Math 523 Final Fall 2001 Solutions

(1) An insurance company has two classes of insureds:

Class	Prob. of Claim	Benefit	Number of Risks
Smoker	.55	1	150
Nonsmoker	.4	1	250

The company charges each smoker a relative security loading that is one and a half times the relative security loading charged to each nonsmoker. The company wants the total premium collected to exceed the 95th percentile of the aggregate claims distribution. Determine the smallest relative security loading for nonsmokers. (Use the fact that $P(Z \leq 1.645) = .95$ for a standard normal variable.) Give your answer as a percentage.

Solution: let S be the aggregate claim amount. Then

$E[S] = 250E[\text{claim of nonsmoker}]+150E[\text{claim of smoker}] = 250(.4)+150(.55) = 182.5$,

$Var(S) = 250Var[\text{nonsmoker}] + 150\text{Var}[\text{smoker}] = 250(.4)(1-.4)+150(.55)(1-.55) = 97.125$.

We assume

$$\frac{S - E[S]}{\sqrt{Var(S)}} = \frac{S - 182.5}{\sqrt{97.125}} = Z$$

is approximately standard normal. If P is the premium collected we want

$$0.95 = \text{Prob}(S < P) = \text{Prob}\left(\frac{S - 182.5}{\sqrt{97.125}} < \frac{P - 182.5}{\sqrt{97.125}}\right)$$

$$\simeq \text{Prob}\left(Z < \frac{P - 182.5}{\sqrt{97.125}}\right) ,$$

whence $(P - 182.5)/\sqrt{97.125} = 1.645$. Let θ be the relative security loading for a non-smoker. Then

$$P = (1 + \theta)250(.4) + (1 + 1.5\theta)150(.55) = 182.5 + 223.75\theta .$$

Hence $223.75\theta = 1.645\sqrt{97.125}$ imples $\theta = .0725$. Security loading is 7.25 percent.

(2) An aggregate claims variable S is compound Poisson with a severity variable X which is exponential with parameter $2/5$. The mean number of claims is 10. Find (a) $Var[S]$, (b) $E[e^{0.1S}]$.

Solution: (a)

$$E[N] = 10, \quad E[X] = 5/2, \quad Var[N] = 10, \quad Var[X] = (5/2)^2,$$

whence we have $E[S] = E[N]E[X] = 10(5/2) = 25$, and

$$Var[S] = E[N]Var[X] + Var[N]E[X]^2 = 10(5/2)^2 + 10(5/2)^2 = 125.$$

(b) Let $\Phi_S(t)$ be the moment generating function for S. Then,

$$\Phi_N(k) = \exp[10(e^k - 1)], \quad \Phi_X(t) = \frac{2/5}{2/5 - t} = \frac{2}{2 - 5t}.$$

Hence,

$$\Phi_S(t) = \Phi_N(\ln \Phi_X(t)) = \exp[10\{\Phi_X(t) - 1\}] = \exp\left[\frac{50t}{2 - 5t}\right] .$$

We conclude,

$$E[\exp(0.1S)] = \Phi_S(0.1) = \exp\left[\frac{5}{1.5}\right] = 28.03.$$

(3) An aggregate claims variable S is compound Poisson with severity variable X. The mean number of claims is 1.5. The severity is either 1 or 2, with $P(X = 1) = 0.6$. Find the pure premium for stop-loss insurance on the risk S with deductible $d = 3.5$.

Solution: We need to compute,

$$E[(S - 3.5)^+] = E[S - 3.5] + .5P(S = 3) + 1.5P(S = 2) + 2.5P(S = 1) + 3.5P(S = 0).$$

Now $E[S] = E[N]E[X] = 1.5E[X]$, and $E[X] = 1(.6) + 2(.4) = 1.4$. Thus $E[S - 3.5] = (1.5)(1.4) - 3.5 = -1.4$. To compute $P(S = m)$ we use the recurrence formula with $\lambda_1 = E[N]P(X = 1) = (1.5)(.6) = .9$ and $\lambda_2 = E[N]P(X = 2) = (1.5)(.4) = .6$. Then we have

$$P(S = 0) = P(N = 0) = e^{-1.5}, \quad P(S = 1) = \lambda_1 P(S = 0) = .9e^{-1.5}.$$

Also we have

$$P(S = 2) = \frac{1}{2}[\lambda_1 P(S = 1) + 2\lambda_2 P(S = 0)] = 1.005e^{-1.5},$$

$$P(S = 3) = \frac{1}{3}[\lambda_1 P(S = 2) + 2\lambda_2 P(S = 1)] = .6615e^{-1.5}.$$

We conclude

$$E[(S - 3.5)^+] = 7.59e^{-1.5} - 1.4 = .29.$$

(4) An insurance company has a full credibility standard of 1500 claims for the frequency of a Poisson claims process. The credibility standard has a confidence interval

of 5%. Suppose the severity variable is uniform on the interval [20, 100]. Find the confidence interval for pure premium of 2000 claims, using the same confidence level as in the full credibility standard for frequency.

