URUGUAY NOTES ON HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. Notes for a 6-hour minicourse in Uruguay on hyperbolic 3-manifolds. The goal is to complete a sketch of Thurston's original proof that 3-manifolds which fiber over the circle are geometrizable.

1. Review of hyperbolic space and Teichmüller space

This section consists of background material on hyperbolic space and Teichmüller space. This material will be covered in Matilda Martinez's lectures, so will not be covered in my lectures. Most of this section is lifted and lightly rewritten from my Informal Lecture Notes on Anosov Representations which are available on my webpage.

1.1. The hyperbolic plane

Recall that the upper half-plane model for the hyperbolic plane is given by

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

with Riemannian metric

$$ds^2_{hyp} = \frac{1}{y^2}\, dxdy.$$

Prosaically, if $\vec{v} \in T_{(x,y)}\mathbb{H}^2$, then its hyperbolic length $||\vec{v}||_{hyp} = \frac{|\vec{v}|}{y}$ where $|\vec{v}|$ is the Eucliden length of $\vec{v}$

One may easily check that the y-axis L is a geodesic in this metric, since if $p : \mathbb{H}^2 \to L$ is Euclidean perpendicular projection, then $||Dp(\vec{v})||_{hyp} \leq ||\vec{v}||_{hyp}$ with equality if and only if $\vec{v}$ is vertical. Moreover, segments of L are the only geodesic joining points on L. One may check that Möbius transformations with real co-efficients act as orientation-preserving isometries of $\mathbb{H}^2$, by a simple calculation. Or you can notice that all such Mobius transformations are generated by maps of the form $z \to a$, $z \to \frac{1}{\bar{z}}$ and $z \to \lambda z$ and checking that each of these maps is an isometry. It follows that all lines and semi-circles perpendicular to the x-axis are geodesics and that these are the only geodesics. Therefore, an orientation-preserving isometry is determined by its action on a single unit tangent vector. Since $\mathsf{PSL}(2,\mathbb{R})$ acts transitively on T^1H^2, we see that

$$\mathrm{Isom}_+(\mathbb{H}^2) = \mathsf{PSL}(2,\mathbb{R}).$$

An ideal triangle is determined by the three geodesics joining any three points in $\partial\mathbb{H}^2$. Since $\mathsf{PSL}(2,\mathbb{R})$ acts transitively on triples of distinct points in $\partial\mathbb{H}^2$, any two ideal triangles are isometric. We say that the angle at an ideal vertex is 0. One may move the end points to 1, -1 and ∞ and compute that the triangle has area π. We say a geodesic triangle is 1/3-ideal if it has two endpoints in $\partial\mathbb{H}^2$ and the other in $\mathbb{H}^2$. We may move the two ideal vertices to

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-1 and ∞ and arrange that the other vertex lies on the unit circle. if the internal angle at the non-ideal vertex is α, then the vertex must lie at the point $(\cos\alpha, \sin\alpha)$. Hence any two 1/3-ideal triangles with internal angle α are isometric and one can compute that they have area $\pi - \alpha$.

In general if a geodesic triangle T has internal angles α, β and γ, we may assume that one vertex lies at $(0,1)$, one edge emanating from $(0,1)$ travels downward and that the other travels to the right of the y-axis. The following picture then proves that $\text{Area}(T) = \pi - (\alpha+\beta+\gamma)$. (I include this mainly because it is one of my favorite picture proofs.)

PICTURE NEEDED

More generally, if P is a geodesic n-gon in $\mathbb{H}^2$ with internal angles $\{\alpha_1, \dots, \alpha_n\}$, then

$$\text{Area}(P) = \pi(n-2) - \sum_{i=1}^{n} \alpha_i.$$

This implies that the hyperbolic plane has constant curvature -1.

Another prominent model for $\mathbb{H}^2$ is the Poincaré Disk model which is the unit disk D^2 with the metric

$$ds^2 = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$$

so if $\vec{v} \in T_{(x,y)}D^2$, then

$$||\vec{v}||_{hyp} = \frac{2|\vec{v}|}{(1-x^2-y^2)}.$$

One may check that any Möbius transformation taking the upper half-plane to D^2 is an isometry with respect to the hyperbolic metrrics. For example, one may take $T(z) = \frac{z-i}{z+i}$. It follows that geodesics in this model are lines and semi-circles perpendicular to $\partial D^2 = S^1$. Moreover, the group of orientation-preserving isometries is the group of Möbius transformation which preserve D^2.

The main advantage of this model is the rotational symmetry about the origin. One can compute that if $r \in (0,1)$ and $z \in S^1$, then

$$d_{hyp}(0, rz) = \log\frac{1+r}{1-r} = 2\tanh^{-1}(r).$$

One may then easily compute that the ball of hyperbolic radius R about the origin is a ball of Euclidean radius $\tanh\left(\frac{R}{2}\right)$. Therefore, one may compute that this ball has hyperbolic circumference

$$2\pi\sinh R = 2\pi\frac{\tanh\left(\frac{R}{2}\right)}{1-\tanh^2\left(\frac{R}{2}\right)}$$

and hyperbolic area

$$2\pi\cosh R - 2\pi = \int_0^R 2\pi \sinh t dt.$$

Since the isometry group of $\mathbb{H}^2$ acts transitively on $\mathbb{H}^2$, every circle of hyperbolic radius R has hyperbolic length $2\pi\sinh R$ and bounds a ball of hyperbolic area $2\pi\cosh R - 2\pi$.

1.1.1. **Life in the hyperbolic plane.** We assume, for simplicity, that the baseball field is a quadrant of a disk with radius 300 feet and that the infield is contained within a disk of radius 100 feet. In Euclidean space, the outfield has area approximately 62,832 square feet. It can be covered by 3 outfielders, so each outfielder covers approximately 20,000 square feet.

In hyperbolic space, the outfield has area $\frac{\pi}{2}\cosh(300) > 10^{100}$. If you assume that each outfielder can still cover 20,000 square feet, you would need more than 10^{94} outfielders to play hyperbolic baseball.

Suppose that you are 300 feet (100 yards) from the pin on the golf course. If you hit the ball exactly 300 feet but one degree off-line, you can use distance along the circle to calculate that you are roughly $\frac{2\pi(300)}{360} = 5.24$ feet from the hole. This estimate is accurate to two significant digits.

In hyperbolic space, the circular estimate would suggest that you are roughly $\frac{2\pi\cosh(300)}{360} > 10^{97}$ feet from the hole, which can't be correct. In fact, you will be over 590 feet from the hole. So hitting it by only one degree off-line is almost as bad as hitting the ball straight backwards.

In Euclidean beachball, a ball of radius one foot which is r feet away takes up roughly $\frac{1}{\pi R}$ of your field of vision (assuming you can see in exactly half the directions). So at 30 feet it takes up roughly 1 percent of your field of vision and at 300 feet it takes up roughly .1 percent of your field of vision.

In hyperbolic beachball, a ball of radius one foot which is r feet away takes up roughly $\frac{1}{\pi\sinh R}$ of your field of vision. So even if you have such good eyesight that you can see things that only take up .01 percent of your field of vision you won't be able to see the beachball if it is more than seven feet from you.

1.2. **Teichmüller space**

A complete orientable Riemannian surface X is said to be **hyperbolic** if it is locally isometric to $\mathbb{H}^2$. In this case, the universal cover $\tilde{X}$ is a simply connected complete Riemannian manifold locally isometric to $\mathbb{H}^2$ and hence can be identified with $\mathbb{H}^2$. Therefore, $X = \mathbb{H}^2/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^2) \cong \mathsf{PSL}(2,\mathbb{R})$. Notice that Γ is only well-defined up to conjugacy, since the identification of $\tilde{X}$ with $\mathbb{H}^2$ is not canonical.

A **marked hyperbolic structure** on a closed orientable surface S is a pair (X, f) where $f : S \to X$ is an orientation-preserving homeomorphism and X is a hyperbolic surface. If $X = \mathbb{H}^2/\Gamma$, then $f_* : \pi_1(S) \to \pi_1(X) \cong \Gamma$ is an isomorphism and hence we obtain a discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$. However, ρ is only well-defined up to conjugation in $\mathsf{PsL}(2,\mathbb{R})$.

One may build a hyperbolic surface of genus two, by starting with a regular hyperbolic octagon, all of whose internal angles are $\frac{\pi}{4}$ and then gluing by the standard gluing pattern. Similarly, one may build a hyperbolic surface of genus g by starting with a regular $(4g-4)$-gon with internal angles $\frac{\pi}{2g}$.

We will choose to formalize Teichmüller space by using representations. Recall that a marked hyperbolic structure on a closed surface S, gives rise to a (conjugacy class of a) discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$. In turn, a discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$ gives rise to a hyperbolic surface $X_\rho = \mathbb{H}^2/\rho(\pi_1(S))$. Since X_ρ is homotopy equivalent to S, it is homeomorphic to S. Moreover, there is homeomorphism $h_\rho : S \to X_\rho$ so that $(h_\rho)_*$ is conjugate to ρ. (Here, we are using a special property of the topology of closed surfaces. The Nielsen-Baer Theorem, see Farb-Margalit [17, Chapter 8], gives that every homotopy equivalence of a closed orientable surface is homotopic to a homeomorphism.) We then let

$$\widetilde{\mathcal{T}}(S) = \{\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R}) \mid \rho \text{ discrete, faithful, and } h_\rho \text{is orientation-preserving}\}$$

and the Teichmüller space of S is the quotient

$$\mathcal{T}(S) = \widetilde{\mathcal{T}}(S)/\mathsf{PSL}(2,\mathbb{R})$$

where $\widetilde{\mathcal{T}}(S)$ inherits a topology as a subset of $\mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$, $\mathsf{PSL}(2,\mathbb{R})$ acts by conjugation and $\mathcal{T}(S)$ inherits the quotient topology.

Alternatively, one may define $\mathcal{T}(S)$ to be the space of marked hyperbolic structure on S up to the equivalence $(X_1, f_1) \sim (X_2, f_2)$ if and only if $f_2 \circ f_1^{-1}$ is homotopic to an isometry. One may think of X as hyperbolic clothing for the naked topological surface S and f as instructions for how to wear the clothing. The equivalence relation allows one to adjust the clothing, but not to wear it backwards or to stick your head through the hole designated for the arm.

It is a classical theorem, going back to the 19th century, that $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$ if $g \geq 2$ is the genus of S. (Notice that $\pi_1(S)$ has a presentation with $2g$ relations and one relation, one would expect that $\mathrm{DF}(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$ has dimension $(2g)3 - 3 = 6g - 3$, so one would predict that Teichmüller space has dimension $6g - 6$.) The *mapping class group* $\mathrm{Mod}(S)$ is the group of (isotopy classes of) self-homeomorphisms of S. Fricke showed that the mapping class group acts properly discontinuously, but not freely, on $\mathcal{T}(S)$ and its quotient is the **moduli space** of unmarked hyperbolic structures on S.

We now give a quick sketch of the Fenchel-Nielsen coordinates on Teichmüller space. Suppose that X is a closed orientable hyperbolic surface of genus $g \geq 2$. Recall that, since X is negatively curved, every homotopically non-trivial closed curve is homotopic to a unique closed geodesic. Morever, if two homotopically non-trivial simple closed curves are disjoint and non-parallel, then their geodesic representatives are also disjoint. Let $\alpha = \{\alpha_1, \ldots, \alpha_{3g-3}\}$ be a maximal collection of disjoint simple closed curves and let α^* be their geodesic representatives on X. The components of $X - \alpha^*$ are a collection of $2g - 2$ hyperbolic pairs of pants with geodesic boundary. (A topological **pair of pants** is a disk with two holes.) Therefore, every closed hyperbolic surface may be built from hyperbolic pairs of pants.

If P is a hyperbolic pair of pants with geodesic boundary and s_1, s_2 and s_3 are the shortest paths joining boundary components (called seams), then $P - \{s_1, s_2, s_3\}$ is a pair of all-right hyperbolic hexagons (i.e. hexagons all of whose interior angles are $\frac{\pi}{2}$). An all-right hexagon is determined by the lengths of any 3 non-consecutive sides. Moreover, any 3 lengths can be achieved. It follows that P is the double of the unique all-right hexagon with alternate sides having lengths agreeing with the lengths of the seams of P. Moreover, we can build a geodesic pair of pants with any collection of boundary lengths and this geodesic pair of pants is entirely determined by its boundary lengths.

