Math 523 Solutions to Final Winter 2004

(1) The future lifetimes of a certain population can be modeled as follows:
(a) Each individual's lifetime is exponentially distributed with parameter Λ. (b) Over the population Λ is uniformly distributed in the interval $[1, 8]$.
Calculate the probability of surviving to time 0.6 for an individual randomly selected at time 0.

Solution: Let T be the individual's lifetime. First note that

$$P(T > 0.6 \mid \Lambda) = \exp(-0.6\Lambda) \ .$$

Hence we have

$$P(T > 0.6) = E[\exp(-0.6\Lambda)] = \frac{1}{7}\int_1^8 \exp(-0.6\lambda)d\lambda \ .$$

The answer is .128.

(2) An investment fund is established to provide benefits on 400 independent lives. (a) On January 1, 2004, each life is issued a 10-year deferred whole life insurance of 1000, payable at the moment of death. (b) Each life is subject to a constant force of mortality of 0.05. (c) The force of interest is 0.07. (d) For the normal variable Z, one has $P(Z < 1.283) = 0.90,\ P(Z < 1.645) = 0.95,\ P(Z < 1.960) = 0.975$.
Calculate the amount needed in the investment fund on January 1, 2004 ,so that the probability, as determined by the normal appoximation, is 0.95 that the fund will be sufficient to provide those benefits.
NOTE: "deferred" means insurance is payable only if death occurs after the deferred period.

Solution: Let τ be the time of death so

$$P(\tau > t) = \exp(-.05t) \ ,$$

where t is in years. If X is the present value of the amount payable to an individual then,

$$X = 1000\exp(-.07\tau)\chi_{10}(\tau) \ ,$$

where $\chi_{10}(t) = 1,\ t > 10,\ \chi_{10}(t) = 0,\ t < 10$. Thus,

$$E[X] = 1000\int_{10}^{\infty} \exp(-.07t)(.05)\exp(-.05t)dt = 125.5 \ ,$$

$$E[X^2] = 1000^2\int_{10}^{\infty} \exp(-.14t)(.05)\exp(-.05t)dt = 39360 \ .$$

Hence $Var[X] = 39360 - 125.5^2 = 23609$. Thus

$$E[S] = 400(125.5), \quad Var[S] = 400(23609) \ .$$

To get the premium P we set

$$P = E[S] + 1.645\sqrt{Var[S]} = 55255 \ .$$

(3) A company provides insurance to a concert hall for losses due to power failure: You are given: (a) The number of power failures in a year has a Poisson distribution with mean 1. (b) The distribution of ground up losses due to a single power failure has possible values $X = 10, 20, 50$ with probabilities $P(X = 10) = 0.3, \ P(X = 20) = 0.3, \ P(X = 50) = 0.4$. (c) The number of power failures and the amounts of losses are independent. (d) There is an annual deductible of 30.
Calculate the expected amount of claims paid by the insurer in one year.

Solution: We need to compute $E[(S-30)^+]$. Now

$$E[S-30] = E[(S-30)^+] - 30P(S=0) - 20P(S=10) - 10P(S=20) \ .$$

Clearly $E[X] = .3(10) + .3(20) + .4(50) = 29$. Since $E[N] = 1$ we conclude $E[S] = E[N]E[X] = 29$. Since $X/10$ takes integer values $1, 2, 5$ we can use the recurrence formula for $S/10$ with $\lambda_1 = .3, \ \lambda_2 = .3, \ \lambda_5 = .4$. We have $P(S=0) = P(N=0) = e^{-1}$. Hence,

$$P(S/10 = 1) = .3P(S=0) = .3e^{-1},$$

$$P(S/10 = 2) = \frac{1}{2}[\lambda_1 P(S/10 = 1) + 2\lambda_2 P(S/10 = 0)] = .345e^{-1}.$$

We conclude that

$$E[(S-30)^+] = 29 - 30 + 30e^{-1} + 20(.3)e^{-1} + 10(.345)e^{-1} = 13.51 \ .$$

(4) For an insurance portfolio, you are given:
(a) For each individual insured, the number of claims follows a Poisson distribution. (b) The mean claim count varies by insured, and the distribution of mean claim counts follows a gamma distribution. (c) Claim sizes follow a Pareto distribution with mean 1500 and variance $6,750,000$. (d) Claim sizes and claim counts are independent. (e) The full credibility standard is to be within 5% of the expected aggregate loss 95% of the time. (f) For the normal variable Z, one has $P(Z < 1.283) = 0.90, \ P(Z < 1.645) = 0.95, \ P(Z < 1.960) = 0.975$. (g) For a random sample of 1000 insureds, the observed claim counts are as follows:

Number of Claims	0	1	2	3	4	5
Number of Insureds	512	307	123	41	11	6

Determine the minimum number of insureds needed for the aggregate loss to be fully credible.

Solution: From classical credibility theory, we have

$$\text{minimum number of insureds} = \left(\frac{y_{.95}}{.05}\right)^2 \frac{Var[S]}{E[S]^2} ,$$

where S is the aggregate loss. S is a compound variable with frequency N and severity X. We are given $E[X] = 1500$, $Var[X] = 6,750,000$. The values of $E[N]$ and $Var[N]$ are estimated from the given data. The unbiased estimate for $E[N]$ is,

$$E[N] = \frac{512(0) + 307(1) + 123(2) + 41(3) + 11(4) + 6(5)}{1000} = .75.$$

The unbiased estimate for $Var[N]$ is,

$$Var[N] = \left\{\frac{512(0^2) + 307(1^2) + 123(2^2) + 41(3^2) + 11(4^2) + 6(5^2)}{1000} - .75^2\right\}/.999 = .9324.$$

Thus we estimate $E[S] = E[N]E[X] = 1125$, and $Var[S] = E[N]Var[X]+Var[N]E[X]^2 = 7,160,400$. Putting $y_{.95} = 1.96$ we conclude minimum number of insureds $= 8694$.

