{d-k}, E is a closed submanifold of $S(\mathbb{R}^d) \cong S^{d-1}$ of dimension $d - 1$, which implies that $E = S(\mathbb{R}^d)$. However, by the classification of sphere fibrations ([3]), this is only possible if $(d - 1, k - 1)$ is $(3, 1)$, $(7, 3)$ or $(15, 7)$. Moreover, in these cases, ρ cannot be projective Anosov, since if ρ is projective Anosov, $\xi_\rho^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^{2k})$ lifts to a section $s : \partial\Gamma \to E$ of p, which is impossible (since $p \circ s = id$, $p_* \circ s_*$ is the identity map on $H_{d-k}(S^{d-k}) \cong \mathbb{Z}$, while, p_* is the zero map on $H_{d-k}(E)$.)

If $\partial\Gamma$ has topological dimension at most $d - k - 1$, then, by Bestvina-Mess [12, Cor. 1.4], Γ has cohomological dimension at most $d - k$. If $\partial\Gamma$ has topological dimension $d - k$, then Γ has cohomological dimension $d - k + 1$, again by [12, Cor. 1.4], and, by the previous paragraph, (d, k) is $(2, 1)$, $(4, 2)$, $(8, 4)$ or $(16, 8)$, $\partial\Gamma \cong S^{d-k}$ and ρ is not projective Anosov if (d, k) is $(4, 2)$, $(8, 4)$ or $(16, 8)$. □

6. BENOIST REPRESENTATIONS

The prototypical example of a Benoist representation is the inclusion of a cocompact discrete subgroup of $\mathsf{SO}_0(n, 1)$ into $\mathsf{SL}(n + 1, \mathbb{R})$. The image acts convex cocompactly on a round disk in $\mathbb{P}(\mathbb{R}^{n+1})$ which is the Beltrami-Klein model for $\mathbb{H}^n$. Johnson and Millson [32] showed that although the inclusion map is rigid in $\mathsf{SO}_0(n, 1)$ if $n \geq 3$, it often admits non-trivial deformations in $\mathsf{SL}(n + 1, \mathbb{R})$. Benoist [7, 8, 9] showed that these deformations are always Benoist representations and developed an extensive theory of groups of projective automorphisms preserving properly convex subsets of $\mathbb{P}(\mathbb{R}^d)$.

If ρ is a Benoist representation, then $\rho(\Gamma)$ has virtual cohomological dimension $d - 1$, so, Theorem 1.3 immediately implies that Benoist representations are only projective Anosov.

Corollary 1.4: *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a Benoist representation, and k is an integer such that $\frac{d}{2} \geq k \geq 2$, then ρ is not P_k-Anosov.*

We characterize Benoist representations in terms of the cohomological dimension of their domain groups.

Theorem 1.5: *If $d \geq 4$, an Anosov representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ of a torsion-free hyperbolic group Γ is a Benoist representation if and only if Γ has cohomological dimension $d - 1$.*

Proof of Theorem 1.5: It is immediate from the definition that if Γ is torsion-free and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a Benoist representation, then Γ has cohomological dimension $d - 1$.

Now suppose that Γ is a torsion-free hyperbolic group of cohomological dimension $d - 1$ and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is an Anosov representation. Notice that, by Theorem 1.3, ρ cannot be P_k-Anosov for any $k \geq 2$, so ρ must be projective Anosov.

There exists a free decomposition $\Gamma = \Gamma_1 * \cdots * \Gamma_s * F_r$ where each Γ_i is one-ended. Since the cohomological dimension of Γ is the maximum of the cohomological dimensions of its one-ended factors, we may assume that Γ_1 has cohomological dimension $d - 1$. Since Γ_1 is a quasiconvex subgroup of Γ (see [15, Prop. 1.2]), the restriction $\rho_1 = \rho|_{\Gamma_1}$ of ρ to Γ_1 is projective Anosov [18, Lemma 2.3]. Moreover, $\xi^1_{\rho_1}$ is the restriction of ξ^1_ρ to $\partial\Gamma_1 \subset \partial\Gamma$.

We first claim that ρ_1 is irreducible. If not, Corollary 2.5 provides a proper $\rho^{ss}(\Gamma_1)$-invariant subspace W of $\mathbb{R}^d$ and an irreducible projective Anosov representation $(\rho_1)^{ss}_W : \Gamma \to \mathsf{SL}^\pm(W)$. However, Theorem 1.3 would then imply that Γ has cohomological dimension at most $\dim(W) - 1 \leq d - 2$, which would be a contradiction.

Since ρ_1 is irreducible and Γ is one-ended and not a surface group, Theorem 2.6 implies that there exists a properly convex open domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ preserved by $\rho(\Gamma_1)$. Proposition 2.7 implies that we may assume that Ω has the form

$$\Omega = \mathbb{P}(\mathbb{R}^d) - \bigcup_{x \in \partial\Gamma_1} \mathbb{P}(\xi^{d-1}_\rho(x)).$$

Since $\Omega/\rho(\Gamma_1)$ is an aspherical $(d - 1)$-manifold and Γ_1 has cohomological dimension $d - 1$, $\Omega/\rho(\Gamma_1)$ must be a closed manifold. Therefore, by Benoist [7, Thm 1.1], Ω is strictly convex, so ρ_1 is a Benoist representation.

If there existed another one-ended or free factor in the free decomposition of Γ, then there would exist an infinite order element t in the other factor. In particular, t^+ does not lie in $\partial\Gamma_1$. Since, by transversality, $\xi^1_\rho(t^+)$ cannot intersect $\xi^{d-1}_\rho(x)$ for any x in $\partial\Gamma_1$, $\xi^1_\rho(t^+)$ must lie in Ω. However, if $\xi^1_\rho(t^+) \in \Omega$, then the hyperplane $\xi^{d-1}_\rho(t^+)$ must intersect the boundary $\partial\Omega$, which again violates transversality of the limit maps. Therefore, $\Gamma = \Gamma_1$ and ρ is a Benoist representation. $\square$

7. BOREL ANOSOV REPRESENTATIONS

The only known examples of Borel Anosov representations into $\mathsf{SL}(d, \mathbb{R})$ have domain groups which contain finite index subgroups which are either free or surface groups. Andres Sambarino asked whether this is always the case.

Sambarino's Question: *If a torsion-free hyperbolic group admits a Borel Anosov representation into $\mathsf{SL}(d, \mathbb{R})$, must it be either a free group or a surface group?*

We do know, by Theorem 1.3, that Borel Anosov representations must have "small" cohomological dimension.

Corollary 7.1. *Suppose that Γ is a torsion-free hyperbolic group, $d \geq 3$ and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is Borel Anosov.*

    (1) *If d is odd, then Γ has cohomological dimension at most $\frac{d+1}{2}$.*
    (2) *If d is even, then Γ has cohomological dimension at most $\frac{d}{2}$.*

Theorem 1.1 answers the question in the positive when $d = 3$. In this section, we handle the case when $d = 4$. We first observe that every Borel Anosov representation of a surface group into $\mathsf{SL}(d, \mathbb{R})$ is irreducible .

Proposition 7.2. *If Γ is a surface group and $\rho : \Gamma \to \mathsf{SL}(4, \mathbb{R})$ is Borel Anosov, then ρ is irreducible.*

Proof. Suppose not. Then we may assume that ρ is a reducible, semisimple, Borel Anosov, representation, since the semisimplification of ρ remains reducible and Borel Anosov. Let W be the subspace spanned by $\xi_\rho^1(\partial\Gamma)$. Then the restriction of ρ to W is irreducible, by Proposition 2.2. Let V be the complementary subspace of $\mathbb{R}^4$ which is also preserved by $\rho(\Gamma)$.

The subspace W cannot be 1-dimensional, since ξ_ρ is injective.

If W is three-dimensional, then V is an eigenline of each $\rho(\gamma)$ so, for all γ, V lies in either $\xi_\rho^2(\gamma^+)$ or in $\xi_\rho^2((\gamma^{-1})^+)$. However, this is impossible since $\xi_\rho^2(\alpha^+)$ and $\xi_\rho^2(\beta^+)$ are transverse for all α and β in distinct maximal cyclic subgroups.