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So the hyperbolic structure on X is determined, up to isometry, by the lengths of the components of α^* and instructions for gluing the pants together. Since the pants are glued along closed geodesic curves, there is a one-dimensional space of ways to glue them. This suggests more forcefully that the space of hyperbolic structures on X has dimension $6g - 6$.

More formally, we get a map $L : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3}$ where

$$L(X, f) = \Big(\ell_X(f(\alpha_i)^*)\Big)_{i=1}^{3g-3}.$$

At each element of α we can define a twist coordinate in $\mathbb{R}$ which records how the geodesic pairs of pants are glued along $f(\alpha_i)^*$, so we obtain $\Theta : \mathcal{T}(S) \to \mathbb{R}^{3g-3}$. It is natural to think at first that the twist should lie in S^1. One way to see that this is not the case is to observe that because we have marked the surface, we can detect the homotopy class of the shortest curve crossing α. When you make a full positive twist, the shortest such curve changes by a full negative twist (at least if it is unique). One can then see that

$$(L, \Theta) : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \cong \mathbb{R}^{6g-6}$$

is a homeomorphism. For a careful discussion of twist coordinates see, for example, Thurston [32, Section 4.6], Farb-Margalit [17, Section 10.6] or Martelli [23, Chapter 7].

References: Farb and Margalit [17] give a nice treatment of Teichmüller space from a modern geometrical/topological viewpoint. Bers' survey paper [5] is a beautiful treatment of the classical complex analytic approach Thurston [32, Section 4.6] gives a concise treatment of the Fenchel-Nielson coordinates. Abikoff [1] gives a treatment of the classical theory with an eye towards the modern viewpoint.

1.3. **Hyperbolic 3-space**

The upper half space model for hyperbolic 3-space is given by

$$\mathbb{H}^3 = \{(z, t) \in \mathbb{C} \times \mathbb{R} | t > 0\}$$

with hyperbolic metric given

$$ds^2 = \frac{dx^2 + \cdots + dy^2 + dt^2}{t^2}$$

where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

It is easy to check, just as in $\mathbb{H}^2$, that the t-axis L is a geodesic and that the only geodesics joining points on L are given by segments of L. Möbius transformations (this time with complex co-efficients) extend to orientation-preserving isometries of $\mathbb{H}^3$. One can do this by writing down a painful formula, or by noting that translations in $\mathbb{C}$ extend to horizontal translations in $\mathbb{H}^3$, dilations $z \to \lambda z$ extend to dilations $(z, t) \to (\lambda z, |\lambda| t)$, inversion in the unit circle extends to inversion in the unit sphere and reflection in the y-axis extends to reflection in the $y-t$-plane. One may then check that each of these extensions is an isometry of $\mathbb{H}^3$. It follows that all geodesics in $\mathbb{H}^3$ are semi-circles or lines perpendicular to $\partial\mathbb{H}^3$ and that an isometry is determined by its action on a single orthonormal frame at a point in $\mathbb{H}^3$. Since, the group generated by these inversions, translations, and dilations, acts transitively on the orthonormal frame bundle of $\mathbb{H}^3$, we see that this group is the full isometry group of $\mathbb{H}^3$. In particular, e can identify $\mathrm{Isom}_+(\mathbb{H}^3)$ with $\mathsf{PSL}(2, \mathbb{C})$. Therefore, $\mathbb{H}^3$ has constant sectional curvature, and since it contains a totally geodesic copy of $\mathbb{H}^2$, the constant is -1.

2. Convex cocompact hyperbolic 3-manifolds

2.1. Basic definitions

For simplicity, we will assume throughout these lectures that all groups are **finitely generated**, **non-abelian** and **torsion-free**. Selberg's Lemma guarantees that every finitely generated subgroup of any linear group contains a finite-index torsion-free subgroup, so the assumption that groups are torsion-free is not very restrictive.

Throughout the lecture course, M will denote a compact, orientable, irreducible 3-manifold, possibly with boundary, with infinite fundamental group. A 3-manifold M is said to be irreducible if every embedded 2-sphere in M bounds a 3-ball in M. These assumptions guarantee that $\pi_1(M)$ is infinite and the universal cover of M is contractible. (If you prefer not to take these facts for granted, you can just add them to the assumptions we are making on M.) In fact, we will mostly be considering the case where $M = S \times [0,1]$ and S is a closed surface. Moreover, S will always denote a closed oriented surface in these notes.

A discrete subgroup Γ of $\mathsf{PSL}(2,\mathbb{C})$ is called a *Kleinian group.* If $\rho : \pi_1(M) \to \mathsf{PSL}(2, C)$ is a discrete, faithful representation, then $\rho(\pi_1(M))$ then we obtain a hyperbolic 3-manifold

$$N_\rho = \mathbb{H}^3/\rho(\pi_1(S)).$$

Since M and N_ρ both have contractible universal cover and ρ gives an identification of the fundamental groups of M and N_ρ, we obtain a homotopy equivalence

$$h_\rho : M \to N_\rho$$

so that $(h_\rho)_* : \pi_1(M) \to \pi_1(N_\rho) = \rho(\pi_1(M))$ is conjugate to ρ. We think of h_ρ as a marking of a hyperbolic 3-manifold (much as in the setting of Teichmüller space) and think of the pair (M_ρ, h_ρ) as a marked hyperbolic 3-manifold.

Remark: It follows from a result of Peter Scott [28] that if Γ is a finitely generated, torsion-free Kleinian group then N_ρ is homotopy equivalent to a compact, irreducible, orientable 3-manifold M, so we may regard Γ as the image of a discrete faithful representation of $\pi_1(M)$.

The *limit set* of a Kleinian group is the set of accumulation points of an orbit in the boundary of $\mathbb{H}^3$, i.e. if $x_0 \in \mathbb{H}^3$

$$\Lambda(\rho) = \overline{\rho(\pi_1(M))(x_0)} - \rho(\pi_1(M))(x_0) \subset \partial\mathbb{H}^3.$$

One may easily check that $\Lambda(\rho)$ does not depend on the choice of basepoint x_0 (EXERCISE). If $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$ is a Fuchsian representation, then $\rho(\pi_1(S))$ acts cocompactly on the isometric copy of $\mathbb{H}^2$ lying above the real axis in $\mathbb{C}$, so $\Lambda(\rho) = \mathbb{R} \cup \{\infty\}$. (EXERCISE) One may characterize $\Lambda(\rho)$ as the smallest non-empty closes subset of $\partial\mathbb{H}^3$ which is invariant under $\rho(\pi_1(M))$.

The complement $\Omega(\rho) = \partial\mathbb{H}^3 \setminus \Lambda(\rho)$ of the limit set is called the domain of discontinuity, since $\rho(\pi_1(M))$ acts properly discontinuously on $\Omega(\rho)$ (EXERCISE). One may characterize $\Omega(\rho)$ as the largest open subset of $\partial\mathbb{H}^3$ on which $\rho(\pi_1(M))$ acts properly discontinuosly. (EXERCISE) We can then consider the *conformal boundary*

$$\partial_c N_\rho = \Omega(\rho)/\rho(\pi_1(M))$$

which has the structure of a Riemann surface. Notice that the conformal boundary can be empty, for example if $\rho(\Gamma)$ is cocompact. We can combine the hyperbolic manifold and its conformal boundary to obtain the *conformal bordification*

$$\widehat{N}_\rho = (\mathbb{H}^3 \cup \Omega(\rho))/\rho(\pi_1(N)) = N_\rho \cup \partial_c N_\rho.$$

The *convex hull* of the limit set $CH(\Lambda(\Gamma)$ is defined to be the smallest closed convex subset of $\mathbb{H}^3$ containing all bi-infinite geodesics with end points in $\Lambda(\Lambda(\Gamma))$. One may check (EXERCISE) that it is the union of all ideal tetrahedra with endpoints in the limit set. (The convex hull clearly contains all such ideal tetrahedra, so it only remains to check that the union of two ideal tetrahedra is contained in the convex hull of their limit points.) The *convex core* is then given by

$$C(N_\rho) = CH(\Lambda(\rho))/\rho(\pi_1(M)) \subset N_\rho$$

Notice that since $CH(\Lambda(\rho))$ is convex, there exists a well-defined retraction, called the *nearest point retraction*,

$$\tilde{r} : \mathbb{H}^3 \to CH(\Lambda(\rho))$$

so that $\tilde{r}(x)$ is the unique point on $CH(\Lambda(\rho))$ closest to x. (Notice that if $x \neq y$ and $d(z,x) = d(z,y)$ then there exists a point $u \in \overline{xy}$ so that $d(z,u) < d(z,x)$.) Since $CH(\Lambda(\rho))$ is $\rho(\Gamma)$-invariant, $\tilde{r}$ is ρ-equivariant, so descends to a retraction, still called the nearest point retraction,

$$r : N_\rho \to C(N_\rho).$$

It is often useful to consider the closed neighborhood $C_1(N_\rho)$ of radius one of the convex core. $C_1(N_\rho)$ is the quotient of the closed neighborhood of radius one of the convex hull of the limit set. One key feature here is that $C_1(N)$ is strictly convex (since if $d(x,w) = 1$ and $d(y,z) = 1$ and u lies in the interior of $\overline{xy}$, then $d(u, \overline{wz}) < 1$). One may also show that $\partial C_1(N_\rho)$ is C^1. One does so by showing that $f : \mathbb{H}^3 - CH(\Lambda(\rho)) \to (0,\infty)$ is a C^1-submersion and applying the regular value theorem. If $x \in \mathbb{H}^3 - CH(\Lambda(\rho))$, then let P_x be the totally geodesic plane through x which is perpendicular to $\overline{xr(x)}$, which is a support plane for $CH(\lambda(\rho)$ if $g(y) = d(y,x)$ and $h(y) = d(y, P_x)$, then $h(y) \le f(y) \le g(y)$ for all y in the same component of $\mathbb{H}^3 - P_x$ as x. Moreover, $f(x) = g(x) = h(x)$ and f and h are differentiable at x and the have the same derivative at x, so f is differentiable at x. Notice that f is a submersion at x, since the restriction of f to $\overrightarrow{r(x)x}$ is a submersion. (With a little more care, one can show that $\partial C_1(N_\rho)$ is $C^{1,1}$.)

Suppose that $x, y \in N_\rho - C_1(N_\rho)$ and $\overrightarrow{r(x)x}$ and $\overline{r(y)y}$ intersect $\partial C_1(N_\rho)$ at the same point z, then $r(z) = r(y) = r(x)$, so $\overrightarrow{r(x)x} = \overrightarrow{r(y)y} = \overrightarrow{r(z)z}$. It follows that there is a homeomorphism

$$f : N_\rho - C(N_\rho) \to \partial C_1(N_\rho) \times (0,\infty) \quad \text{given by} \quad f(x) = \left(\overrightarrow{r(x)x} \cap \partial C_1(N), d(r, r(x))\right).$$

(Notice that if $r(x) = r(y)$ it need not be the case that $\overrightarrow{r(x)x} = \overrightarrow{r(y)y}$, which is another main reason we work mostly with $C_1(N_\rho)$.)

One may continuously extend $\tilde{r}$ to a map $\partial\tilde{r} : \Omega(\rho) \to CH(\Lambda(\rho))$ by letting $\tilde{r}(z)$ be the first point of contact of an expanding family of horospheres based at z with $CH(\Lambda(\rho))$. Again, $\tilde{r}$ descends to a map

$$\partial r : \partial_c N_\rho \to C(N_\rho).$$

We can then define a homeomorphism

$$\partial f : \partial_c N_\rho \to \partial C_1(N_\rho) \quad \text{given by} \quad \partial f(z) = \overrightarrow{r(z)z} \cap \partial C_1(N).$$

One may combine f and ∂f into a homeomorphism

$$\bar{f} : \widehat{N}_\rho - C(N_\rho) \to \partial C_1(N_\rho) \times (0,1]$$

where if $x \in N_\rho - C(N_\rho)$ then $\bar{f}(x) = \left(\overrightarrow{r(x)x} \cap \partial C_1(N), 1 - \frac{1}{1+d(r,r(x))}\right)$ and if $z \in \partial_c N_\rho$, then $\bar{f}(z) = \left(\overrightarrow{r(z)z} \cap \partial C_1(N), 1\right)$. So we see:

Proposition 2.1. *If $\rho : \pi_1(M) \to \mathsf{PSL}(2,\mathbb{C})$ is a discrete, faithful representation, then $\widehat{N}_\rho$ is homeomorphic to $C_1(N_\rho)$.*

We say that ρ is *convex cocompact* if $C_1(N_\rho)$ is compact. Notice that this is equivalent to requiring that either $\widehat{N}_\rho$ or $C(N_\rho)$ is compact.