(5) For a portfolio of independent risks, you are given:
(a) The risks are divided into two classes, Class A and Class B. (b) Equal numbers of risks are in Class A and Class B. (c) For each risk, the probability of having exactly one claim during the year is 20% and the probability of having no claims is 80%. (d) All claims for Class A are of size 2. (e) All claims for Class B are of size c, an unknown but fixed quantity.
One risk is chosen at random, and the total loss for one year for that risk is observed. You wish to estimate the expected loss for that same risk in the following year. Determine the limit of the Buhlmann credibility factor as c goes to infinity.

Solution: Θ has the two values A, B with equal probability. For class A the claim is compound with frequency N a Bernoulli variable with $P(N = 1) = .2$. The severity is deterministic, $X = 2$. Thus $\mu(A) = E[N]E[X] = .2(2) = .4$. We also have the variance, $\sigma^2(A)$ given by,

$$\sigma^2(A) = E[N]Var[X] + Var[N]E[X]^2 = .2(0) + [.2(1 - .2)](2^2) = .64.$$

Similarly we have $\mu(B) = .2c$, $\sigma^2(B) = .16c^2$. Thus

$$E[\mu(\Theta)] = \frac{1}{2}[.4 + .2c], \quad E[\mu(\Theta)^2] = \frac{1}{2}[.4^2 + (.2c)^2] .$$

We conclude $Var[\mu(\Theta)] = .04 + .01c^2 - .04c$. We also have $E[\sigma^2(\Theta)] = .5[.64 + .16c^2]$. Hence

$$k = \frac{E[\sigma^2(\Theta)]}{Var[\mu(\Theta)} = \frac{.32 + .08c^2}{(.2 - .1c)^2} \ .$$

Thus k converges to 8 as $c \to \infty$. Since $Z = 1/(1+k)$ it follows that Z converges to 1/9 as $c \to \infty$.

(6) Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins per minute. The denominations are randomly distributed as follows: 60% of the coins are worth 1, 20% of the coins are worth 5 and 20% of the coins are worth 10.
Calculate the conditional expected value of the coins Tom found during his one hour walk today, given that among the coins he found exactly 10 were worth 5 each.

Solution: Let $S(t)$ be the value of coins found up to time t where t is in minutes. Then $S(t)$ is compound Poisson with severity variable X where $P(X = 5) = .2$, $P(X = 10) = .2$, $P(X = 1) = .6$. Let $\mathcal{N}(t)$ be the frequency variable. Then $\mathcal{N}(t)$ is a Poisson process with frequency 0.5. We can write

$$\mathcal{N}(t) = \mathcal{N}_1(t) + \mathcal{N}_5(t) + \mathcal{N}_{10}(t),$$

$$S(t) = \mathcal{N}_1(t) + 5\mathcal{N}_5(t) + 10\mathcal{N}_{10}(t),$$

where $\mathcal{N}_1(t)$ is the number of coins of value 1 found up to time t etc. Then $\mathcal{N}_1(t)$ is a Poisson process with frequency $0.5P(X = 1) = .3$. Similarly $\mathcal{N}_5(t)$ is an independent Poisson process with frequency $0.5P(X = 5) = .1$ and $\mathcal{N}_{10}(t)$ with frequency $0.5P(X = 10) = .1$. We need to compute

$$E[S(60) \mid \mathcal{N}_5(60) = 10] = E[\mathcal{N}_1(60)] + 50 + 10E[\mathcal{N}_{10}(60)] = (.3)(60) + 50 + (10)(.1)(60) = 128.$$

(7) In the state of Elbonia all adults are drivers. It is illegal to drive drunk. If you are caught, your driver's license is suspended for the following year. Driver's licenses are suspended only for drunk driving. If you are caught driving with a suspended license, your license is revoked and you are imprisoned for one year. Licenses are reinstated upon release from prison. Every year 5% of adults with an active license have their license suspended for drunk driving. Every year 40% of drivers with suspended licenses are caught driving. Assume that all changes in driving status take place on January 1, all drivers act independently, and the adult population does not change.
Suppose a person is driving legally on April 28, 2004. Find the probability he will also be driving legally three years from now i.e. on April 28, 2007.

Solution: Let state 1 denote a legal driver, 2 a suspended driver and 3 a driver in prison. Then the transition matrix P is given by,

$$P = \begin{bmatrix} .95 & .05 & 0 \\ .6 & 0 & .4 \\ 1 & 0 & 0 \end{bmatrix} .$$

We need to compute $P^3_{1,1}$. Now we have

$$P^2 = \begin{bmatrix} .9325 & .0475 & .02 \\ * & * & * \\ * & * & * \end{bmatrix} .$$

Thus $P^3_{1,1} = .9325(.95) + .0475(.6) + .02(1) = .9344$.

(8) An insurer's claims follow a compound Poisson claims process with two claims expected per period. Claim amounts can be only $1, 2$ or 3 and these are equal in probability.
Calculate the continuous premium rate that should be charged each period so that the adjustment coefficient will be 0.5.

Solution: We use the equation for the adjustment coefficient R,

$$cR + \lambda = \lambda \Phi_X(R) .$$

Now we have for the moment generating function of X, the formula,

$$\Phi_X(R) = \frac{1}{3}[e^R + e^{2R} + e^{3R}] .$$

Thus $\Phi_X(.5) = 2.95$. Since $\lambda = 2$ we conclude $c = 2(1.95)/.5 = 7.8$.