If W is two-dimensional, then we may pass to a subgroup of index at most 4, still called Γ, so that $\rho(\gamma)|_V$ and $\rho(\gamma)|_W$ both have positive determinant for all $\gamma \in \Gamma$. Let

$$a(\gamma) = \sqrt{\det(\rho(\gamma)|_W)}$$

for all $\gamma \in \Gamma$ and define $\rho_1 : \Gamma \to \mathsf{SL}(W)$ by $\rho_1(\gamma) = a(\gamma)^{-1}\rho(\gamma)|_W$ and $\rho_2 : \Gamma \to \mathsf{SL}(V)$ by $\rho_2(\gamma) = a(\gamma)\rho(\gamma)|_V$. Since

$$\frac{\lambda_1(\rho_1(\gamma))}{\lambda_2(\rho_1(\gamma))} = \frac{\lambda_1(\rho(\gamma))}{\lambda_4(\rho(\gamma))} \quad \text{and} \quad \frac{\lambda_1(\rho_2(\gamma))}{\lambda_2(\rho_2(\gamma))} = \frac{\lambda_2(\rho(\gamma))}{\lambda_3(\rho(\gamma))}$$

for all $\gamma \in \Gamma$ and ρ is Borel Anosov, we see that ρ_1 and ρ_2 are Fuchsian.

Since ρ is projective Anosov, there exists $s > 0$ so that

$$\lambda_1(\rho(\gamma)) > e^{s||\gamma||}\lambda_2(\rho(\gamma))$$

where $||\gamma||$ is the cyclically reduced word length of γ. Observe that

$$\lambda_1(\rho(\gamma)) = a(\gamma)\lambda_1(\rho_1(\gamma)) \quad \text{and} \quad \lambda_2(\rho(\gamma)) = a(\gamma)^{-1}\lambda_1(\rho_2(\gamma))$$

for all $\gamma \in \Gamma$, so

$$\lambda_1(\rho_1(\gamma)) > e^{s||\gamma||}a(\gamma)^{-2}\lambda_2(\rho(\gamma)).$$

Now, since $\lambda_1(\rho_i(\gamma^{-1})) = \lambda_1(\rho_i(\gamma))$ and $a(\gamma^{-1}) = a(\gamma)^{-1}$, we see that

$$\lambda_1(\rho_1(\gamma)) > e^{s||\gamma||}\lambda_2(\rho(\gamma))$$

for all $\gamma \in \Gamma$. However, this is impossible, since

$$h(\rho_i) = \lim \frac{1}{T} \log \, \#\{[\gamma] \in [\Gamma] \mid 2\log \lambda_1(\rho_i(\gamma)) \leq T\} = 1$$

for both $i = 1, 2$ (see Patterson [45]). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We are now ready to answer Sambarino's question when $d = 4$.

Theorem 1.6: *If Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(4, \mathbb{R})$ is Borel Anosov, then Γ is either a surface group or a free group.*

Proof of Theorem 1.6: By Theorem 1.2, we know that Γ is isomorphic to a convex cocompact subgroup Δ of $\mathsf{PO}(3, 1)$. Moreover, by Theorem 1.3, Γ has cohomological dimension at most 2, so Δ is not cocompact. Therefore, if Γ is not free or a surface group, Γ contains infinitely many quasiconvex surface subgroups with mutually disjoint boundaries in $\partial\Gamma$ (see Abikoff-Maskit [2]).

Let H and J be quasiconvex surface subgroups of Γ so that ∂H and ∂J are disjoint in $\partial\Gamma$. Choose $z \in \partial\Gamma - (\partial J \cup \partial H)$. By transversality, both $\xi_\rho^1(\partial H)$ and $\xi_\rho^1(\partial J)$ are disjoint from the projective plane $\mathbb{P}(\xi_\rho^3(z))$, so are contained in the affine chart $A = \mathbb{P}(\mathbb{R}^4 - \xi_\rho^3(z))$.

Since $\rho|_H$ is irreducible, by Proposition 7.2, Corollary 2.9 implies that

$$\Omega_H = \mathbb{P}\left(\mathbb{R}^4 - \bigcup_{x \in \partial H} \xi_\rho^3(x)\right)$$

is a properly convex domain which is $\rho(H)$-invariant and $\xi_\rho^1(\partial H) \subset \partial\Omega_H$. Let

$$T_H = \mathbb{P}\left(\bigcup_{x \in \partial H} \xi_\rho^2(x)\right).$$

By transversality, T_H is a disjoint union of projective lines, so it is a S^1-bundle over the circle ∂H. It follows, that T_H is a Klein bottle or a torus. Since the Klein bottle does not embed in $\mathbb{P}(\mathbb{R}^4)$ (see [16]), T_H is a torus. Notice that T_H separates since $H^2(\mathbb{P}(\mathbb{R}^4)) = 0$.

If $x \in \partial H$, the projective line $\mathbb{P}(\xi_\rho^2(x))$ intersects the projective plane $\mathbb{P}(\xi_\rho^3(z))$ in exactly one point, so $C_H = T_H \cap \mathbb{P}(\xi_\rho^3(z))$ is a simple closed curve. Since, by transversality, C_H is disjoint from the projective line $\mathbb{P}(\xi_\rho^2(z))$ in $\mathbb{P}(\xi_\rho^3(z))$, C_H bounds a disk D_H in the disk $\mathbb{P}(\xi_\rho^3(z)) \backslash \mathbb{P}(\xi_\rho^2(z))$. Notice that D_H is unique, since the other component of $\mathbb{P}(\xi_\rho^3(z)) - C_H$ is an open Möbius band.

The boundary of the regular neighborhood of $D_H \cup T_H$ has a spherical component S_H contained in A, which bounds a ball B_H in A, since A is irreducible. Therefore, T_H bounds an open solid torus V_H which contains B_H and intersects $\mathbb{P}(\xi_\rho^3(z))$ exactly in D_H. Since $\xi_\rho^1(\partial H)$ is homotopic to C_H it also bounds a disk in V_H. However, since $\xi_\rho^1(\partial H)$ is homotopically non-trivial in T_H, it can't also bound a disk in $\mathbb{P}(\mathbb{R}^4) - V_H$. (If it bounds a disk both in V_H and in its complement, then the sphere S_H made from the two disks intersects each projective line in T_H exactly once, which contradicts the fact that every sphere in $\mathbb{P}(\mathbb{R}^4)$ bounds a ball.)

Since Ω_H is disjoint from T_H and $\xi_\rho^1(\partial H)$ bounds a disk in Ω_H, we must have Ω_H contained in V_H.

Now consider the torus

$$T_J = \mathbb{P}\left(\bigcup_{x \in \partial J} \xi_\rho^2(x)\right),$$

simple closed curve $C_J = T_J \cap \mathbb{P}(\xi_\rho^3(z))$, disk $D_J \subset \mathbb{P}(\xi_\rho^3(z))$ bounded by C_J, and open solid torus V_J bounded by T_J so that

$$\Omega_J = \mathbb{P}\left(\mathbb{R}^4 - \bigcup_{x \in \partial J} \xi_\rho^3(x)\right)$$

and D_J are both contained in V_J.

Since $\xi_\rho^1(\partial J) \subset \Omega_H \subset V_H$ and T_J is disjoint from $T_H = \partial V_H$, by transversality, T_J is contained in $\mathrm{int}(V_H)$. Therefore, C_J is contained in D_H which implies that D_J is contained in D_H. So the

regular neighborhood of $D_J \cup T_J$ can be taken to have a spherical component S_J contained in B_H, so S_J bounds a ball B_J contained in B_H. Putting this all together, we see that V_J must be contained in V_H. Therefore, $\xi^1_\rho(\partial H)$ is contained in the complement of V_J and hence in the complement of Ω_J. It follows that

$$\xi^1_\rho(\partial H) \subset \bigcup_{x \in \partial J} \mathbb{P}(\xi^3_\rho(x))$$

which contradicts transversality. Therefore, Γ is either a surface group or a free group. $\square$

8. Hyperconvexity

Labourie [40] introduced the theory of Anosov representations in his study of Hitchin representations. Recall that a representation is *d-Fuchsian* if it is the composition of a Fuchsian representation of a surface group into $\mathsf{SL}(2, \mathbb{R})$ with an irreducible representation of $\mathsf{SL}(2, \mathbb{R})$ into $\mathsf{SL}(d, \mathbb{R})$. *Hitchin representations* [29] are representations of a surface group into $\mathsf{SL}(d, \mathbb{R})$ which can be continuously deformed to a *d*-Fuchsian representation. Labourie showed that Hitchin representations are irreducible and Borel Anosov.