2.2. **The viewpoint from geometric group theory**

Quasi-isometries and quasi-isometric embeddings are natural classes of mappings in the context of geometric group theory. They are generalizations of bilipschitz homeomorphisms and embeddings which ignore the local structure. However, they need not even be continuous. For example, an infinite line is quasi-isometric to both an infinite Euclidean cylinder and to $\mathbb{Z}$ and all compact metric spaces are quasi-isometric. One justification for working in this looser context, is that the natural geometric structure on a group, given by a word metric associated to some (finite) generating set, is only well-defined up to quasi-isometry.

We will always work in the setting of proper length spaces. A metric space is **proper** if all closed metric balls are compact. A proper metric space X is a **length space** if given any $x, y \in X$, then there exists a rectifiable path joining x to y of length $d(x,y)$. If J is an interval in $\mathbb{R}$ and $\alpha : J \to X$ is a path so that $d(\alpha(s), \alpha(t)) = |t-s|$ for all $s, t \in J$, then we say that α is a **geodesic**. Notice that in this case $\alpha([s,t])$ has length $t-s$ if $t > s$. An action of a group Γ on X is **properly discontinuous** if whenever $K \subset X$ is compact, $\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\}$ is finite. (I include this definition since some standard texts in general topology include the non-standard assumption that the group acts freely to the definition of proper discontinuity.)

A map $f : Y \to Z$ between metric spaces is a **quasi-isometric embedding** if there exists $K \geq 1$ and $C \geq 0$ such that

$$\frac{1}{K} d_Y(a,b) - C \leq d_Z(f(a), f(b)) \leq K d_Y(a,b) + C$$

for all $a, b \in Y$. If we want to remember the constants, we say that f is a (K,C)-quasi-isometric embedding. We say that $f : X \to Y$ is a **quasi-isometry** if there exists $K \geq 1$ and $C \geq 0$ so that f is a (K,C)-quasi-isometric embedding and if $y \in Y$, then there exists $x \in X$ so that $d(f(x), y) \leq C$, i.e. f is a quasi-isometric embedding which is coarsely surjective. One may think of quasi-isometric embeddings as bilipschitz embeddings "in the large," where you don't care at all what happens on the "scale" of the additive constant C.

If $f : X \to Y$ is a quasi-isometry, one may define a **quasi-inverse** $g : Y \to X$, i.e. a quasi-isometry so that there exists $\hat{C}$ so that $d_X(x, g(f(x)) \leq \hat{C}$ and $d_Y(y, f(g(y)) \leq \hat{C}$ for all $x \in X$ and $y \in Y$. There is only one sensible way to construct g. Given $y \in Y$, there exists some $x \in X$ so that $d(f(x), y)) \leq C$, and we set $g(y) = x$. If you haven't done so before, I recommend checking the claim that g is a quasi-inverse for yourself. Notice that the quasi-inverse is far from canonical.

The Milnor-Svarc lemma assures us that if a group acts properly discontinuously and cocompactly on two spaces, then the spaces are quasi-isometric. This allows one to freely study finitely presented groups by studying their actions on spaces, since any such space is quasi-isometric to the Cayley graph of the group. Moreover, any two Cayley graphs for a group (with respect to different finite generating sets) are quasi-isometric.

Lemma 2.2. (Milnor-Svarc Lemma) *If Γ acts properly discontinuously, cocompactly and by isometries on a proper, length space X, then Γ is finitely generated and the orbit map $\Gamma \to X$ given by $\gamma \mapsto \gamma(x_0)$, for all $\gamma \in \Gamma$ and some $x_0 \in X$, is a quasi-isometry.*

We will make use of a special case of this lemma which is tailored to our situation. If $\rho : \pi_1(M) \to \mathsf{PSL}(2,\mathbb{C})$ is a convex cocompact representation, we pick $x_0 \in CH(\Lambda(\Gamma))$ and define the orbit map $\tau_\rho : \Gamma \to \mathbb{H}^3$ by $\tau_\rho(\gamma) = \gamma(x_0)$. If we choose a generating set S for $\pi_1(M)$, we can construct a Cayley graph C_M for $\pi_1(M)$ and extend τ_ρ to a map $\hat{\tau}_\rho : C_M \to \mathbb{H}^3$ by mapping each edge of C_M to a geodesic segment. Recall that we metrize C_M by giving each edge length 1.

Lemma 2.3. (Specialized Milnor-Svarc Lemma) *If $\rho : \pi_1(M) \to \mathsf{PSL}(2,\mathbb{C})$ is a convex cocompact representation, $x_0 \in CH(\Lambda(\Gamma))$ and S is a finite generating set for $\pi_1(M)$, then there exists $K \geq 1$ and $C > 0$ so that $\hat{\tau}_\rho : C_M \to CH(\Lambda(\rho))$ is K-bilipschitz and a (K,C)-quasi-isometry.*

Proof. Let $A = \max_{s\in S} d(x_0, \rho(s)(x_0))$. Then $\hat{\tau}_\rho$ is A-Lipschitz by construction.

Let R be the diameter of $C(N)$ and let $T = \{\gamma \in \pi_1(M) \mid \rho(\gamma)(D(3R,x_0)) \cap D(3R,x_0) \neq \emptyset\}$, where $D(3R,x_0)$ is the closed ball of radius $3R$ about x_0. Since $\rho(\pi_1(M))$ acts properly discontinuously on $\mathbb{H}^3$, T is finite.

Let $\gamma \in \pi_1(M)$ and let L be a geodesic segment in X joining x_0 to $\rho(\gamma)(x_0)$. Divide L up into

$$n = \left\lfloor \frac{d(x_0, \rho(\gamma()x_0))}{R} \right\rfloor + 1$$

segments of equal length, with endpoints $\{x_0, x_1, \ldots, x_n\}$. Notice that each segment has length less than R. Since $C(N_\rho)$ has diameter R and $x_0 \in CH(\Lambda(\rho))$, there exists, for each i, $\gamma_i \in \pi_1(M)$ so that $d(x_i, \rho(\gamma_i)x_0)) \leq R$ where we may choose $\gamma_0 = id$ and $\gamma_n = \gamma$. Then, since $d(\rho(\gamma_i)(x_0), \rho(\gamma_{i+1})(x_0)) \leq 3R$ (by the triangle inequality), $\gamma_i^{-1}\gamma_{i+1} \in T$. Notice that

$$\gamma = \gamma_0(\gamma_0^{-1}\gamma_1)(\gamma_1^{-1}\gamma_2)\cdots(\gamma_{n-1}^{-1}\gamma_n).$$

Notice that since T is finite and S generates $\pi_1(M)$, there exists K_1 such that $d_{C_M}(1,t) \leq K_1$ for all $t \in T$. Therefore,

$$d_{C_M}(id,\gamma) \leq K_1 n = K_1\left(\left\lfloor \frac{d(x_0,\gamma(x_0)}{R}\right\rfloor + 1\right) \leq \frac{K_1}{R} d(x_0,\gamma(x_0)) + K_1$$

so

$$\frac{R}{K_1} d_{C_M}(id,\gamma) - K_1 \leq d(x_0,\gamma(x_0))$$

and, since τ is Γ-equivariant,

$$\frac{R}{K_1} d_{C_M}(\alpha,\beta) - K_1 \leq d(\alpha(x_0),\beta(x_0))$$

for all $\alpha, \beta \in \Gamma$.

Finally, notice that, every point in $CH(\Lambda(\Gamma)$ lies within R of $\hat{\tau}_\rho(C_M)$. Therefore, $\hat{\tau}_\rho$ is a $\left(\max\{A, \frac{K_1}{R}\}, \max\{R, K_1\}\right)$-quasi-isometry. $\square$

Since the inclusion map of any convex subset of $\mathbb{H}^3$ into $\mathbb{H}^3$ is an isometric embedding, we obtain the following immediate consequence.

Corollary 2.4. *f $\rho : \pi_1(M) \to \mathsf{PSL}(2,\mathbb{C})$ is a convex cocompact representation, $x_0 \in CH(\Lambda(\Gamma))$ and S is a finite generating set for $\pi_1(M)$, then $\hat{\tau}_\rho : C_M \to \mathbb{H}^3$ is a quasi-isometric embedding.*

2.3. **The fellow traveller property**

We will say that a proper length space X is (Gromov) **δ-hyperbolic** if whenever T is a geodesic triangle in X with sides s_1, s_2 and s_3 and $y \in s_1$, then $d(y, s_2 \cup s_3) \leq \delta$. If X is δ-hyperbolic for some δ, we often simply say that it is **Gromov hyperbolic** or simply **hyperbolic**.

The simplest examples of Gromov hyperbolic spaces are trees, which are 0-hyperbolic. The name is motivated, in part, by the observation that $\mathbb{H}^3$ is hyperbolic.

Lemma 2.5. *Hyperbolic space $\mathbb{H}^3$ is $\cosh^{-1}(2)$-hyperbolic for any n.*

Proof. Let T be a geodesic triangle in $\mathbb{H}^3$ with sides s_1, s_2 and s_3. Since any three points in $\mathbb{H}^3$ are contained in a totally geodesic, isometrically embedded copy of $\mathbb{H}^2$, we may assume that $n = 2$.

By the Gauss-Bonnet Theorem, T has area at most π. If $y \in s_1$ and $r = d(y, s_2 \cup s_3)$, then T contains a half-disk D of hyperbolic radius r. Since D has area $\pi \cosh r - \pi$, we see that

$$\pi \cosh r - \pi \leq \pi,$$

so $r \leq \cosh^{-1}(2) \approx 1.317$. $\square$

Remarks: 1) Actually, $\mathbb{H}^3$ is δ-hyperbolic for $\delta = \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 0.8814$.

2) A stronger notion of negative curvature is given by considering CAT(-1)-spaces. One says that a proper length space is CAT$(-k)$, for some $k \geq 0$, if every geodesic triangle is at least as thin as the triangle with the same lengths in a simply connected, complete Riemannian surface of curvature $-k$. The Comparision Theorem in Riemannian geometry implies that any simply connected Riemannian manifold with sectional curvature $\leq -k$ is CAT$(-k)$. The above lemma implies that CAT$(-k)$ spaces are $\cosh^{-1}(2)/k^2$-hyperbolic if $k > 0$.

The key property of Gromov hyperbolic spaces which we will need is the Fellow Traveller Property which tells us that quasi-geodesics remain a bounded distance from actual geodesics in a hyperbolic space. Notice that this is far from true in Euclidean geometry.

Theorem 2.6. (Fellow Traveller Property*) Given (K, C) and δ there exists R so that if X is δ-hyperbolic and $f : [a,b] \to X$ is a (K,C)-quasi-isometric embedding and L is a geodesic joining $f(a)$ to $f(b)$, then the Hausdorff distance between L and $f([a,b])$ is at most R.*

Suppose that C and D are closed subsets of a metric space Y . We say that the **Hausdorff distance** between C and D is at most R if both

1. $d(c, D) \leq R$ for all $c \in C$, and
2. $d(d, C) \leq R$ for all $d \in D$.

Alternatively, one can say that C lies in the (closed) metric neighborhood of radius R of D and D lies in the (closed) metric neighborhood of radius R of C. The Hausdorff distance is symmetric, satisfies the triangle inequality, and equals 0 if and only if $C = D$, but is not truly a distance, since two closed sets can fail to be a finite Hausdorff distance apart.

We sketch the proof in the case when f is a K-Lipschitz, (K, C)-quasi-isometric embedding. Notice that in this case f is rectifiable and its image has length at most $K|b-a|$. This situation contains all the key ideas of the general proof.

Sketch of Proof: The key observation is that it is "exponentially inefficient" for a path to wander far from the geodesic joining the endpoints. One manifestation of this principle is that if β is a path joining the endpoints of a geodesic of length $2A$ in $\mathbb{H}^3$ and lies entirely outside the ball of radius A about the midpoint x_0, then β has length at least $\pi \sinh A$ (which is the length of the shortest such path in the sphere of radius A about x_0).