Solution: We have $(y_p/.05)^2 = 1500$. We find the confidence interval for pure premium by solving

$$\left(\frac{y_p}{r}\right)^2 \left[1 + \frac{Var[X]}{E[X]^2}\right] = 2000.$$

Now $E[X] = 60, \ Var[X] = 80^2/12$. Hence

$$r = \left(\frac{3}{4}\right)^{1/2} (.05) \left[1 + \frac{Var[X]}{E[X]^2}\right]^{1/2} = .046.$$

(5) An auto insurance company divides its clients into two classes. It classifies 30% of its clients as Class I and the remainder Class II. It models the claim experience of its clients by a compound Poisson claims variable. For a Class I client the mean number of annual claims is 0.8. The severity is Gamma with parameters $r = 3, \ \lambda = 2$. For a Class II client the mean number of annual claims is 0.1. The severity is Gamma with parameters $r = 2, \ \lambda = 0.8$.
Suppose a client has made total claims of 2 in the last 5 years. Using Buhlmann credibility, estimate the client's claim in the next year.

Solution: Now $\Theta = I$ or $\Theta = II$ with $P(\Theta = I) = .3$. We also have

$$\mu(I) = (.8)(3/2) = 1.2, \quad \sigma^2(I) = (.8)(3/2^2) + (.8)(3/2)^2 = 2.4,$$

$$\mu(II) = (.1)(2/.8) = .25, \quad \sigma^2(II) = (.1)(2/.8^2) + (.1)(2/.8)^2 = .9375.$$

Hence

$$\mu = .3\mu(I) + .7\mu(II) = .535, \quad E[\sigma^2(\Theta)] = .3(2.4) + .7(.9375) = 1.376.$$

Further,

$$Var[\mu(\Theta)] = .3[1.2 - .535]^2 + .7[.25 - .535]^2 = .1895.$$

We conclude then that

$$k = \frac{E[\sigma^2(\Theta)]}{Var[\mu(\Theta)]} = \frac{1.376}{.1895} = 7.26, \quad Z = \frac{5}{5+k} = \frac{5}{12.26}.$$

The estimated claim in the next year is

$$\frac{2}{5}Z + (1-Z)\mu = \frac{2}{5}\frac{5}{12.26} + \frac{7.26}{12.26}(.535) = .48.$$

(6) Planes arrive at an airport at a Poisson rate of 12 per hour. You have just arrived at the airport and been told that 18 planes arrived in the past hour.
(a) Find the probability that 5 planes will arrive in the next 20 minutes.
(b) Find the probability that 5 planes arrived in the last 20 minutes.

Solution: Let $\mathcal{N}(t)$ be the number of planes that arrive up to time t with t in hours and $t = 0$ corresponding to one hour ago. Hence $\mathcal{N}(1) = 18$ and $E[\mathcal{N}(t)] = 12t$.
(a) We need to compute

$$P(\mathcal{N}(4/3) = 23|\mathcal{N}(1) = 18) = P(\mathcal{N}(4/3)-\mathcal{N}(1) = 5|\mathcal{N}(1) = 18) = P(\mathcal{N}(4/3)-\mathcal{N}(1) = 5).$$

Now $\mathcal{N}(4/3) - \mathcal{N}(1)$ is a Poisson variable with parameter $12(4/3 - 1) = 4$. Hence

$$P(\mathcal{N}(4/3) - \mathcal{N}(1) = 5) = \frac{4^5}{5!}e^{-4} = .156.$$

(b) We need to compute

$$P(\mathcal{N}(1)-\mathcal{N}(2/3) = 5 \mid \mathcal{N}(1) = 18) = P(\mathcal{N}(1)-\mathcal{N}(2/3) = 5,\ \mathcal{N}(1) = 18)/P(\mathcal{N}(1) = 18).$$