Labourie [40] and Guichard [27] proved that a representation $\rho : \pi_1(S) \to \mathsf{SL}(d, \mathbb{R})$ is Hitchin if and only if there exists a hyperconvex limit map, i.e. a ρ-equivariant map $\xi^1_\rho : \partial \pi_1(S) \to \mathbb{P}(\mathbb{R}^d)$ so that if $\{x_1, \ldots, x_d\}$ are distinct points in $\partial \Gamma$, then $\xi^1_\rho(x_1) \oplus \cdots \oplus \xi^1_\rho(x_d) = \mathbb{R}^d$. Labourie further shows that if ρ is Hitchin, $n_1, \ldots, n_k \in \mathbb{N}$, $n_1 + \cdots + n_k = d$ and $\{x_1, \ldots, x_k\}$ are distinct points in $\partial \pi_1(S)$, then $\xi^{n_1}_\rho(x_1) \oplus \cdots \oplus \xi^{n_k}_\rho(x_k) = \mathbb{R}^d$ and that if $\{(y_n, z_n)\}$ is a sequence in $\partial \Gamma \times \partial \Gamma$, with $y_n \neq z_n$ for all n, converging to (x, x), and $p, q, r \in \{1, \ldots, p-1\}$ with $p + q = r$, then $\{\xi^p_\rho(y_n) \oplus \xi^q_\rho(z_n)\}$ converges to $\xi^r_\rho(x)$.

We use these hyperconvexity properties to show that Hitchin representations cannot be extended to representations of larger groups which are P_1-Anosov and P_2-Anosov. We consider this to be more evidence for a positive answer to Sambarino's question.

Proposition 8.1. *Suppose Γ contains a surface subgroup Γ_0, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is projective Anosov and $\rho|_{\Gamma_0}$ is Hitchin.*

    (1) *If d is even, then Γ_0 has finite index in Γ.*

    (2) *If d is odd, and ρ is also P_2-Anosov, then Γ_0 has finite index in Γ.*

Proof. Suppose that Γ_0 has infinite index in Γ. Since $\rho|_{\Gamma_0}$ is projective Anosov, Γ_0 is quasiconvex in Γ (see [18, Lemma 2.3]), so $\partial \Gamma_0$ embeds in $\partial \Gamma$. Choose $z \in \partial \Gamma - \partial \Gamma_0$ and let A be the affine chart $\mathbb{P}(\mathbb{R}^d - \xi^{d-1}_\rho(z))$. By transversality, $\xi^1_\rho(\partial \Gamma_0) \subset A$.

If d is even, we have obtained a contradiction to Lemma 12.3 in Danciger-Guéritaud-Kassel [21], which asserts that if $\rho|_{\Gamma_0}$ is a Hitchin representation, then $\xi^1_\rho(\partial \Gamma_0)$ cannot lie in any affine chart.

Now suppose that d is odd, and that ρ is P_2-Anosov. There exists a continuous map $h : D^2 \to A$ and a homeomorphism $g : \partial \Gamma_0 \to S^1$ so that $\xi^1_\rho|_{\partial \Gamma_0} = h \circ g$. Let $V = \xi^{d-2}_\rho(z)^\perp$ and define the continuous map

$$F : D^2 \to \mathbb{P}(V) \cong S^1$$

by letting

$$F(x) = \left[\left(h(x) \oplus \xi^{d-2}_\rho(z) \right) \cap V \right].$$

We now claim that $F|_{\partial \Gamma_0}$ is locally injective. If not there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\partial \Gamma_0$ so that $x_n \neq y_n$ (for any n), $\lim x_n = q = \lim y_n$ and $F(x_n) = F(y_n)$ for all n. Since $\rho|_{\Gamma_0}$ is Hitchin, the sequence $\{\xi^1_\rho(x_n) \oplus \xi^1_\rho(y_n)\}$ converges to $\xi^2_\rho(q)$. So, for all n, we may choose vectors

u_n, v_n and w_n in $\xi_\rho^1(x_n)$, $\xi_\rho^1(y_n)$ and $\xi_\rho^{d-2}(z)$ so that $u_n + v_n = w_n$ and w_n is unit length. Up to subsequence, $\{w_n\}$ converges to a unit vector w, but then $w \in \xi_\rho^2(q)$, since $w_n \in \xi_\rho^1(x_n) \oplus \xi_\rho^1(y_n)$ for all n, and $w \in \xi_\rho^{d-2}(z)$, since $w_n \in \xi_\rho^{d-2}(z)$ for all n. However, this violates transversality, since $q \neq z$. Therefore, $F|_{\partial \Gamma_0}$ is a covering map, which is impossible since $(F|_{\partial \Gamma_0})_*$ is trivial on π_1. We have obtained contradictions when d is either even or odd, which completes the proof. $\qquad\square$

Pozzetti, Sambarino and Wienhard [46] recently introduced the notion of (p, q, r)-hyperconvex representations which share specific transversality properties with Hitchin representations. A representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is said to be (p, q, r)-*hyperconvex*, where $p + q \leq r$, if ρ is P_p, P_q and P_r (or P_{d-r})-Anosov and whenever $x, y, z \in \partial \Gamma$ are distinct,

$$\left(\xi_\rho^p(x) \oplus \xi_\rho^q(y) \right) \cap \xi^{d-r}(z) = \{0\}.$$

One may view the following as a generalization of Corollary 6.6 of Pozzetti-Sambarino-Wienhard [46] which asserts that if $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is $(1, 1, r)$-hyperconvex and $x_0 \in \partial \Gamma$, then there is a continuous injection of $\partial \Gamma - \{x_0\}$ into $\mathbb{P}(\mathbb{R}^r)$, see also Lemma 4.10 in Zhang-Zimmer [51]. (Pozzetti, Sambarino and Wienhard's result [46, Corollary 6.6] also applies to representations into $\mathsf{SL}(d, K)$ where K is any local field.)

Proposition 8.2. *Suppose that Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_p-Anosov. If there exists a $(d - r)$-plane V such that*

$$V \cap (\xi^p(x) \oplus \xi^p(y)) = \{0\}$$

if $x, y \in \partial \Gamma$, then Γ has cohomological dimension at most $r - p + 1$. If Γ has cohomological dimension $r - p + 1$, then $\partial \Gamma \cong S^{r-p}$ and $(r - p, p)$ is either $(1, 1)$, $(2, 2)$, $(4, 4)$ or $(8, 8)$.

Moreover, if ρ is (p, p, r)-hyperconvex, then Γ has cohomological dimension at most $r - p + 1$ and if Γ has cohomological dimension $r - p + 1$, then $\partial \Gamma \cong S^{r-p}$.

Notice that if $p \leq q$, then (p, q, r)-hyperconvex representations are (p, p, r)-hyperconvex, so we may conclude that if $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is (p, q, r)-hyperconvex, then Γ has cohomological dimension at most $r + 1 - \min\{p, q\}$. Pozzetti, Sambarino and Wienhard [46, Cor. 7.6] observe that if $k \geq 2$ and $\rho : \Gamma \to \mathsf{PO}(d, 1)$, then the kth symmetric power $S^k \rho : \Gamma \to \mathsf{PGL}(S^k(\mathbb{R}^{d+1}))$ is $(1, 1, d)$-hyperconvex, so one obtains no additional topological restrictions in the case where ρ is $(1, 1, d)$-hyperconvex and Γ has maximal cohomological dimension d.