We first bound how far any point on L can lie from $f([a, b])$. Choose a point $x_0 \in L$ which lies furthest from $f([a, b])$, i.e.

$$D = d\big(x_0, f([a, b])\big) = \sup\{d\big(x, f([a, b])\big) \mid x \in L\}.$$

Choose a point y on L so that y lies between $f(a)$ and x_0 and $d(y, x_0) = 2D$ (or $y = f(a)$ if $d(f(a), y) \leq 2D$). Choose $s \in [a, b]$ so that $d(f(s), y) \leq D$ (or $s = a$ if $y = f(a)$). Choose a point z on L which lies between x_0 and $f(b)$ and and $d(z, x_0) = 2D$ (or $z = f(b)$ (if $d(f(b), x_0) \leq 2D$.) Choose $t \in [a, b]$ so that $d(f(t), y) \leq D$ (or $t = b$ if $z = f(b)$). We then concatenate a geodesic joining y to $f(s)$, $f([s, t])$ and the geodesic joining $f(t)$ to z to produce a path γ joining y to z. Since $d(f(s), f(t)) \leq 6D$, $|s - t| < 6KD + KC$, and since f is K-lipschitz, $\ell(f([s, t])) \leq 6DK^2 + K^2C$, so

$$\ell(\gamma) \leq 6DK^2 + 2KD + K^2C.$$

Let $\hat{y}$ be the point between x_0 and y so that $d(x_0, \hat{y}) = D$ and let $\hat{z}$ between x_0 and z so that $d(x_0, \hat{z}) = D$, and form a path joining $\hat{y}$ to $\hat{z}$ by appending to γ segments in L joining y to $\hat{y}$ and joining z to $\hat{z}$. Then

$$\ell(\hat{\gamma}) \leq 6DK^2 + K^2C + 4D$$

and $\hat{\gamma}$ lies entirely outside of the ball of radius D about x_0. Therefore,

$$\ell(\hat{\gamma}) \geq \pi \sinh D$$

so

$$D \leq \sinh^{-1}\left(\frac{6DK^2 + KC + 4D}{\pi}\right) = D_0.$$

We now bound the distance from any point on $f([a, b])$ to L. Let $f([s, t])$ be maximal subsegment of $f([a, b])$ which stays outside of an open neighborhood of L of radius D_0. Notice that the subset of L consisting of points within D_0 of $f([a, s])$ is closed and the subset of L consisting of points within D_0 of $f([t, b])$ is closed. On the other hand their union is all of L, by the previous paragraph, so, since L is connected, their intersection is non-empty. So, there exists $r \in [a, s]$, $u \in [t, b]$ and $w \in L$ so that $d(w, f(r)) \leq D_0$ and $d(w, f(u)) \leq D_0$.

Since $d(f(r), f(u)) \leq 2D_0$, we see that $|r - u| \leq 2KD_0 + KC$ and, since f is K-lipschitz,

$$\ell(f([u, r])) \leq 2K^2D_0 + K^2C$$

so if $q \in [s,t] \subset [r,u]$, then

$$d(f(q),L) \le D_0 + K^2 D_0 + \frac{K^2 C}{2} = R.$$

Therefore, the Hausdorff distance between $f([a,b])$ and L is at most R. □

We say that a group is *Gromov hyperbolic* if its Cayley graph, with respect to some finite generating set, is a Gromov hyperbolic metric space. It is a consequence of the Fellow Traveler Property that if two spaces X and Y are quasi-isometric, then X is Gromov hyperbolic if and only if Y is Gromov hyperbolic. (EXERCISE) Since Cayley graphs of a fixed group, with respect to different finite generating sets are quasi-isometric, this notion is well-defined independent of the (finite) choice of generating set. So we obtain the following consequence:

Proposition 2.7. *If $\rho : \pi_1(M) \to \mathsf{PSL}(2,\mathbb{C})$ is a convex cocompact representation, then $\pi_1(M)$ is Gromov hyperbolic.*

2.4. Alternative characterizations

We see that a representation is convex cocompact if and only if its orbit map is a quasi-isometric embedding.

Proposition 2.8. *A discrete faithful representation $\rho : \pi_1(M) \to \mathsf{PSL}(2,\mathbb{C})$ is convex cocompact if and only if its extended orbit map $\hat\tau_\rho : C_M \to \mathbb{H}^3$ is a quasi-isometric embedding.*

Proof. We have already established the forward direction.

So, suppose $\hat\tau_\rho$ is a quasi-isometric embedding into $\mathbb{H}^3$. Recall that $\mathrm{CH}(\Lambda(\rho))$, is the union of all ideal tetrahedra in $\mathbb{H}^3$ with endpoints in the limit set. We may assume that x_0 has been chosen to lie in $CH(\Lambda(\Gamma))$, which implies that $\hat\tau_\rho(C_M) \subset CH(\Lambda(\rho)$.

Suppose that $\hat\tau_\rho$ is a (K,C)-quasi-isometric embedding. The Fellow Traveller Property implies that there exists $R = R(K,C,\cosh^{-1}(2))$ so that if $[a,b]]$ is a geodesic segment in C_M, then $\hat\tau_\rho([a.b])$ is a Hausdorff distance at most R apart from the geodesic $\overline{\hat\tau_\rho(a))\hat\tau_\rho(b)}$ joining $\hat\tau_\rho(a)$ to $\hat\tau_\rho(b)$.

If $z \neq w \in \Lambda(\rho)$, then there exists $\{\gamma_n\}$ and $\{\beta_n\}$ in $\pi_1(M)$ so that $\tau_\rho(\gamma_n) \to z$ and $\tau_\rho(\beta_n) \to w$. Then $\overline{\tau_\rho(\gamma_n)\tau_\rho(\beta_n)}$ lies in the (closed) neighborhood $\mathcal{N}_R(\hat\tau_\rho(C_M)$ of $\hat\tau_\rho(C_M)$ of radius R, for all n. Since $\overline{\tau_\rho(\gamma_n)\tau_\rho(\beta_n)} \to \overline{zw}$ we see that $\overline{zw} \subset \mathcal{N}_R(\hat\tau_\rho(C_\Gamma))$.

There exists B so that if T is an ideal tetrahedra in $\mathbb{H}^s$, then every point in T lies within B of an edge of T (EXERCISE) Therefore, every point in $\mathcal{C}H(\Lambda(\rho))$ lies with $R+B$ of a point in $\hat\tau_\rho(C_M)$. It follows that if D is the diameter of the bouquet of circles $\hat\tau_\rho(C_M)/\rho(\pi_1(M)) \subset N_\rho$, then $C(N_\rho)$ has diameter at most $R+B+D$. Since $C(N_\rho)$ is a closed subset of the complete Riemannian manifold N_ρ, this implies that $C(N_\rho)$ is compact, and hence that ρ is convex cocompact. □

Beardon and Maskit [4] showed that a Kleinian group is convex cocompact if and only every point in the limit set is a conical limit point. We say that $z \in \Lambda(\rho)$ is a *conical limit point* if whenever $\overrightarrow{xz}$ is a geodesic ray ending at z, there exists R and a sequence $\{\gamma_n\} \subset \pi_1(M)$ so that $\gamma_n(x) \to z$ and $d(\gamma_n(x), \overrightarrow{xz}) \le R$ for all n.

Medium Exercise: Prove that if ρ is convex cocompact, then every limit point is conical.

Hard Exercise: Prove that if every point in $\Lambda(\rho)$ is conical, then ρ is convex cocompact. Medium

Marden [22] showed that a Kleinian group is convex cocompact if and only if it has a finite-sided convex fundamental domain and contains no parabolic elements. (EXERCISE)

Remark: In one of my first conversations with Bill Thurston, before he became my advisor, I tried to explain how to prove some fact using fundamental domains. He immediately told me that was the wrong way to think about hyperbolic manifolds.

2.5. Stability

Let

$$\widetilde{CC}(M) \subset \mathrm{Hom}(\pi_1(M), \mathsf{PSL}(2,\mathbb{C}))$$

be the set of convex cocompact representations and let $CC(M)$ be its image in the quotient character variety

$$X(M) = \mathrm{Hom}(\pi_1(M), \mathsf{PSL}(2,\mathbb{C})//\mathsf{PSL}(2,\mathbb{C}).$$

(The double back-slash indicates that we are taking the geometric invariant theory quotient which gives $X(M)$ the structure of the variety. We will not need to worry about this construction but there is an open neighborhood of the set of discrete faithful representation on which the quotient is simply the usual quotient and the image of the neighborhood lies in the smooth part of the character variety.)

It is a crucial property of convex cocompact representations, known as stability, that $\widetilde{CC}(M)$ is open in $\mathrm{Hom}(\Gamma, \mathsf{PSL}(2,\mathbb{C}))$. This was first established by Marden [22]. Informally, if you wiggle a convex cocompact representation a little bit it remains convex cocompact.

Theorem 2.9. *If $\rho : \pi_1(M) \to \mathsf{PSL}(2,\mathbb{C})$ is convex cocompact, then there exists a neighborhood U of ρ in $\mathrm{Hom}(\pi_1(M), \mathsf{PSL}(2,\mathbb{C}))$ such that if $\sigma \in U$, then σ is convex cocompact.*

Theorem 2.9 was first establshed by Marden [22, Theorem 10.1]. Thurston [31, Proposition 8.3.3] observed that it followed from argument of the form due to Weyl, see also Canary-Epstein-Green [10, Section I.2.5].

Proof. Suppose that the orbit map $\hat{\tau}_\rho : C_M \to \mathbb{H}^3$ is a (K,C)-quasi-isometric embedding, where C_M is constructed from a finite generating set S for $\pi_1(M)$. The local-to-global principle, see Theorem 2.13, implies that there exists A, $\hat{K}$, and $\hat{C}$ so that if $f : J \to \mathbb{H}^3$ (where J is an interval in $\mathbb{R}$) is a $(K+C+2, C+2)$-quasi-isometry on all segments of length at most A, then f is a $(\hat{K},\hat{C})$-quasi-isometry.

Let U be an open neighborhood of ρ in $\mathrm{Hom}(\pi_1(M), \mathsf{PSL}(2,))$ so that if $\sigma \in U$, $\gamma \in \pi_1(M)$ and $d_{C_M}(1,\gamma) \le A+1$, then $d(\rho(\gamma(x_0)), \sigma(\gamma)(x_0)) < 1$. (We may do so since there are only finitely many elements of γ within $A+1$ of id.)

If $\sigma \in U$, let τ_σ be the orbit map of σ. We see that if $d_{C_M}(1,\gamma) \le A+1$, then

$$\frac{1}{K} d_{C_M}(id,\gamma) - C - 1 \le d(\tau_\sigma(id), \tau_\sigma(\gamma)) \le K d_{C_M}(id,\gamma) + C + 1.$$

Therefore, the extended orbit map $\hat{\tau}_\sigma$ is a $(K+C+2, C+2)$-quasi-isometry on all geodesic segments in C_M of length at most $A+1$ emanating from the origin. However, every segment of length A in C_M may be translated by an element of $\pi_1(M)$ to a subsegment of a geodesic segment in C_M of length at most $A+1$ emanating from the origin. Since $\hat{\tau}_\sigma$ is σ-equivariant, $\hat{\tau}_\sigma$ is a $(K+2, C+2)$-quasi-isometric embedding on all geodesic segments in C_M of length at most

A. Therefore, $\hat{\tau}_\sigma$ is a $(\hat{K},\hat{C})$-quasi-isometric embedding on all geodesic segments in C_M, which implies that $\hat{\tau}_\sigma$ is a $(\hat{K},\hat{C})$-quasi-isometric embedding. Therefore, σ is convex cocompact. □

Since the set $CC(M)$ is invariant under conjugation, we immediately see that both $\widetilde{CC}(M)$ and its quotient $CC(M)$ are open.