Now

$$P(\mathcal{N}(1) - \mathcal{N}(2/3) = 5,\ \mathcal{N}(1) = 18) = P(\mathcal{N}(1) - \mathcal{N}(2/3) = 5,\ \mathcal{N}(2/3) = 13),$$

and

$$P(\mathcal{N}(1) - \mathcal{N}(2/3) = 5,\ \mathcal{N}(2/3) = 13) = P(\mathcal{N}(1) - \mathcal{N}(2/3) = 5)\ P(\mathcal{N}(2/3) = 13).$$

Hence

$$P(\mathcal{N}(1)-\mathcal{N}(2/3) = 5 \mid \mathcal{N}(1) = 18) = P(\mathcal{N}(1)-\mathcal{N}(2/3) = 5)P(\mathcal{N}(2/3) = 13)/P(\mathcal{N}(1) = 18).$$

Since,

$$P(\mathcal{N}(1) = 18) = \frac{12^{18}}{18!}e^{-12},$$
$$P(\mathcal{N}(2/3) = 13) = \frac{8^{13}}{13!}e^{-8},$$
$$P(\mathcal{N}(1) - \mathcal{N}(2/3) = 5) = \frac{4^5}{5!}e^{-4},$$

we conclude that

$$P(\mathcal{N}(1) - \mathcal{N}(2/3) = 5 \mid \mathcal{N}(1) = 18) = .18.$$

(7) A new car dealer is commencing business. The dealer has neither assets nor liabilities. In exchange for continuous payments at the rate of expected sales, the car

dealer receives cars on demand from the manufacturer. Car sales occur in accordance with a compound Poisson process at the rate of 12 cars per month. The price of each car sold is either 20,000 or 30,000, with equal probability. The dealer is in a positive position when cumulative sales exceed cumulative payments to the car manufacturer.
Given that the dealer will attain a positive position, calculate the conditional expected value of the first positive position.

Solution: The severity variable X takes just the two values $X = 20,000$ or $X = 30,000$ with equal probability. Hence $E[X] = 25,000$. We need to compute $E[L_1]$. The pdf of L_1 is $P(X > w)/E[X]$, $w \geq 0$. Now $P(X > w) = 1$ if $0 < w < 20,000$, $P(X > w) = 1/2$ if $20,000 < w < 30,000$, and $P(X > w) = 0$ if $w > 30,000$. Thus,

$$E[L_1] = \int_0^{20,000} \frac{y}{25,000} dy + \int_{20,000}^{30,000} \frac{y}{50,000} dy = 13,000.$$

(8) A nervous investor has gotten into the habit of moving his assets from dollars into sterling and back. If the assets are in sterling he waits an exponential time with a mean of 10 days. He then tosses a coin. If the coin comes down heads he transfers his assets to dollars. Otherwise he lets his assets remain in sterling. If the assets are in dollars he waits an exponential time with a mean of 20 days. He then tosses the same coin. If the coin comes down heads he transfers his assets to sterling. Otherwise he lets his assets remain in dollars. Suppose that today he has his assets in dollars. Find the probability that his assets will be in sterling 15 days from now. Assume the probability of the coin coming down heads is 50%.

Solution: Let sterling be state 1 and dollars be state 2. Then,

$$\Lambda = \begin{bmatrix} 1/10 & 0 \\ 0 & 1/20 \end{bmatrix}, \quad P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Hence we have,

$$\Lambda[I - P] = \begin{bmatrix} 1/20 & -1/20 \\ -1/40 & 1/40 \end{bmatrix}.$$

We need to find the eigenvalues and eigenvectors of $\Lambda[I - P]$. Evidently $\lambda = 0$ is an eigenvalue with eigenvector $[1, 1]^T$. To find the other eigenvalue we solve,

$$\det \begin{bmatrix} 1/20 - \lambda & -1/20 \\ -1/40 & 1/40 - \lambda \end{bmatrix} = 0 .$$

This is a quadratic equation with solutions $\lambda = 0,\ 3/40$. The eigenvector for $\lambda = 3/40$ is found by solving

$$\begin{bmatrix} 1/20 & -1/20 \\ -1/40 & 1/40 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \frac{3}{40} \begin{bmatrix} x \\ 1 \end{bmatrix},$$

which has solution $x = -2$. Since

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

it follows that

$$\exp\{-\Lambda[I-P]t\}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3}\exp\left\{-\frac{3t}{40}\right\}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Hence we have that if $X(t),\ t > 0$, is the Markov chain then

$$\text{Prob}(X(t) = 1 \mid X(0) = 2) = \frac{1}{3} - \frac{1}{3}\exp\left\{-\frac{3t}{40}\right\}.$$

Substituting $t = 15$ in the previous formula gives the probability .225 of the assets being in sterling 15 days from now.