Proof. Let $p : E \to \partial \Gamma$ be the fibre bundle provided by Lemma 2.1 where

$$E = \bigcup_{x \in \partial \Gamma} S(\xi_\rho^p(x)) \subset S(\mathbb{R}^d).$$

If $\partial \Gamma$ has topological dimension m, then E has topological dimension $m + p - 1$.

Let $\pi : \mathbb{R}^d \to V$ be orthogonal projection (with respect to some fixed background metric on $\mathbb{R}^d$). Let

$$f : \mathbb{R}^d - V \to S(V^\perp) \cong S^{r-1}$$

be the continuous map given by

$$f(u) = \frac{u - \pi(u)}{||u - \pi(u)||}.$$

Notice that if $f(u) = f(v)$, then $||u - \pi(u)||u - ||v - \pi(v)||v \in V$. Suppose $u \in S(\xi_\rho^p(x))$ and $v \in S(\xi_\rho^p(y))$ and $f(u) = f(v)$, then since $(\xi_\rho^p(x) \oplus \xi_\rho^p(y)) \cap V = \{0\}$, it must be the case that $u = v$ and, since $\xi_\rho^p(x) \cap \xi_\rho^p(y) = \{0\}$ if $x \neq y$, it must be the case that $x = y$. Therefore, the restriction $f|_E$ of f to E is injective and hence a topological embedding. It follows, since E has

topological dimension $m + p - 1$, that $m + p - 1 \leq r - 1$, so $m \leq r - p$, which implies, by [12, Cor. 1.4], that Γ has cohomological dimension at most $r - p + 1$.

Suppose that $m = r - p$. We first show that $\partial\Gamma \cong S^m = S^{r-p}$. Fix $x_0 \in \partial\Gamma$. Choose a $(r - p)$-dimensional subspace $W_1 \subset \mathbb{R}^d$ so that $\xi_\rho^p(x_0) \oplus W_1 \oplus V = \mathbb{R}^d$. Pick a line $L \subset \xi_\rho^p(x_0)$ and let $W = W_1 \oplus L$. There exists a neighborhood U of x_0 in $\partial\Gamma$ so that $(\xi_\rho^p(x) \oplus V) \cap W$ is a line if $x \in U$. Consider the continuous map $F : U \to \mathbb{P}(W)$ given by

$$F(x) = \left[\left(\xi_\rho^p(x) \oplus V \right) \cap W \right].$$

We argue as above to show that F is injective. If $F(x_1) = [u_1 + v_1]$ and $F(x_2) = [u_2 + v_2]$, where $u_i \in \xi_\rho^p(x_i)$ and $v_i \in V$, and $F(x_1) = F(x_2)$, we may assume that $u_1 + v_1 = u_2 + v_2$, so $u_1 - u_2 = v_2 - v_1 \in (\xi_\rho^p(x_1) \oplus \xi_\rho^p(x_2)) \cap V = \{0\}$. Therefore, $u_1 = u_2$ (and each u_i is non-zero, since $V \cap W = \{0\}$), which implies that $\xi_\rho^p(x_1) \cap \xi_\rho^p(x_2) \neq \{0\}$, so $x_1 = x_2$. Therefore, F is a topological embedding. Since U and $\mathbb{P}(W)$ both have topological dimension $m = r - p$, U and hence $\partial\Gamma$ contains a manifold point. Thus, by Kapovich-Benakli [33, Th. 4.4], $\partial\Gamma \cong S^m$. Moreover, $f|_E$ is one of the four Hopf fibrations, so the only possibilities for $(r - p, p)$ are $(1, 1)$, $(2, 2)$, $(4, 4)$ and $(8, 8)$.

If ρ is (p, p, r)-hyperconvex, we choose $x_0 \in \partial\Gamma$ and let $V = \xi_\rho^{d-r}(x_0)$ and apply the same argument to conclude that $E - S(\xi_\rho^p(x_0))$ has topological dimension $m + p - 1$ and every compact subset embeds in a sphere of dimension $r - 1$. We again conclude that Γ has cohomological dimension at most $r - p + 1$. Similarly, we may show, that if Γ has cohomological dimension $r - p + 1$, then a neighborhood of a point $z_0 \neq x_0 \in \partial\Gamma$ embeds in a projective space of dimension $r - p$, so $\partial\Gamma \cong S^{r-p}$. $\qquad\square$

9. Characterizing Benoist representations by limit maps

In this section we obtain characterizations of Benoist representations purely in terms of limit maps. We first work in the setting where the domain group doesn't split over a cyclic subgroup.

Theorem 9.1. *Suppose that $d \geq 4$ and Γ is a torsion-free hyperbolic group. A representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a Benoist representation if and only if Γ has cohomological dimension $d - 1$, Γ does not split over a cyclic subgroup and there is a ρ-equivariant continuous non-constant map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$.*

Proof. If ρ is a Benoist representation then Γ has cohomological dimension $d - 1$, ρ is projective Anosov and ξ_ρ^1 is a ρ-equivariant continuous non-constant map. Moreover, since $\partial\Gamma \cong S^{d-2}$ and $d \geq 4$, Γ does not split over a cyclic subgroup ([15, Thm 6.2]). So, the bulk of our work is in establishing the converse.

We first prove a general result about representations admitting a limit map whose image spans. We recall that ρ is said to be P_1-*divergent* if whenever $\{\gamma_n\}$ is a sequence of distinct elements in Γ, then

$$\lim \frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} = 0$$

where $\sigma_i(\rho(\gamma_n))$ is the ith singular value of $\rho(\gamma_n)$.

Lemma 9.2. *Suppose that Γ is a non-elementary hyperbolic group, $d \geq 2$, and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ admits a continuous ρ-equivariant map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ such that $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$. Then the representation ρ is P_1-divergent and if $\rho(\gamma) \in \rho(\Gamma)$ is proximal, then $\xi(\gamma^+)$ is the attracting eigenline of $\rho(\gamma)$. In particular, $\rho(\Gamma)$ is discrete and ρ has finite kernel.*

Moreover, if in addition, ρ is irreducible, then $\rho(\Gamma)$ contains a biproximal element.

Proof. If ρ is not P_1-divergent, then there exists a sequence $\{\gamma_n\}$ of distinct elements of Γ so that $\lim \frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} = C > 0$. Since Γ acts a convergence group on $\partial\Gamma$, we may pass to another subsequence so that there exist η and η' such that if $x \neq \eta'$ then $\lim \gamma_n x = \eta$. Since $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$ and $\partial\Gamma$ is perfect, there exist $x_1, ..., x_d \neq \eta'$ such that $\mathbb{R}^d = \oplus\xi(x_i)$. Suppose that $\xi(x_i) = [l_i e_1]$ for each i and $\xi(\eta) = [l_\eta e_i]$ where $l_i, l_\eta \in \mathsf{O}(d)$. We write each $\rho(\gamma_n) = k_n a_n k_n'$ in the Cartan decomposition where $k_n, k_n' \in \mathsf{O}(d)$ and a_n is the diagonal matrix so that $(a_n)_{ii} = \sigma_i(\rho(\gamma_n))$ for all n. We may pass to another subsequence so that $\{k_n\}$ and $\{k_n'\}$ converge to k and k'. Then, since ξ is ρ-equivariant, $\lim \rho(\gamma_n)\xi(x_i) = \xi(\eta)$, so $\{[k_n a_n k_n' l_i e_1]\}$ converges to $[l_\eta e_1]$, which implies that $\{[a_n k_n' l_i e_1]\}$ converges to $[k^{-1} l_\eta e_1]$ in $\mathbb{P}(\mathbb{R}^d)$. So, perhaps after replacing l_i with $-l_i$,

$$\lim \frac{a_n k_n' l_i e_1}{||a_n k_n' l_i e_1||} = k^{-1} l_\eta e_1$$

for all i. We pass to a subsequence so that

$$\nu_i = \lim \frac{||a_n k_n' l_i e_1||}{\sigma_1(\rho(\gamma_n))}$$

exists for all i. Then,

$$
\begin{aligned}
\nu_i \left\langle k^{-1} l_\eta e_1, e_1 \right\rangle &= \lim \left(\frac{||a_n k_n' l_i e_1||}{\sigma_1(\rho(\gamma_n))} \right) \left\langle \frac{a_n k_n' l_i e_1}{||a_n k_n' l_i e_1||}, e_1 \right\rangle \\
&= \lim \left\langle a_n k_n' l_i e_1, \frac{e_1}{\sigma_1(\rho(\gamma_n))} \right\rangle \\
&= \lim \left\langle k_n' l_i e_i, e_1 \right\rangle = \left\langle k' l_i e_1, e_1 \right\rangle
\end{aligned}
$$