Corollary 2.10. *$\widetilde{CC}(M)$ is open in $\mathrm{Hom}(\pi_1(M),\mathsf{PSL}(2,\mathbb{C}))$ and $CC(M)$ is open in $X(M)$.*

2.6. Extra for experts: the topology of $DF(\pi_1(M),\mathsf{PSL}(2,\mathbb{C}))$

Let

$$\mathrm{DF}(\pi_1(M),\mathsf{PSL}(2,\mathbb{C}))\subset \mathrm{Hom}(\pi_1(M),\mathsf{PSL}(2,\mathbb{C}))$$

be the set of discrete, almost faithful, representations and let $\mathrm{AH}(M)$ be its image in $X(M)$. The following is a standard consequence of the Margulis-Zassenhaus Lemma. (GET REFS)

Theorem 2.11. *With our assumptions on M, $\mathrm{DF}(\pi_1(M),\mathsf{PSL}(2,\mathbb{C}))$ is a closed subset of $\mathrm{Hom}(\pi_1(M),\mathsf{PSL}(2,\mathbb{C}))$ and $AH(M)$ is a closed subset of $X(M)$. Similarly, if Γ is any torsion-free finitely generated group, $\mathrm{DF}(\Gamma,\mathsf{PSL}(2,\mathbb{R}))$ is a closed subset of $\mathrm{Hom}(\Gamma,\mathsf{PSL}(2,\mathbb{R}))$.*

Since $\widehat{\mathcal{T}}(S)$ is also open and connected, this immediately implies that $\widehat{\mathcal{T}}(S)$ is a component of $DF(\pi_1(S),\mathsf{PSL}(2,\mathbb{R}))$.

The same logic would imply that if M is closed 3-manifold, then $AH(M)$ is a collection of components of $X(M)$. However, Mostow's rigidity theorem implies that if M is a closed 3-manifold and $AH(M)$ is non-empty, then it consists of exactly two points (one for each orientation on the unique hyperbolic 3-manifold homotopy equivalent to M.

In general, $AH(M)$ will not be open and $\widehat{CC}(M)$ will not be closed. We describe a simple instance of this phenomenon in dimension 2.

Proposition 2.12. *If F_2 is the free group with 2 generators, then $CC(F_2,\mathsf{PSL}(2,\mathbb{R}))$ is open, but not closed, in $\mathrm{Hom}(F_2,\mathsf{PSL}(2,\mathbb{R}))$. Moreover, $DF(F_2,\mathsf{PSL}(2,\mathbb{R}))$ is closed, but not open, in $\mathrm{Hom}(F_2,\mathsf{PSL}(2,\mathbb{R}))$.*

We first describe the classical Schottky construction of convex cocompact representations of free groups. If $\{C_1,C_2,\ldots,C_{2n-1},C_{2n}\}$ is a family of disjoint geodesics in $\mathbb{H}^2$ bounding disjoint (closed) half-spaces $\{D_1,D_2,\ldots,D_{2n-1},D_{2n}\}$ (whose closures are disjoint in $\mathbb{H}^2\cup\partial\mathbb{H}^2$), then we may construct a convex cocompact representation $\rho:F_n\to\mathsf{PSL}(2,\mathbb{R})$ by letting $\rho(a_i)$ be a Möbius transformation taking D_{2i-1} to $\mathbb{H}^2-\mathrm{int}(D_{2i})$ for all i, where $F_n=<a_1,\ldots,a_n>$. If $P=\mathbb{H}^2-\bigcup\mathrm{int}(D_i)$, then one may form a complete hyperbolic surface from P by identifying C_{2i-1} to C_{2i} by $\rho(a_i)$ for all i. Covering space theory then allows us to conclude that P is a fundamental domain for the action of $\rho(F_n)$ and that orbits of P tesselate $\mathbb{H}^2$. (One may also verify these facts, using the Ping Pong Lemma.)

If we choose $x_0\in\mathrm{int}(P)$ and let $\delta=\min\{d(C_i,C_j)\mid i\neq j\}$, then one may easily check that

$$d(x_0,\gamma(x_0))\geq \delta d(1,\gamma).$$

On the other hand, if $K=\max\{d(x_0,\rho(a_i)(x_0))\}$, then

$$d(x_0,\gamma(x_0))\leq K d(1,\gamma).$$

Therefore, τ_ρ is a quasi-isometric embedding, so ρ is convex cocompact. Notice that, in this case, one may easily see that all representations near to ρ are also convex cocompact, since wiggling the representation, just amounts to wiggling the C_i.

We now observe that not all discrete, faithful representations of F_2 are convex cocompact. Suppose that C_1 is the x-axis, C_2 is the line $Re(z) = 1$, C_3 is a semi-circle based at $1/4$ with radius $1/8$ and C_4 is a semi-circle based at $3/4$ with radius $1/8$. Let $\rho_0(a_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and let $\rho_0(a_2)$ be a Möbius transformation taking the half-space "below" C_3 to the half-space "above" C_4 and preserving the height of points on C_3. Let P be the (closure of the) region between C_1 and C_2 and above C_3 and C_4. Consider the hyperbolic surface X obtained from identifying C_1 with C_2 by $\rho_0(a_1)$ and identifying C_3 with C_4 by $\rho_0(a_2)$ and the sequence of regions X_n given by the quotient of $\{z \in P \,|e^{-n} \leq Im(z) \leq e^n\}$. Notice that X_n contains the ball of radius n about the quotient of $i + 1/2$ and exhausts X. It follows that X is complete. Covering space theory, then guarantees that $\rho_0 : F_2 \to \mathsf{PSL}(2, \mathbb{R})$ is discrete and faithful and that P is a fundamental domain for the action of $\rho_0(F_2)$ on $\mathbb{H}^2$. However, τ_ρ is not a quasi-isometric embedding, since if we choose $x_0 = i + 1/2$, then

$$d(x_0, \rho_0(a^n)(x_0)) = 2 \log \frac{n + \sqrt{n^2 + 4}}{2} \sim 2 \log n$$

so ρ_0 is not convex cocompact.

Notice that it is important to be careful in checking completeness. Suppose that we choose C_1, C_2, C_3 and C_4 as for ρ_0 but then let $\hat{\rho}_0(a_1)$ to be given by $z \mapsto \frac{1}{2}z + 1$ and $\hat{\rho}_0(a_2) = \rho_0(a_2)$. The region X_n is not preserved by the gluings $\hat{\rho}(a_i)$, the quotient of P is not complete and, in fact, $\hat{\rho}_0$ is convex cocompact. One may see that the quotient of P is not complete, by considering the path in the quotient which is the union of horizontal segments in P of height 2^n for all n. This path has finite length but leaves every compact subset of the quotient of P. Notice that the lines $\{\hat{\rho}(a_1^n)(C_0)\}$ accumulates at the line $Re(z) = \sum_{i=0}^n \frac{1}{2^n} = 2$, so the translates of P do not tesselate $\mathbb{H}^2$. A fundamental domain for the action of $\hat{\rho}(F_2)$ is given by looking at the region below the circle of radius 2 about $z = 2$ and above the circle of radius 1 about $z = 2$ and above C_3 and C_4. One may then use this picture, just as above, to show that $\hat{\rho}_0$ is convex cocompact.

We now observe that ρ is a limit of a sequence $\{\rho_n\}$ of representations whose image is not discrete and faithful. For all $n \geq 2$, let $\rho_n(a)$ be an element of $\mathsf{PSL}(n, \mathbb{R})$ which fixes $ni + \frac{1}{2}$ and takes i to $i + 1$ and let $\rho_n(b) = \rho(b)$. It is then easy to check that $\{\rho_n\}$ converges to ρ and that $\rho_n(F_2)$ is either indiscrete or not faithful (since either $\rho_n(a)$ has finite order, or $< \rho_n(a) >$ is indiscrete). Similarly, we choose $\hat{\rho}_n(a) \in \mathsf{PSL}(2, \mathbb{R})$ to take the interior of the circle R_{-n} of radius n about $-n$ to the exterior of the circle R_n of radius n about $n + 1$, so that the "height" (i.e. the imaginary component) of points on R_{-n} is preserved and let $\hat{\rho}_n(b) = \rho(b)$. Then $\hat{\rho}_n$ is convex cocompact for all n and $\lim \hat{\rho}_n = \rho$.

2.7. **Extra for Experts: the proof of the Local-to-global principle**

We will only use this fact for bi-infinite quasigeodesics in $\mathbb{H}^3$, where the proof is easier, but we state the general fact here. See Coornaert-Delzant-Papadopoulos [14, Thm. 3.1.4] for a complete proof.

Theorem 2.13. (Local to Global Principle) *Given $K \geq 1$, $C \geq 0$ and $\delta \geq 0$, there exists $\hat{K}$, $\hat{C}$ and A so that if J is an interval in $\mathbb{R}$, X is δ-hyperbolic and $h : J \to X$ is a (K, C)-quasi-isometric embedding restricted to every connected subsegment of J with length $\leq A$, then h is a $(\hat{K}, \hat{C})$-quasi-isometric embedding.*

We will give a sketch of the proof in the case when $X = \mathbb{H}^3$ and $J = \mathbb{R}$ which is based on an argument of Minsky [25]. (The assumption that $J = \mathbb{R}$ is simply for convenience, while the restriction to $X = \mathbb{H}^3$ significantly simplifies the proof).

Proof. We will make use of an elementary lemma in hyperbolic geometry.

Lemma 2.14. *Given $S > 0$, $T > 0$, there exists $B = B(S,T) > 0$ so that if P and Q are totally geodesic hyperplanes in $\mathbb{H}^3$, $p \in P$, $q \in Q$ and $x \in \mathbb{H}^3$, and $\overline{px}$ and $\overline{qx}$ are geodesic segments perpendicular to P and Q respectively, so that $d(p,x) \geq B$, $d(q,x) \geq B$, and $d(x, \overline{pq}) \leq S$, then $d(P,Q) \geq T$.*

(The idea of the proof of the lemma is that if B is large enough, then $\overline{px}$ and $\overline{pq}$ are nearly tangent, so $\overline{pq}$ is nearly orthogonal to P. Similarly, $\overline{pq}$ is nearly orthogonal to Q. So we choose B large enough that the angles between $\overline{pq}$ and both P and Q is at least .75. Notice that $\overline{pq}$ has length at least $2B - 2S$. Let $\mathcal{C} = \{(P,Q,L)\}$ be the set of triples where P and Q are geodesic hyperplanes which are joined by a geodesic segment L which makes angle at least .75 with each of P and Q and $d(P,Q) \leq T$. If we also assume $\overline{pq}$ passes through a fixed point, then $\mathcal{C}$ is a compact set of configurations. Therefore, there is an upper bound R on the length of L. So, if we also choose B large enough that $2B - 2S > R$, then our assumptions guarantee that $d(P,Q) \geq T$.)

Given $K \geq 1$ and $C \geq 0$, let $R = R(K, C, \cosh^{-1}(2))$ be the constant provided by the Fellow Traveller property and let $B = B(2R, 2R)$ be the constant provided by Lemma 2.14. Choose $A \geq 4K(B + C + R)$.

For all $i \in \mathbb{Z}$, let $t_i = \frac{iA}{2}$ and $y_i = h(t_i)$. Let $G_i = \overline{y_i y_{i+1}}$ be the geodesic segment with vertices y_i and y_{i+1} and midpoint m_i. Notice that $d(y_i, y_{i+1}) \geq \frac{A}{2K} - C$, so

$$d(m_i, y_{i+1}) \geq \frac{A}{4K} - \frac{C}{2} \geq B.$$

Similarly, $d(m_{i+1}, y_i) \geq L$.

By the Fellow Traveller Property, there exists $s_i \in [t_i, t_{i+1}]$ such that $d(f(s_i), m_i) \leq R$. The Fellow traveller property, then implies that $d\left(y_i, \overline{h(s_i)h(s_{i+1})}\right) \leq R$. Choose $z_i \in \overline{h(s_i), h(s_{i+1})}$, so that $d(z_i, y_i) \leq R$. Since $\overline{m_i, m_{i+1}}$ are $\overline{h(s_i)h(s_{i+1})}$ are geodesics whose endpoints are a distance at most R apart, the convexity of the distance function implies that the Hausdorff distance between $\overline{m_i, m_{i+1}}$ are $\overline{h(s_i)h(s_{i+1})}$ is at most R. Therefore, $d(z_i, \overline{m_i m_{i+1}}) \leq R$, so $d(y_i, \overline{m_i m_{i+1}}) \leq 2R$. Lemma 2.14 then implies that

$$d(P_i, P_{i+1}) \geq 2R \qquad \text{for all } \ i \in \mathbb{Z}.$$

We next claim that P_{i-1} and P_{i+1} lie on opposite sides of P_i. If not, then y_{i-1} and y_{i+1} lie on the same side of P_i, so the geodesic segment $\overline{y_{i-1}y_{i+1}}$ lies on the opposite side of P_i from y_i, but

$$d(y_i, P_i) = d(y_i, m_i) \geq \frac{A}{4K} - C > 2R, \qquad \text{so} \qquad d\big(f(s_i), \overline{y_{i-1}, y_{i+1}}\big) > R$$

which would contradict the Fellow Traveller Property. It follows that P_{i-1} lies on the opposite side of P_i as P_{i+1}. Therefore, since $d(P_i, P_{i+1}) \geq 2R$ for all i and are ordered monotonically, we see that

$$d(y_m, y_n) \geq (|m - n| - 1)2R \qquad \text{for all } \ m, n \in \mathbb{Z},$$

If $a, b \in \mathbb{R}$, choose $m, n \in \mathbb{Z}$ so that $a \in [t_{m-1}, t_m]$ and $b \in [t_n, t_{n+1}]$. Then

$$|a - t_m| < \frac{A}{2}, \quad |b - t_n| < \frac{A}{2}, \quad d(f(a), f(t_m)) \leq \frac{KA}{2} + C \quad d(f(b), f(t_n)) \leq \frac{KA}{2} + C$$

so

$$\begin{aligned} d(f(a), f(b)) &\geq 2R(|m-n| - 1) - KA - 2C \\ &= \frac{4R}{A}|t_m - t_n| - 2R - KA - 2C \\ &\geq \frac{4R}{A}|b - a| - 6R - KA - 2C \end{aligned}$$

Since, $|t_i - t_{i+1}| = \frac{A}{2}$, $d(f(t_i), f(t_{i+1})) \leq \frac{KA}{2} + C$, so we see that

$$d(f(t_m), f(t_n)) \leq |m - n| \left(\frac{KA}{2} + C \right).$$

Therefore,

$$\begin{aligned} d(f(a), f(b)) &\leq |m - n| \left(\frac{KA}{2} + C \right) + KA + 2C \\ &= \frac{2}{A}|t_m - t_n| \left(\frac{KA}{2} + C \right) + KA + 2C \\ &\leq |b - a| \left(K + \frac{2C}{A} \right) + A \left(K + \frac{2C}{A} \right) + KA + 2C \end{aligned}$$

We conclude that f is a $(\hat{K}, \hat{C})$-quasi-isometry where

$$\hat{K} = \max \left\{ \frac{A}{R}, K + \frac{2C}{A} \right\} \quad \text{and} \quad \hat{C} = A \left(K + \frac{2C}{A} \right) + 6R + KA + 2C.$$

□

3. **Quasifuchsian groups**

If S is a closed oriented surface, we consider *quasifuchsian space*

$$QF(S) = CC(S \times [0, 1]).$$

Representations in $QF(S)$ are called quasifuchsian representations and their images are called quasifiuchsian groups.

If $\rho \in QF(S)$, then $C_1(N_\rho)$ is a closed, irreducible, orientable 3-manifold homotopy equivalent to S. It follows from basic results on the topology of 3-manifolds (REF) that $C_1(N_\rho)$ is homeomorphic to $S \times [0, 1]$, and hence that $\widehat{N}_\rho$ is homeomorphic to $S \times [0, 1]$ and N_ρ is homeomorphic to $S \times (0, 1)$. Moreover, $h_\rho : S \times [0, 1] \to \widehat{N}_\rho$ is homotopic to a homeomorphism. Since $S \times [0, 1]$ admits an orientation-reversing homeomorphism, we may assume that h_ρ is orientation-preserving. Therefore, we get a map

$$B : QF(S) \to \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \quad \text{given by} \quad B(\rho) = (\partial_c N_\rho, h_\rho)$$

where $\bar{S}$ is S with the opposite orientation. The main goal of this section is to show that B is a homeomorphism.

3.1. **Quasiconformal maps and Beltrami differentials**

We begin with a brief survey of the theory of quasiconformal maps (without proof). Roughly, quasiconformal maps are orientation preserving homeomorphisms which distort the conformal structure by a bounded amount. One may view them as the conformal analogue of bilipschitz maps, which distort the metric structure a bounded amount. Good references for the theory of quasiconformal maps are the books of Lehto-Virtanen [21] and Lehto [20]. This section is plagiarized from the research monograph of Canary-McCullough [12].

Given a function $f\colon D \to \overline{\mathbb{C}}$ defined on a domain D in $\overline{\mathbb{C}}$, we may write it as $f(x,y) = u(x,y) + iv(x,y)$. We say f is *ACL* (absolutely continuous on lines) if given any rectangle $R = [a,b] \times [c,d]$ in D both u and v are absolutely continuous restricted to almost every vertical and almost every horizontal line segment in R. If f is ACL then the partial derivatives of u and v exist almost everywhere and we define $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$. Then, we let $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar z} = \frac{1}{2}(f_x + if_y)$. (Recall that the Cauchy-Riemann equations assert that if f is analytic then $f_{\bar z} = 0$ for all $z \in D$.) We define the *Beltrami differential* of f to be $\mu_f = \dfrac{f_{\bar z}}{f_z}$ Notice that if f is differentiable at a point z and $Jf(z)$ is its Jacobian, then the image of the unit circle (in the tangent space $T_z(D)$) under $Jf(z)$ is an ellipse, the ratio of the lengths of the axes is given by $K(z) = \dfrac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}$ and the angle that the preimage of the (longer) axis makes with the x-axis is $\frac{1}{2}\arg(\mu_f(z))$.

One says that an orientation-preserving homeomorphism $f\colon D \to D'$ is K*-quasiconformal* if f is ACL and $|\mu_f| \le \frac{K-1}{K+1}$ almost everywhere. This says that, typically, very small circles are taken to curves very much like ellipses with eccentricity at most K. One way of formalizing this is by defining

$$H(z) = \limsup_{r\to 0} \frac{\max_\theta |f(z + re^{i\theta}) - f(z)|}{\min_\theta |f(z + re^{i\theta}) - f(z)|} \, .$$

An orientation-preserving homeomorphism $f\colon D \to \mathbb{C} \cup \{\infty\}$ is K-quasiconformal if and only if H is bounded on $D - \{\infty, f^{-1}(\infty)\}$ and $H(z) \le K$ almost everywhere in D (see pages 177 and 178 in Lehto [20]). If one uses the spherical metric on $\overline{\mathbb{C}}$, then one need not exclude ∞ and $f^{-1}(\infty)$ from consideration.

One may check that the composition of a K_1-quasiconformal map and a K_2-quasiconformal map is a K_1K_2-quasiconformal map. Another useful fact is:

Proposition 3.1. (Lehto-Virtanen [21, Thm. 1.5.1]) *A quasiconformal map is conformal if and only if it is* 1*-quasiconformal.*

The most fundamental result concerning quasiconformal maps is the Measurable Riemann Mapping Theorem (see Ahlfors-Bers [3] or Lehto [20]) which asserts that the Beltrami differential determines the quasiconformal map (up to normalization) and that every Beltrami differential (of norm less than 1) determines a quasiconformal map.

Measurable Riemann Mapping Theorem: *Suppose that* $\mu \in L_\infty(\overline{\mathbb{C}}, \mathbb{C})$ *and* $||\mu||_\infty < 1$. *Then there exists a unique quasiconformal map* $\phi_\mu\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *whose Beltrami differential is* μ *and such that* ϕ_μ *fixes* 0, 1, *and* ∞. *Moreover,* ϕ_μ *depends analytically on* μ.

Notice that one may combine the Measurable Riemann Mapping Theorem and the traditional Riemann Mapping Theorem to observe that the same result holds for the upper half-plane $\mathbb{H}^2$. This version of the result is used in traditional Teichmüller theory and also plays a role in our proof of the Quasiconformal Parameterization Theorem.

Measurable Riemann Mapping Theorem (Disk version): *Suppose that $\mu \in L_\infty(\mathbb{H}^2, \mathbb{C})$ and $||\mu_\infty|| < 1$. Then there exists a unique quasiconformal map $\phi_\mu\colon \mathbb{H}^2 \to \mathbb{H}^2$ whose Beltrami differential is μ and such that ϕ_μ fixes i, $2i$, and $3i$. Moreover, ϕ_μ depends analytically on μ.*

An alternative characterization of quasiconformal mappings of $\overline{\mathbb{C}}$ is obtained by considering bilipschitz homeomorphisms of $\mathbb{H}^3$. The Fellow Traveller property may be easily used to show that any bilipschitz homeomorphism of $\mathbb{H}^3$ to itself extends continuously to a homeomorphism of $\partial\mathbb{H}^3$ to itself. (EXERCISE). One must work a little harder (although not too much harder) to show that this extension is quasiconformal. (I like how this is written up in Thurston's notes [31], but one may find this argument many places.) It is a deeper fact that any quasiconformal automorphism of $\partial\mathbb{H}^3$ extends to a bilipschitz map of $\mathbb{H}^3$. (One place to read an exposition is in Matsuzaki-Taniguchi [24, Thm 5.3.1].)

Proposition 3.2. *Let $\phi\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an orientation-preserving homeomorphism. Then ϕ is quasiconformal if and only if it extends to a homeomorphism $\Phi\colon \mathbb{H}^3 \cup \overline{\mathbb{C}} \to \mathbb{H}^3 \cup \overline{\mathbb{C}}$ whose restriction to $\mathbb{H}^3$ is bilipschitz (with respect to the hyperbolic metric).*

Notice that one can show that a bilipschitz map extends to a well-defined map of the boundary using only the Fellow Traveller Property. One must work a little harder (although not too much harder0 to show that this extension is quasiconformal. The fact that quasiconformal maps extend to bilipschitz maps is a deeper fact.

It will also be useful to know that quasiconformal homeomorphisms take sets of measure zero to sets of measure zero. (FIND REF)

Theorem 3.3. *If $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a quasiconformal map and $E \subset \mathbb{C}$ has measure zero, then $f(E)$ has measure zero.*

3.2. **The geometry of quasifuchsian hyperbolic 3-manifolds**

If H is a totally geodesic hyperplane in $\mathbb{H}^3$, then one may parameterize a component of $\mathbb{H}^3 \setminus H$ by $H \times (0, \infty)$ where the first coordinate associated to a point x is its nearest point retraction onto H and the second coordinate is its distance from H. In these coordinates, the metric takes the form $ds^2 = \cosh^2 t ds_H^2 + dt^2$, where ds_H^2 is the intrinsic hyperbolic metric on H and t is the real coordinate.

If ρ is convex cocompact, then we identified $N_\rho - C_1(N_\rho)$ with $C_1(N_\rho) \times (1, \infty)$. We may then show that in these coordinates $N_\rho - C_1(N_\rho)$ is bilipschitz to the metric

$$\cosh^2 t ds^2_{\partial C_1(N_\rho)} + dt^2$$

where $ds^2_{\partial C_1(N_\rho)}$ is the intrinsic metric on $\partial C_1(N_\rho)$ and t is the real coordinate. (The argument is relatively elementary, but tedious, so we will not give it. I promise not to assign it as an exercise if you promise not to ask me to do it.)

We can use this to show that any two quasifuchsian representations are quasiconformally conjugate.

Theorem 3.4. *If $\rho_1, \rho_2 \in QF(S)$, then there exists a quasiconformal map $\phi : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$, so that $\rho_2(\gamma) = \phi \circ \rho_2(\gamma) \circ \phi^{-1}$ for all $\gamma \in \Gamma$.*

Proof. Let $h_{\rho_i} : S \times [0,1] \to C_1(N_{\rho_i})$ be orientation-preserving homeomorphism in the homotopy class of ρ_i. Since $C_1(N_{\rho_i})$ is a compact C^1-manifold, we may assume that h_{ρ_i} is bilipschitz. So

$$h = h_{\rho_2} \circ h_{\rho_1}^{-1} : C_1(N_{\rho_1}) \to C_1(N_{\rho_2})$$

is a bilipschitz homeomorphism. We may then extend h radially in the coordinates to obtain an orientation-preserving bilipschitz homeomorphism

$$H : N_{\rho_1} \to N_{\rho_2}$$

where if $x \in N_{\rho_1} - C_1(N_{\rho_1})$ has coordinates (y,t), then $H(x)$ has coordinates $(h(y),t)$.