A similar calculation, and the fact that $\lim \frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} = C > 0$, yields

$$\left\langle k' l_i e_1, e_2 \right\rangle = \frac{\nu_i}{C} \left\langle k^{-1} l_\eta e_1, e_2 \right\rangle.$$

Since the vectors $\{k' l_i e_1\}_{i=1,..,d}$ span $\mathbb{R}^d$, there exists i_0 such that $\nu_{i_0} \neq 0$. Then we further observe that

$$\left\langle k' l_i e_1 - \frac{\nu_i}{\nu_{i_0}} k' l_{i_0} e_1, e_1 \right\rangle = \left\langle \nu_i k^{-1} l_\eta e_1 - \frac{\nu_i}{\nu_{i_0}} \nu_{i_0} k^{-1} l_\eta e_1, e_1 \right\rangle = 0$$

and similarly that

$$\left\langle k' l_i e_1 - \frac{\nu_i}{\nu_{i_0}} k' l_{i_0} e_1, e_2 \right\rangle = 0.$$

Therefore, each $k' l_i e_1$ lies in the subspace of $\mathbb{R}^d$ spanned by $k' l_{i_0} e_1$ and $e_1^\perp \cap e_2^\perp$, which has codimension at least one. This however contradicts the fact that $\{k' l_i e_1\}_{i=1,..,d}$ spans $\mathbb{R}^d$. Therefore, ρ is P_1-divergent. Notice that if $\rho(\gamma) \in \rho(\Gamma)$ is proximal, then there exists $x \in \partial\Gamma$ so that $\xi(x)$ does not lie in the repelling hyperplane of $\rho(\gamma)$, so $\rho(\gamma^n)(x)$ converges to the attracting eigenline of $\rho(\gamma)$. Since ξ is ρ-equivariant and $\lim \gamma^n(x) = \gamma^+$, $\xi(\gamma^+)$ is the attracting eigenline of $\rho(\gamma)$.

We now assume that ρ is also irreducible. Theorem 2.3 provides a finite subset A of Γ and $M > 0$ so that if $\gamma \in \Gamma$, then there exists $\alpha \in A$ so that

$$|\log \lambda_i(\rho(\gamma\alpha)) - \log \sigma_i(\rho(\gamma))| \leq M$$

for all i. Let $\{\gamma_n\}$ be an infinite sequence of distinct elements in Γ and let $\{\alpha_n\}$ be the associated sequence of elements of A. Since ρ is P_1-divergent, $\frac{\lambda_1(\rho(\gamma_n \alpha_n))}{\lambda_2(\rho(\gamma_n \alpha_n))} \to \infty$ and $\frac{\lambda_{d-1}(\rho(\gamma_n \alpha_n))}{\lambda_d(\rho(\gamma_n \alpha_n))} \to \infty$, so $\rho(\gamma_n \alpha_n)$ is biproximal for all large enough n. $\qquad\square$

We now complete the proof of Theorem 9.1 in the case where ρ is irreducible.

Proposition 9.3. *Suppose that Γ is a torsion-free hyperbolic group of cohomological dimension $m \geq d - 1 \geq 2$ which does not admit a cyclic splitting. If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is irreducible and there exists a ρ-equivariant continuous map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$, then $m = d - 1$ and ρ is a Benoist representation.*

Proof. Since ρ is irreducible, $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$, since $\rho(\Gamma)$ preserves the space spanned by $\xi(\partial\Gamma)$. Lemma 9.2 allows us to choose $\gamma_0 \in \Gamma$ so that $\rho(\gamma_0)$ is biproximal. We may assume that the attracting eigenlines of $\rho(\gamma_0)$ and $\rho(\gamma_0^{-1})$ are $\langle e_1 \rangle$ and $\langle e_d \rangle$ respectively and the corresponding attracting hyperplanes are $e_d^\perp$ and $e_1^\perp$. In particular, $\xi(\gamma_0^+) = [e_1]$ and $\xi(\gamma_0^-) = [e_d]$. Suppose that $x \in \partial\Gamma - \{\gamma_0^+, \gamma_0^-\}$. Since $\lim \gamma_0^n(x) = \gamma_0^+$ and $\lim \gamma_0^{-n}(x) = \gamma_0^-$, $\xi(x)$ cannot lie in either $\mathbb{P}(e_1^\perp)$ or $\mathbb{P}(e_d^\perp)$. Since the group Γ does not split over a cyclic subgroup, the set $\partial\Gamma - \{\gamma_0^\pm\}$ is connected (see Bowditch [15, Thm 6.2]), so we may assume that $\xi(\partial\Gamma - \{\gamma_0^\pm\})$ is contained in the connected component $\{[1 : x_2 : ... : x_d] \mid x_d > 0\}$ of $\mathbb{P}(\mathbb{R}^d) - \mathbb{P}(e_1^\perp) \cup \mathbb{P}(e_d^\perp)$. It follows that $\xi(\partial\Gamma)$ lies in the affine chart $\mathbb{P}(\mathbb{R}^d) - \mathbb{P}(V)$ where $V = \{(x_1, ..., x_d) \in \mathbb{R}^{n+1} \mid x_1 = -x_d\}$. Lemma 2.8 then implies that $\rho(\Gamma)$ preserves a properly convex domain Ω in $\mathbb{P}(\mathbb{R}^d)$. Since $\rho(\Gamma)$ is P_1-divergent, it is discrete and faithful, so it must acts properly discontinuously on Ω (see [9, Fact 2.10]). Since $\rho(\Gamma)$ has cohomological dimension $m \geq d - 1$, it must have compact quotient. Hence, by Benoist [7, Thm 1.1], Ω is strictly convex, so ρ is a Benoist representation and $m = d - 1$. $\qquad\square$

It remains to rule out the case where ρ is reducible. We first deal with the case where $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$.

Proposition 9.4. *Suppose that Γ is a torsion-free hyperbolic group of cohomological dimension $m \geq d - 1 \geq 2$ which does not admit a cyclic splitting. If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation and there exists a ρ-equivariant continuous non-constant map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ so that $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$, then ρ is irreducible.*

Proof. If ρ is reducible, one may conjugate it to have the form

$$\begin{bmatrix} \rho_1 & * & * & * \\ 0 & \rho_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & \rho_k \end{bmatrix}$$

where $k \geq 2$, each $\rho_i : \Gamma \to \mathrm{GL}(V_i)$ is a d_i-dimensional irreducible representation and $\mathbb{R}^d = \oplus_{i=1}^k V_i$. Notice that if $x \in \partial\Gamma$ and $\xi(x)$ lies in $\hat{V} = \left(\oplus_{i=1}^{k-1} V_i \right) \times \{0\}^{d_k}$ then, since Γ acts minimally on $\partial\Gamma$ and $\rho(\Gamma)$ preserves $\hat{V}$, $\xi(\partial\Gamma)$ would be contained in the proper subspace $\hat{V}$, which would contradict our assumption that $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$. It follows that there exists a ρ_k-equivariant map $\xi_k : \partial\Gamma \to \mathbb{P}(V_k)$, obtained by letting $\xi_k(x)$ denote the orthogonal projection of $\xi(x)$ onto V_k. Notice that $\xi_k(\partial\Gamma)$ spans V_k, since $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$. Let $\pi_k : \mathsf{GL}(V_k) \to \mathsf{SL}^\pm(V_k)$ be the obvious projection map and let Γ_1 be the finite index subgroup of Γ given by $(\pi \circ \rho_k)^{-1}(\mathrm{SL}(V_k))$. Since ξ_k is $(\pi_k \circ \rho_k)$-equivariant, Proposition 9.3, applied to the representation $(\pi_k \circ \rho_k)|_{\Gamma_1}$, implies that Γ_1, and hence Γ, has cohomological dimension $d_k - 1$, which is a contradiction. $\qquad\square$