Then H lifts to an orientation-preserving bilipschitz homeomorphism $\tilde{H} : \mathbb{H}^3 \to \mathbb{H}^3$ which conjugates the action of $\rho_1(\pi_1(S))$ to the action of $\rho_2(\pi_1(S)$, so, perhaps after precomposition by an element of $\rho_1(\pi_1(S))$,

$$\tilde{H} \circ \rho_1(\gamma) = \rho_2(\gamma) \circ \tilde{H}$$

for all $\gamma \in \pi_1(S)$. Then $\tilde{H}$ extends to a quasiconformal homeomorphism $\phi : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ so that

$$\phi \circ \rho_1(\gamma) = \rho_2(\gamma) \circ \phi$$

for all $\gamma \in \pi_1(S)$. □

If ρ_1 is Fuchsian, then its limit set has measure zero, which implies that $\Lambda(\rho_2) = \phi(\Lambda(\rho_1))$ also has measure zero.

Corollary 3.5. *If $\rho \in QF(S)$, then $\Lambda(\rho)$ has measure zero.*

3.3. Simultaneous uniformization

Bers [6] showed that B gives a complete parametrization of $QF(S)$.

Theorem 3.6. *The map*

$$B : QF(S) \to \mathcal{T}(S) \times \mathcal{T}(\bar{S})$$

is a homeomorphism.

Proof. We first prove that B is injective. If $B(\rho_1) = B(\rho_2)$, then there exists a conformal map $\psi : \partial_c N_{\rho_1} \to \partial_c N_{\rho_2}$ in the homotopy class of $h_{\rho_2} \circ h_{\rho_1}^{-1}$. This map lifts to a conformal homeomorphism $\tilde{\psi} : \Omega(\rho_1) \to \Omega(\rho_2)$ so that $\rho_2(\gamma) = \tilde{\phi} \circ \rho_1(\gamma) \circ \tilde{\phi}^{-1}$ for all $\gamma \in \pi_1(S)$.

We may use the radial coordinates on $\widehat{N}_{\rho_i} - C_1(N_{\rho_i})$ to extend ψ to a differentiable bilipschitz homeomorphism Ψ from $N_{\rho_1} \setminus \mathrm{int}C_1(N_{\rho_1})$ to $N_{\rho_2} \setminus \mathrm{int}C_1(N_{\rho_2})$. Since $C_1(N_{\rho_1})$ and $C_1(N_{\rho_2})$ are homeomorphic, we may then extend Ψ to a differentiable bilipschitz diffeomorphism from N_{ρ_1} to N_{ρ_2}. The map Ψ lifts to a bilipschitz map from $\mathbb{H}^3$ to $\mathbb{H}^3$ which admits a continuous extension to a quasiconformal homeomorphism C of $\partial\mathbb{H}^3$. By construction, C agrees with $\tilde{\psi}$ on $\Omega(\rho_1)$. (Bers uses a more analytic argument to construct the extension C.)

Since $\Lambda(\rho_1)$ has measure zero and C is conformal on $\Omega(\rho_1)$, we conclude that C is conformal and that $\rho_2(\gamma) = C \circ \rho_1(\gamma) \circ C^{-1}$ for all $\gamma \in \pi_1(S)$. Therefore $\rho_1 = \rho_2 \in QF(S)$.

The fact that B is surjective is an application of the Measurable Riemann Mapping Theorem. Suppose that $(Y, \bar{Z}) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$. Let ρ_0 be a Fuchsian group so that $B(\rho_0) = (X, \bar{X})$. Let $\psi : X \cup \bar{X} \to Y \cup \bar{Z}$ be an orientation-preserving diffeomorphism (in the correct homotopy class). We may lift ψ to a map $\tilde{\psi}:\Omega(\rho_0)\to\mathbb{H}^2\cup\mathbb{H}^2$ and compute its Beltrami differential $\mu_{\tilde{\psi}}$. We may then extend $\mu_{\tilde{\psi}}$ to a Beltrami differential on $\partial\mathbb{H}^3$ by setting it equal to 0 on $\Lambda(\rho_0)$. The Measurable Riemann Mapping Theorem implies that there exists a quasiconformal map $\phi:\partial\mathbb{H}^3\to\partial\mathbb{H}^3$ with Beltrami differential μ.

By construction, if $\gamma\in\pi_1(S)$, then $\phi\circ\rho_0(\gamma)$ and ϕ have the same Beltrami differentials on $\Omega(\rho_0)$, since they differ by postcomposition by a conformal automorphism. Therefore, $\tilde{\psi}\circ\rho_0(\gamma)$ and $\tilde{\psi}$ have the same Beltrami differential on $\Omega(\rho_0)$ and hence on $\hat{C}$ (since $\Omega(\rho)$ has full measure). The uniqueness portion of the Measurable Riemann Mapping Theorem, that $\phi\circ\rho_0(\gamma)\circ\phi^{-1}$ is conformal on $\phi(\Omega_(\rho_0))$. Since $\phi(\Omega(\rho_0))$ is full measure and $\phi\rho_0(\gamma)\phi^{-1}$ is quasiconformal then implies that they differ by post-composition by a conformal automorphism, so $\phi\circ\rho_0(\gamma)\circ\phi^{-1}$ is conformal. Therefore, we obtain a faithful representation $\rho:\pi_1(S)\to\mathsf{PSL}(2,\mathbb{C})$ defined by $\rho(\gamma)=\phi\circ\rho_0(\gamma)\circ\phi^{-1}$ for all $\gamma\in\Gamma$. ρ is discrete since $\rho(\pi_1(S))$ acts properly discontinuously on $\phi(\Omega(\rho_0))$.

Then $\widehat{N}_\rho$ is an irreducible, orientable manifold homotopy equivalent to S. It has two boundary components which are homotopic, hence homologous, which implies that $\widehat{N}_\rho$ is compact, and hence that ρ is convex cocompact. Since ϕ and $\tilde{\psi}$ have the same Beltrami differential on $\Omega(\rho_0)$, we see that $\phi\circ\tilde{\psi}^{-1}$ is a conformal homeomorphism from $\mathbb{H}^2\cup\mathbb{H}^2$ to $\Omega(\rho)$ which descends to a conformal homeomorphism from $Y\cup\bar{Z}$ to $\partial_c N_\rho$. Therefore, $B(\rho)=(Y,\bar{Z})$. (If one prefers a more geometric argument, one may use work of Douady and Earle [15] to extend the quotient of ϕ to a homeomorphism of $\widehat{N}_{\rho_0}$ to $\widehat{N}_\rho$ which is bilipschitz on N_{ρ_0}.) Therefore, $\rho\in QF(S)$ and $B(\rho)=(Y,\bar{Z})$.) We have now shown that B is surjective which completes the proof.

□

An excellent, analytically oriented, survey of the quasiconformal deformation theory of Kleinian groups is given in a paper of Bers [8]. A full treatment from a more topological viewpoint is given in Canary-McCullough [12].

3.4. **Extra for experts: limit sets of convex cocompact Kleinian groups**

4. **Laminations and pleated surfaces**

4.1. **Geodesic laminations**

A *geodesic lamination* on a hyperbolic surface X is a closed set which is a disjoint union of simple complete geodesics, i.e. geodesics in the disjoint union are either simple closed geodesics or bi-infinite simple geodesics. The simplest examples are disjoint unions of simple closed geodesics. We will primarily be interested in maximal finite-leaved geodesics. In fact every geodesic lamination is a (Gromov-Hausdorff) limit of a sequence of a sequence of finite-leaved geodesic laminations.

We now explain how to obtain a maximal finite-leaved geodesic lamination whose complement is a finite collection of ideal triangles. We begin with the case of a pair of pants with geodesic boundary. Consider the three common perpendicular geodesic segments which join pairs of sides. These geodesic segments decompose the pair of pants into two all-right angled hexagons. One can spin the vertices along the geodesic, reducing the angle and lengthening each segment. If one takes a limit as one spins a larger and larger amount, each geodesic segment will converge to a bi-infinite geodesic each of whose ends spirals about one of the closed geodesics. The complement of the three leaves is a union of two ideal triangles.

If one prefers one may add more edges (but not vertices) to obtain a triangulation of the pair of pants. If one spins this larger configuration each of the added edges converges to one of the limits of the original three edges.

Given a geodesic pants decomposition of a hyperbolic surface, one may perform the same operation on each pair of pants to obtain a finite leaved lamination contains the original pants decomposition suc h that each component of its complement is an ideal triangle. More generally, given a collection C of disjoint simple closed geodesics on the surface, one may place one vertex on each component of C and complete C to a triangulation T. If we spin T about C we obtain a finite-leaved geodesic lamination whose closed geodesics are exactly C and so that all other geodesics are bi-infinite geodesics each of whose ends spiral about a component of C. All maximal finite-leaved laminations may be obtained in this manner.

It is much harder to draw a picture of a "typical" geodesic lamination has uncountably many leaves and the intersection with a short goedesic segment transverse to the lamination is typically a Cantor set.

If $X = \mathbb{H}^2/\Gamma_X$, then a geodesic lamination λ lifts to a Γ_X-invariant geodesic lamination $\tilde{\lambda}$. Notice that the collection C_λ of pairs of points in $\partial\mathbb{H}^2\times\partial\mathbb{H}^2$ which arise as endpoints of geodesics in $\tilde{\lambda}$. If $h : X \to Y$ is a homeomorphism and $Y = \mathbb{H}^2/\Gamma_Y$, then h lifts to a homeomorphism $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ conjugating the action of Γ_X to Γ_Y. $\tilde{h}$ extends to a homeomorphism $\partial h : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ conjugating the action of Γ_X to Γ_Y. One way then transport λ on X to a geodesic lamination on Y, we simply look at the geodesic lamination determined by the pairs of points given by $(\partial H \times \partial h)(C_\lambda)$. The set $(\partial h \times \partial h)(C_\lambda)$ gives rise to a Γ_Y-invariant geodesic lamination on $\mathbb{H}^2$, which descends to a lamination on Y. Therefore, it makes sense to talk about the space$GL(S)$ of geodesic laminations on S.

More complete discussion of geodesic laminations is contained in the notes of Canary-Epstein-Green [10] or the book of Casson-Bleiler [13].

4.2. **Pleated surface**

A pleated surface in a quasifuchsian hyperbolic manifold is a pair (X, f) where X is a hyperbolic surface, N is a quasifuchsian hyperbolic manifold and $f : X \to N$ is a homotopy equivalence, so that

(1) f is a pathwise isometry, i.e. if α is a rectifiable path in X then $f(\alpha)$ is a rectifiable path in N of the same length at α.
(2) If $x \in X$, then some geodesic segment with x in its interior maps to a geodesic segment in N.

We define the pleating locus of (X, f) to be the set of points on X so that there any two geodesics with x in their interior lie inside a common geodesic segment through x.

It is an EXERCISE to show the the pleating locus of (X, f) is always a geodesic lamination and that f is a totally geodesic map on the complement of the pleating locus. (We will not need to do this exercise, since the pleated surfaces we construct will have these properties by construction.

The most fundamental example of a pleated surface is the boundary of the convex core. This makes intuitive sense, but is non-trivial to check. We won't need it, so we will not attempt a proof (although we may wave our hands). See Epstein-Marden [16] for a carful proof.

Let $h : S \to N$ be a homotopy equivalence and let λ be a maximal finite-leaved geodesic lamination on S. We will construct a (probably new) hyperbolic structure on X and pleated surface (Y, f) whose pleating locus agrees with λ (after accounting for the change in hyperbolic structure). One may assume that λ is obtained by starting wlth a collection C of closed geodesics, extending to a triangulation T and then spinning T about C. First, alter h so that it maps each component c of C to the closed geodesic in the homotopy class of $h(c)$. Then if e is an edge of T (which does not lie in a component of C), then alter h so that it maps e to the geodesic segment homopic $h(e)$ rel endpoints. Then if F is a face of T, its edges map to a geodesic triangle in N, so we may alter h to be totally geodesic on F. The resulting map $\hat{h} : S \to N$ is called a simplicial hyperbolic surface, since its pull-back metric is hyperbolic except along the vertices of C. Moreover, the total angle of the singular hyperbolic structure at any vertex is at least 2π.