In the final case of the proof of Theorem 9.1, $W = \langle \xi(\partial\Gamma) \rangle$ is a proper subspace of $\mathbb{R}^d$. Let $\pi_W : \mathsf{GL}(W) \to \mathsf{SL}^\pm(W)$ be the obvious projection map. Consider $\hat{\rho} = \pi_W \circ \rho|_W : \Gamma \to \mathsf{SL}^\pm(W)$ and the non-constant $\hat{\rho}$-equivariant map $\hat{\xi} : \partial\Gamma \to \mathbb{P}(W)$ (which is simply ξ with the range regarded as $\mathbb{P}(W)$). Since ξ is non-constant, W has dimension at least 2. If W has dimension 2,

then, by Lemma 9.2, ρ is discrete and faithful, which implies that Γ is a free group or surface group, contradicting our assumptions on Γ. If W has dimension at least 3, then Proposition 9.4 implies that $\hat{\rho}$ is irreducible, while Proposition 9.3 provides a contradiction in this case. $\qquad\square$

We next observe that if $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ has a non-constant spanning limit map, then the restriction to the boundary of any non-abelian quasiconvex subgroup is also non-constant.

Lemma 9.5. *Suppose that Γ is a torsion-free hyperbolic group and Γ_0 is a non-abelian quasiconvex subgroup of Γ. If $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ admits a continuous ρ-equivariant map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ so that $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$, then the restriction of ξ to $\partial\Gamma_0$ is non-constant.*

Proof. Lemma 9.2 implies that ρ is discrete and faithful. Suppose that ξ is constant on $\partial\Gamma_0$. By conjugating, we may assume $\xi(\partial\Gamma_0) = \{[e_1]\}$. Then $\rho|_{\Gamma_0}$ has the form

$$\rho(\gamma) = \begin{bmatrix} \varepsilon(\gamma) & u(\gamma) \\ 0 & |\varepsilon(\gamma)|^{-1/(d-1)}\rho_0(\gamma) \end{bmatrix}$$

for some homomorphism $\varepsilon : \Gamma \to \mathbb{R}^*$ and some representation $\rho_0 : \Gamma_0 \to \mathsf{SL}(d-1, \mathbb{R})$. Notice that the representation of $\hat{\rho} : \Gamma_0 \to \mathsf{SL}(d, \mathbb{R})$ given by

$$\hat{\rho}(\gamma) = \begin{bmatrix} \varepsilon(\gamma) & 0 \\ 0 & |\varepsilon(\gamma)|^{-1/(d-1)}\rho_0(\gamma) \end{bmatrix}$$

is the limit of the discrete faithful representations $\{Q_n^{-1} \circ \rho|_{\Gamma_0} \circ Q_n\}$, where Q_n is a diagonal matrix with $a_{11} = n$ and all other diagonal entries equal to 1, so $\hat{\rho}$ is discrete and faithful (see Kapovich [34, Thm. 8.4]).

We next show that if $\gamma \in \Gamma_0$ and $\varepsilon(\gamma) = 1$, then $\lambda_i(\hat{\rho}(\gamma)) = 1$ for all i. If not, consider the Jordan normal formal for $\rho_0(\gamma)$, regarded as a matrix in $\mathsf{SL}(d-1, \mathbb{C})$, i.e.

$$\rho_0(\gamma) = P \begin{bmatrix} J_{q_1,k_1} & & \\ & \ddots & \\ & & J_{q_r,k_r} \end{bmatrix} P^{-1}$$

where $P \in \mathrm{SL}(d-1, \mathbb{C})$ and $J_{q,k}$ is the k-dimensional Jordan block with the value $q \in \mathbb{C}$ along the diagonal. We may assume that $|q_1| \geqslant \cdots \geqslant |q_r|$ and that if $|q_i| = |q_{i+1}|$, then $k_i \geq k_{i+1}$. Notice that, if n is sufficiently large, the co-efficient of $J_{q,k}^n$ with largest modulus has modulus exactly $\binom{n}{k-1}|q|^{n-k+1}$ It follows that there exists $C > 1$ so that

$$\frac{1}{C}\binom{n}{k_1 - 1}|q_1|^{n-k_1+1} \leq \|\rho_0(\gamma^n)\| \leq C\binom{n}{k_1 - 1}|q_1|^{n-k_1+1}$$

for all $n \in \mathbb{N}$. Therefore, $\left\{ \left(\binom{n}{k_1-1}|q_1|^{n-k_1+1} \right)^{-1} \hat{\rho}(\gamma^n) \right\}$ has a subsequence which converges to a non-zero matrix A_∞. One may then show that if $w \in \mathbb{R}^d$, does not lie in the kernel K of A_∞, then $\{[\rho(\gamma)^n(w)]\}$ does not converge to $[e_1]$. Since $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$ and K is a proper subspace of $\mathbb{R}^d$, there exists $x \in \partial\Gamma - \partial\Gamma_0$ so that $\xi(x)$ does not lie in K. Since ξ is ρ-equivariant, $\{\rho(\gamma)^n(\xi(x))\}$ must converge to $\xi(\gamma^+) = [e_1]$, which is a contradiction.

Notice that if N is the commutator subgroup of Γ_0, then $\varepsilon(N) = \{1\}$. Since Γ_0 is a non-abelian torsion-free hyperbolic group, N contains a free subgroup Δ of rank 2. Let $\psi = \hat{\rho}|_\Delta^{ss}$ be a semisimplification of $\hat{\rho}|_\Delta$. Since ψ is a limit of conjugates of $\hat{\rho}|_\Delta$ and $\hat{\rho}|_\Delta$ is discrete and faithful, ψ is also discrete and faithful [34, Thm 8.4]. Since $\log \lambda_i(\psi(\gamma)) = \log \lambda_i(\hat{\rho}(\gamma)) = 0$ for all $\gamma \in \Delta$ and all i, Theorem 2.3 guarantees that there exists M so that $\|\log \sigma_i(\psi(\gamma))\| \leq M$ for all $\gamma \in \Delta$ and all i. Therefore, $\psi(\Delta)$ is bounded which contradicts the fact that ψ is discrete and faithful and that Δ is infinite. $\qquad\square$

The work of Louder-Touikan [42] allows us to find cohomologically large quasiconvex subgroups which do not split.

Proposition 9.6. *If Γ is a torsion-free hyperbolic group of cohomological dimension $m \geq 3$ which splits over a cyclic subgroup, then Γ contains an infinite index, quasiconvex subgroup of cohomological dimension m which does not split over a cyclic subgroup.*

Proof. One first considers a maximal splitting of Γ along cyclic subgroups. One of the factors, say Δ has cohomological dimension m (see Bieri [13, Cor. 4.1] and Swan [48, Thm. 2.3]). A result of Bowditch [15, Prop. 1.2], implies that Δ is a quasiconvex subgroup of Γ. If Δ itself splits along a cyclic subgroup, we consider a maximal splitting of Δ along cyclic subgroups. We then again find a factor Δ_1 of this decomposition which has cohomological dimension m and is quasiconvex in Δ, hence in Γ. Louder and Touikan [42, Cor. 2.7] implies that this process terminates after finitely many steps, so one obtains the desired quasiconvex subgroup of cohomological dimension m. □

We now combine the above results to establish Theorem 1.7.

Theorem 1.7: *If $d \geq 4$ and Γ is a torsion-free hyperbolic group, a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a Benoist representation if and only if Γ has cohomological dimension $d - 1$ and there is a non-constant ρ-equivariant continuous map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$.*

Proof. If ρ is a Benoist representation then Γ has cohomological dimension $d - 1$ and ξ^1_ρ is a continuous, non-constant ρ-equivariant map.

Now suppose that Γ has cohomological dimension $d-1$ and there is a non-constant ρ-equivariant map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$. If Γ does not split over a cyclic subgroup, then Theorem 9.1 implies that ρ is a Benoist representation. If Γ does split over a cyclic group, let Γ_1 be an infinite index, quasiconvex subgroup of Γ of cohomological dimension $d - 1$ which does not split over a cyclic subgroup.

We next observe that $\xi(\partial\Gamma)$ must span $\mathbb{R}^d$. If it doesn't, let W be the subspace spanned by $\xi(\partial\Gamma)$. We obtain a representation $\hat{\rho} : \Gamma \to \mathsf{SL}^\pm(W)$, given by $\pi_W \circ \rho|_W$ and a continuous $\hat{\rho}$-equivariant map $\hat{\xi} : \partial\Gamma \to \mathbb{P}(W)$, which is simply ξ with the range regarded as $\mathbb{P}(W)$, so that $\hat{\xi}(\partial\Gamma)$ spans W. There exists a subgroup Γ_2 of index at most two in Γ_1, so that $\hat{\rho}(\Gamma_2)$ lies in $\mathsf{SL}(W)$. Notice that Γ_2 also has cohomological dimension $d - 1$ and does not split over a cyclic subgroup. By Proposition 9.5, $\hat{\xi}|_{\partial\Gamma_2}$ is non-constant, so Propositions 9.3 and 9.4 imply that $\hat{\rho}|_{\Gamma_2}$ is a Benoist representation and that W has dimension d, which is a contradiction.

Since $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$, Proposition 9.5 implies that $\xi|_{\partial\Gamma_1}$ is non-constant, so Theorem 9.1 implies that $\rho_1 = \rho|_{\Gamma_1}$ is a Benoist representation. Therefore, $\rho(\Gamma_1)$ acts properly discontinuously and cocompactly on

$$\Omega = \mathbb{P}\left(\mathbb{R}^d - \bigcup_{x \in \partial\Gamma_1} \xi^{d-1}_{\rho_1}(x)\right)$$

where $\xi^{d-1}_{\rho_1}$ is the limit map for ρ_1. Moreover, $\xi(\partial\Gamma_1) = \partial\Omega$.

Suppose that $\alpha \in \Gamma - \Gamma_1$ and $\rho(\alpha)$ is biproximal. Let $V(\alpha)$ be the repelling hyperplane of $\rho(\alpha)$. Since ξ is equivariant, if $x \in \partial\Omega$, then $\{\rho(\alpha^n)(x)\}$ converges to $\xi(\alpha^+)$. Therefore, $V(\alpha)$ is disjoint from $\partial\Omega$. It follows that $\mathbb{P}(\mathbb{R}^d) - \Omega$ is the closure of the set of repelling hyperplanes of biproximal elements of $\rho(\Gamma)$. Therefore, the complement of Ω, and hence Ω itself, is invariant under the full group $\rho(\Gamma)$. Lemma 9.2 implies that ρ is discrete and faithful. Since $\rho(\Gamma)$ is discrete and $\rho(\Gamma_1)$ acts cocompactly on Ω, $\rho(\Gamma_1)$ must have finite index in $\rho(\Gamma)$ which contradicts the fact that ρ is faithful. □

Remarks: (1) In the 3-dimensional case, one may show that if Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(3, \mathbb{R})$ admits a non-constant continuous ρ-equivariant map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^3)$, then Γ is a surface group or a free group. If the space W spanned by $\xi(\partial\Gamma)$ is 2-dimensional, then it follows from Lemma 9.2 that $\rho|_W : \Gamma \to \mathsf{SL}(W)$ is discrete and faithful, so Γ is a surface group or a free group. Thus, we may assume that $\xi(\partial\Gamma)$ spans $\mathbb{R}^3$, so, again by Lemma 9.2, ρ is discrete and faithful and $\rho(\Gamma)$ contains a biproximal element.

Corollary B of Wilton [50] gives that if Γ is not free or a surface group, then Γ contains either an infinite index quasiconvex surface subgroup or a quasiconvex group which does not split over a cyclic subgroup. If Γ contains a quasiconvex subgroup Δ which does not split over a cyclic subgroup, then Propositions 9.3 and 9.4 imply that $\rho|_\Delta$ is a Benoist representation, and thus, by Theorem 1.1, Δ is a surface group, which is a contradiction. If Γ contains a quasiconvex surface subgroup Γ_0 of infinite index, then, by Lemma 9.5, $\xi|_{\partial\Gamma_0}$ is non-constant. There exists a biproximal element $\rho(\alpha) \in \rho(\Gamma) - \rho(\Gamma_0)$, and, since ξ is ρ-equivariant, $\xi(\partial\Gamma_0)$ cannot intersect $\mathbb{P}(V(\alpha))$ where $V(\alpha)$ is the repelling hyperplane of $\rho(\alpha)$, which implies that $\rho(\partial\Gamma_0)$ lies in an affine chart. If the span W_0 of $\xi(\partial\Gamma_0)$ is a proper subspace of $\mathbb{R}^3$, then, since $(\rho|_{\Gamma_0})|_{W_0} : \Gamma_0 \to \mathsf{SL}(W_0)$ is discrete and faithful, $\xi(\partial\Gamma_0) = \mathbb{P}(W_0)$ intersects $\mathbb{P}(V(\alpha))$, which is a contradiction. We then argue, just as in the proof of Proposition 9.3, that $\rho|_{\Gamma_0}$ is a Benoist representation. We further argue, as in the proof of Theorem 1.7, that this is impossible if Γ_0 has infinite index in Γ.

(2) One may use similar techniques to show that in three of the four exceptional cases in Theorem 1.3 one does not even have a non-constant limit map into $\mathbb{P}(\mathbb{R}^d)$. More precisely, if k is 2, 4 or 8, Γ is torsion-free hyperbolic group, $\partial\Gamma \cong S^k$ and $\rho : \Gamma \to \mathsf{SL}(2k, \mathbb{R})$ is P_k-Anosov, then there does not exist a non-constant, continuous ρ-equivariant map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^{2k})$.

Suppose that $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^{2k})$ is a non-constant, continuous ρ-equivariant map. If ρ is not irreducible, then let W be a proper $\rho(\Gamma)$-invariant subspace of $\mathbb{R}^d$. One may show that the dimension $W \cap \xi^k(x)$ is constant, say r, over $\partial\Gamma$. Let $p = p_{\rho,k} : E \to \partial\Gamma$ be the fibre bundle given by Lemma 2.1. Recall, from the proof of Theorem 1.3, that in these exceptional cases $E = S(\mathbb{R}^{2k})$. The restriction $q = p|_{S(W)} : S(W) \to \partial\Gamma$ is then a fibre bundle with base space S^k, fibres homeomorphic to S^{r-1} and total space $S(W)$. However, since $\dim(W) < 2k$, this is impossible, by the classification of sphere fibrations ([3]). So, we may assume that ρ is irreducible. Lemma 9.2 then implies that ρ is P_1-divergent and that there exists a biproximal element $\rho(\gamma) \in \rho(\Gamma)$ and $\xi(\gamma^+)$ is the attracting eigenline of $\rho(\gamma)$. Therefore, since ρ is P_k-Anosov, $\xi(\gamma^+) \subset \xi_\rho^k(\gamma^+)$. Since ξ and ξ_ρ^k are both ρ-equivariant and Γ acts minimally on $\partial\Gamma$, we see that $\xi(x) \subset \xi_\rho^k(x)$ for all $x \in \partial\Gamma$. Therefore, ξ lifts to a section of the spherical fibration p, which we have already seen is impossible.

10. Examples and Questions

In this section, we collect examples related to our results and discuss questions that arise.