If one spins T about C in X and simultaneously spin $\hat{h}(T)$ about $\hat{h}(C)$, the limiting map will be a pleated surface $h_\infty : Y \to N_\rho$, where Y is the limit of the singular hyperbolic structures on the spun surfaces, and the pleating lamination is contained in λ. (If one wants to think about this more abstractly, then we could have just considered the limit map $L_\rho : \partial_\infty \pi_1(S) \to \Lambda(\rho)$ which is a ρ-equivariant homeomorphism. Here $\partial_\infty \pi_1(S)$ is the Gromov boundary of the hyperbolic group $\pi_1(S)$. Then the pleating locus of $\tilde{h}_\infty$ is the set of geodesic spanned by pairs of points in $L_\rho(C_\lambda)$, one then fills in all the ideal triangles in $S - \lambda$ by totally geodesic ideal triangles in $\mathbb{H}^3$. The construction is equvariant so descends to the pleated surface h_∞.)

4.3. **Uniform injectivity**

Thurston proved a uniform injectivity theorem for pleated surfaces which will be crucial in the next section. The proof simply involves studying limits of pleated surfaces, so is a typical Thurstonian compactness argument.

If $\rho \in QF(S)$ and $f : X \to N_\rho$ is a pleated surface in the homotopy class of ρ, then one obtains a map

$$P_f : \lambda \to \mathbb{P}(T^1 N_\rho)$$

where $\mathbb{P}(T^1 N_\rho)$ is the projective unit tangent bundle. Here, $P_f(x)$ is the direction of the pleating locus through $f(x)$.

Uniform Injectivity Theorem: (Thurston [30]) *Given $\epsilon_0 > 0$ and S, if $\rho \in QF(S)$ and $f : X \to N_\rho$ is a pleated surface, then P_f is uniformly injective, i.e. for all $\epsilon > 0$, there exists $\delta > 0$ so that if $x, y \in \lambda$, $\mathrm{inj}_X(x) \geq \epsilon > 0$, $\mathrm{inj}_X(y) \geq \epsilon > 0$ and $d(x, y) \geq \epsilon$, then $d(P_f(x), P_f(y)) \geq \delta$.*

Roughly, if x and y are both in the thick part of X and are not nearby in X, then the pleating loci through $f(x)$ and $f(y)$ are not both nearby and nearly parallel.

4.4. **Alternation number**

If γ is a simple closed geodesic on X and λ is finite leaved lamination on X, then one can perturb γ to find a representative $\hat{\gamma}$ which consists of portions of λ and geodesic jumps of length at most ϵ_0. We denote by $a(\gamma, \lambda)$ the minimal number of jumps one needs here. (I will draw a picture on the board).

EXERCISE: There exists $C > 0$ so that length($\hat{\gamma}$) $\leq$ length(γ) $+ Ca(\gamma, \lambda)$.

The upshot is that we don't mind giving up a multiple of $a(\lambda, \gamma)$ it is always fine to work with $\hat{\gamma}$. One may imagine, and you would be correct, that this approximation will allow us to make use of the uniform injectivity theorem.

5. The double limit theorem and the geometrization of closed 3-manifolds which fibre over the circle

5.1. Efficiency of pleated surfaces

Thurston showed that pleated surfaces with finite-leaved pleating loci record lengths in the hyperbolic manifold up to a combinatorially bounded error.

Theorem 5.1. *Given ϵ_0 and S, there exists $C > 0$ so that if $\rho \in QF(S)$ and $f : X \to N_\rho$ is a pleated surface with finite-leaved pleating locus λ (so that X is homeomorphic to S and $f_* = \rho$) and no closed leaf of λ has length less than ϵ and α is a simple closed geodesic on X, then*

$$\ell_\rho(\alpha^*) \leq \ell_X(\alpha) \leq \ell_\rho(\alpha^*) + Ca(\gamma, \lambda)$$

where $\ell_\rho(\alpha^*)$ is the length in N_ρ of the closed geodesic α^* in N_ρ in the homotopy class of $f(\alpha)$.

Idea of proof: Let's assume for simplicity that every closed geodesic in N_ρ has length at least ϵ_0. Let $\hat{\alpha}$ be the piecewise geodesic approximation to α constructed in the last section. Then $f(\hat{\alpha})$ has $2a(\gamma, \lambda)$ geodesic segments. Consider the annulus A which is the domain of the homotopy from $f(\hat{\alpha})$ to α^*. We may triangulate A so it has vertices at (the preimage of) the intersection of the geodesic segments of $f(\alpha^*)$ and one vertex on (the pre-image of) α^*. We may pull this triangulation tight to obtain a simplicial hyperbolic surface $\hat{A}$ realizing the homotopy. The surface $\hat{A}$ has area at most $2\pi a(\gamma, \lambda)$.

Therefore, in $\hat{A}$ only a bounded portion of $f(\hat{\alpha})$ can't run parallel to either itself or α^*. This bound depends on the area of $\hat{A}$ which depends on $a(\gamma, \lambda)$. However, the uniform injectivity theorem says that different segments of $f(\hat{\alpha})$ can't run nearly parallel to each other. Therefore, all but a bounded portion of $f(\hat{\alpha})$ runs nearly parallel to α^*. This implies that the difference between the length of $f(\hat{\alpha})$ and α^* is bounded above by $Ca(\gamma, \lambda)$. But $\ell_X(\alpha) \leq \ell_X(\hat{\alpha})$ so we are done. □

So suppose that α and β *bind* S, i.e. all components of $S \setminus (\alpha \cup \beta)$ are simply connected. One first consequence of this result is that if you can bound the length of α and β uniformly over the sequence, then the sequence has a convergent subsequence.

Corollary 5.2. *Suppose that α and β bind S, $\{\rho_n\}$ is a sequence in $QF(S)$ and there exists K so that $\ell_{\rho_n}(\alpha^*) \leq K$ and $\ell_{\rho_n}(\beta^*) \leq K$ for all n. Then $\{\rho_n\}$ has a subsequence which converges in $AH(S)$.*

Proof. Let λ be a finite-leaved lamination of S. For simplicity we assume that there is a lower bound on the length of the simple closed curves in λ which holds in all N_{ρ_n}. (One can always arrange this by passing to a subsequence and choosing λ carefully.)

For all n, let $f_n : X_n \to N_{\rho_n}$ be a pleated surface with pleating locus (contained in) λ. Then

$$\ell_{X_n}(\alpha) \leq K + Ca(\alpha, \gamma) \quad \text{and} \quad \ell_{X_n}(\beta) \leq K + Ca(\beta, \gamma).$$

Since α and β bind the X_n all lie in a compact portion of $\mathcal{T}(S)$.

Therefore, if we choose $x_0 \in f_n(X_n)$ and lift x_0 to $\tilde{x}_0 \in \mathbb{H}^3$ (appropriately, there are conjugacy issues here) if $\gamma \in \pi_1(S)$, there is a uniform upper bound on $d(\tilde{x}_0, \rho_n(\gamma)(\tilde{x}_0))$, determined by its translation distance in X_n (which is bounded). Therefore, one can pass to a subsequence so that $\rho_n(\gamma)$ converges. Since $\pi_1(S)$ is finitely generated, we may pass to a subsequence so that ρ_n converges. □

5.2. **The conformal boundary and the internal geometry**

Bers [7] proved the following lemma using extremal length.

Lemma 5.3. *If $\rho \in QF(S)$ and α is any simple closed curve in S, then*

$$\ell_\rho(\alpha^*) \leq 2\ell_{\partial_c N_\rho}(\alpha).$$

There is a simple geometric proof of this fact, which only uses basic facts about the Poincaré metric, see [11, Lemma 2.1] but I won't have enough time to to give it. (I may try to add it later if you look for these notes in a few weeks).

Sullivan (see [16]) proved the following much deeper fact. (We are only stating a special case of his result.)

Theorem 5.4. *There exists $K > 1$ so that if $\rho \in QF(S)$, then $\partial r : \partial_c N_\rho \to \partial C(N_\rho)$ is homotopic to a K-quasiconformal map.*

Either of these results has the following immediate consequence.

Corollary 5.5. *Suppose that α and β bind S, $\{\rho_n\}$ is a sequence in $QF(S)$ and there exists K so that $\ell_{\partial_C N_\rho}(\alpha^*) \leq K$ and $\ell_{\partial_C N_\rho}(\beta^*) \leq K$ for all n. Then $\{\rho_n\}$ has a subsequence which converges in $AH(S)$.*

5.3. **Measured laminations and the Double Limit Theorem**

A *measured lamination* is a geodesic lamination which admits a transverse measure, i.e. a measure on arcs transverse to the lamination which is invariant under homotopies respecting the lamination. The only finite-leaved laminations which admit transverse measures are weighted collections of simple closed curves. The space $ML(S)$ of measured laminations is the closure of the set of weighted multicurves and is homeomorphic to $\mathbb{R}^{6g-6}$.

One may extend the length and alternation functions to continuous function

$$\ell : ML(S) \times AH(S) \to \mathbb{R} \quad \text{and} \quad a_\lambda : ML(S) \to \mathbb{R}$$

so that if γ is a simple closed curve with weight one, then $\ell(\rho, \gamma) = \ell_\rho(\gamma^*)$ and $a_\lambda(\gamma) = a(\gamma, \lambda)$. (The continuity of length is not as simple to prove as one might think, see Brock [**?**].)

Thurston showed how to compactify $\mathcal{T}(S)$ by $\mathbb{P}(ML(S))$. It has the following crucial property: if $X_n \to [\mu]$, there exists $\mu_n \in ML(S)$ so that $[\mu_n] \to [\mu]$, $\ell_{X_n}(mu_n)$ is bounded, but $\ell_X(\mu_n) \to \infty$ for any fixed $X \in \mathcal{T}(S)$.

We say that a pair of measured lamination $(\mu^+, .\mu^-)$ *bind* S if they do not share any leaves and every simple closed geodesic intersects one of them transversely. Putting this all together and taking limits, Thurston proves:

Double Limit Theorem: (Thurston [33]) *Suppose that $\{\rho_n\}$ is a sequence in $QF(S)$ and $B(\rho_n) = (Y_n, \bar{Z}_n) \to (\mu^+, \mu^-)$ and μ^+ and μ^- bind S. Then $\{\rho_n\}$ has a subsequence which converges in $AH(S)$.*

5.4. **Hyperbolization**

We say that $\psi : S \to S$ is *pseudo-Anosov* if there does not exists a finite collection C of (non-parallel) simple closed closed curves, so that $\psi(C)$ is homotopic to C. One easily sees that if ψ is not pseudo-Anosov, then its mapping torus M_ψ is not hyperbolizable. In theses cases, you must decompose iM_ψ into geometric pieces by the JSJ decomposition and geometrize each piece separately. Not all these geometric pieces will be hyperbolic, some of them will be Seifert-fibred instead.

Thurston proved that if ψ is pseudo-Anosov and $X \in \mathcal{T}(S)$, then $(\psi^n(X), \psi^{-n}(X)) \to (\mu_\psi^+, \mu_\psi^+)$. It then follows from the Double Limit Theorem that if $B(\rho_n) = (\psi^n(X), \psi^{-n}(X))$, then ρ_n has a subsequence converging to $\rho \in AH(S)$.

There exists a uniformly quasiconformal map $\phi_n : \partial_c N_{\rho_n} \to \partial_c N_{\rho_n}$ such that $(\phi_n)_* = \psi_*$. Then ϕ_n lifts (and extends) to a uniformly quasiconformal map $\tilde{\phi}_n : \partial\mathbb{H}^3 \to \mathbb{H}^3$ which conjugates $\rho_n(\pi_1(S))$ to itself and induces ψ_*, i.e. $\phi_n \circ \rho_n(\gamma) \circ \phi^{-1} = \rho_n(\psi_*(\gamma))$ for all $\gamma \in \pi_1(S)$.

Finally, one observes that up to subsequence $\phi_n \to \phi$ which is quasiconformal and conjugates $\rho(\pi_1(S))$ and induces ψ_*. One then checks that $\Lambda(\rho) = \partial\mathbb{H}^3$. Sullivan's Rigidity Theorem [29] then implies that ϕ is conformal. So $\Gamma = \langle \rho(\pi_1(S)), \phi \rangle$ is a discrete subgroup of $\mathsf{PSL}(2, \mathbb{C})$ (since the group of isometries of a hyperbolic 3-manifold always acts discretely on the manifold). So $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold homotopy equivalent M_ψ. Classic results in 3-manifold topology then imply that N is homeomorphic to M_ψ.

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