It is natural to ask when the cohomological dimension bounds provided by Theorem 1.3 are sharp. Cocompact lattices in $\mathsf{SO}(d, 1) \subset \mathsf{SL}(d + 1, \mathbb{R})$ have cohomological dimension d and the inclusion map is a projective Anosov, so our results are sharp whenever $k = 1$. Cocompact lattices in $\mathsf{SU}(d, 1) \subset \mathsf{SL}(2d + 2, \mathbb{R})$ have cohomological dimension $2d$ and are P_2-Anosov, so our results are sharp when $k = 2$ and $d > 4$ is even. Similarly, cocompact lattices in $\mathsf{Sp}(n, 1) \subset \mathsf{SL}(4n + 4, \mathbb{R})$ demonstrate sharpness when $k = 4$ and $d = 4n + 4 > 8$

We also note that all the exceptional cases in part (2) of Theorem 1.3 occur. If $\rho : \Gamma \to \mathsf{SL}(2, \mathbb{R})$ is Fuchsian and Γ is a surface group, then Γ has cohomological dimension 2 and ρ is projective Anosov. If Γ is a cocompact lattice in $\mathsf{SL}(2, \mathbb{C}) \subset \mathsf{SL}(4, \mathbb{R})$, then Γ has cohomological dimension

3 and the inclusion map is P_2-Anosov. Similarly, cocompact lattices in $\mathsf{SL}(2,\mathcal{Q}) \subset \mathsf{SL}(8,\mathbb{R})$ and $\mathsf{SL}(2,\mathcal{O}) \subset \mathsf{SL}(16,\mathbb{R})$ have cohomological dimension 5 and 9 and are P_4-Anosov and P_8-Anosov, respectively, where $\mathcal{Q}$ is the quaternions and $\mathcal{O}$ is the octonions. (Notice that $\mathsf{PSL}(2,\mathcal{Q})$ may be identified with $\mathsf{SO}_0(5,1)$, in such a way that $\partial\mathbb{H}^5 \cong S^4$ is identified with $\mathcal{Q}\mathbb{P}^1$, hence if Γ is a cocompact lattice in $\mathsf{SL}(2,\mathcal{Q})$, then Γ is hyperbolic, $\partial\Gamma \cong S^4$ and there is an equivariant homeomorphism from $\partial\Gamma$ to $\mathcal{Q}\mathbb{P}^1 \subset \mathrm{Gr}_4(\mathbb{R}^8)$. Similarly, $\mathsf{PSL}(2,\mathcal{O})$ is identified with $\mathsf{SO}_0(9,1)$ and one may make a similar analysis. See Baez [4] for more details.)

Question 1: *For what values of d and k are the estimates in Theorem 1.3 sharp?*

If Γ is a convex cocompact subgroup of $\mathsf{PSL}(2,\mathbb{C}) \cong \mathsf{SO}_0(3,1)$, the inclusion map lifts to a representation $\rho : \Gamma \to \mathsf{SL}(2,\mathbb{C}) \subset \mathsf{SL}(4,\mathbb{R})$. In light of Theorem 1.2 it is natural to ask:

Question 2: *If Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(4,\mathbb{R})$ is P_2-Anosov, must Γ be isomorphic to a convex cocompact subgroup of $\mathsf{PO}(3,1)$?*

We know of no examples of Borel Anosov representations of surface groups in even dimensions which are not Hitchin. Proposition 7.2 assures us that every Borel Anosov representation of a surface group into $\mathsf{SL}(4,\mathbb{R})$ is irreducible. Together, they suggest the following ambitious question.

Question 3: *Is every Borel Anosov representation of a surface group into $\mathsf{SL}(4,\mathbb{R})$ Hitchin?*

Notice that Danciger and Zhang [23, Thm. 1.3] proved that Hitchin representations into $\mathsf{SO}(n,n)$ are not P_n-Anosov, if you regard them as representations into $\mathsf{SL}(2n,\mathbb{R})$.

We characterize Borel Anosov subgroups in dimensions 3 and 4. We note that it is easy to show that a cocompact lattice Γ in $\mathsf{Sp}(n,1)$ does not admit a Borel Anosov representation into $\mathsf{SL}(d,\mathbb{R})$ for any d. Suppose that $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is Borel Anosov. By Corlette [19] and Gromov-Schoen's [25] superrigidity theorem, see also [24], there exists $\rho_1 : \mathsf{Sp}(n,1) \to \mathsf{SL}(d,\mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ with compact closure so that $\rho = (\rho_1|_\Gamma)\rho_2$ and $\rho_1|_\Gamma$ and ρ_2 commute. Since ρ does not have compact closure, the representation ρ_1 has discrete kernel (in fact central). Let $\gamma \in \Gamma$ have infinite order. Then, the centralizer Z of γ in $\mathsf{Sp}(n,1)$ is non-abelian, which implies that the centralizer of $\rho(\gamma)$ in $\mathsf{SL}(d,\mathbb{R})$ contains $\rho_1(Z)$ and is hence non-abelian. However, $\rho(\gamma)$ is diagonalizable with distinct eigenvalues, hence has abelian centralizer, so we have arrived at a contradiction.

Question 4: *What other classes of hyperbolic groups can be shown not to admit Borel Anosov representations in any dimension?*

It is expected that not all linear hyperbolic group admit linear Anosov representations, but we know of no explicit examples. See also the discussion in Kassel [38, Section 8].

Question 5: *Can one exhibit explicit examples of linear hyperbolic groups which do not admit Anosov representations into $\mathsf{SL}(d,\mathbb{R})$ for any d?*

One can exhibit a sequence $\{\Gamma_k\}$ of hyperbolic groups so that each Γ_k has virtual cohomological dimension 2, admits a faithful representation into $\mathsf{SL}(2k,\mathbb{R})$, but admits no projective Anosov representation into $\mathsf{SL}(2k+1,\mathbb{R})$. Let $\Gamma_1 = \pi_1(S) * \mathbb{Z}$ where S is a closed orientable surface of genus at least two. In general, we define $\Gamma_k = (\Gamma_{k-1} \oplus \mathbb{Z}_3) * \mathbb{Z}$ and notice that Γ_k can be realized as a subgroup of $\hat{\Gamma}_k = (\Gamma_{k-1} * \mathbb{Z}) \oplus (\mathbb{Z}_3 * \mathbb{Z})$. It is not difficult to check that there is a faithful representation $\rho_1 : \Gamma_1 \to \mathsf{SL}(2,\mathbb{R})$. Theorem 1.1 implies that Γ_1 does not admit a projective Anosov representation into $\mathsf{SL}(3,\mathbb{R})$. Since Γ_{k-1} contains a subgroup isomorphic to $\Gamma_{k-1} * \mathbb{Z}$, there exists a faithful representation $\hat{\rho}_k : \Gamma_{k-1} * \mathbb{Z} \to \mathsf{SL}(2k-2,\mathbb{R})$. If $\sigma : \mathbb{Z}_3 * \mathbb{Z} \to \mathsf{SL}(2,\mathbb{R})$ is a faithful representation and π_i is the projection of $\hat{\Gamma}_k$ onto the ith

summand, then $(\rho_{k-1} \circ \pi_1) \oplus (\sigma \circ \pi_2)$ is a faithful representation of $\hat{\Gamma}_k$ into $\mathsf{SL}(2k, \mathbb{R})$, which restricts to a faithful representation $\rho_k : \Gamma_k \to \mathsf{SL}(2k, \mathbb{R})$. Suppose $\rho : \Gamma_k \to \mathsf{SL}(2k+1, \mathbb{R})$ is projective Anosov. Let c be the generator of $\mathbb{Z}_3$ in the first factor of Γ_k. Since the element c fixes $\partial\Gamma_{k-1}$ pointwise, $V = \langle \xi^1_\rho(\partial\Gamma_{k-1}) \rangle$ is contained in the kernel of $\rho(c) - I$, which has dimension at most $2k - 2$. (Notice that $\rho(c) \neq I$, since ρ has finite kernel, and c is not contained in a finite normal subgroup of Γ_k.) However, the restriction $\rho|_V : \Gamma_{k-1} \to \mathrm{GL}(V)$ would then be projective Anosov, which is impossible by our inductive assumption.

It is a consequence of the Geometrization Theorem that any hyperbolic group which admits a discrete faithful representation into $\mathsf{PO}(3,1)$ also admits a convex cocompact representation into $\mathsf{PO}(3,1)$. One might ask by extension:

Question 6: *Are there hyperbolic groups which admit discrete faithful linear representations, but do not admit Anosov representations?*

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