Math 523 Midterm Winter 2004 Solutions

(1) A company insures a risk which has a pdf $f(x)$ given by

$$f(x) = \frac{x}{10,000} \exp\left[-\frac{x}{100}\right], \qquad 0 < x < \infty.$$

(a) Find the pure premium if the policy on the risk is total loss insurance.
(b) Find the pure premium if the policy is stop-loss insurance with deductible 100.

**Solution:** $X$ is a Gamma variable with parameters $(r, \lambda) = (2, 1/100)$. Hence the answer to (a) is,

$$E[X] = \frac{r}{\lambda} = 200 \ .$$

For (b) we need to compute

$$E[(X-100)^+] = \int_{100}^{\infty} (x-100)x \exp\left[-\frac{x}{100}\right] \frac{dx}{10,000} =$$

$$\int_{100}^{\infty} (x-100)^2 \exp\left[-\frac{x}{100}\right] \frac{dx}{10,000} + 100 \int_{100}^{\infty} (x-100) \exp\left[-\frac{x}{100}\right] \frac{dx}{10,000}$$
$$= 100e^{-1}\Gamma(3) + 100e^{-1}\Gamma(2) = 300e^{-1} = 110.36 \ ,$$

where $\Gamma(r)$ is the $\Gamma$ function, $\Gamma(r) = (r-1)!$ if $r$ is a non-negative integer.

(2) A life-insurance company covers 300 independent lives for 1-year term life insurance as follows:

| Benefit amount | Number of policies | Prob. of claim per policy |
|---|---|---|
| 1 | 140 | 0.20 |
| 2 | 90 | 0.15 |
| 3 | 70 | 0.10 |

(a) Find the mean and the variance of the aggregate claim to the company.
(b) The insurance company wants to be 95 percent sure that claims will be less than the premiums it collects. Find the security loading the company should have. Give your answer as a percentage. [Use $P(Z < 1.645) = 0.95$ for a standard normal variable.]

**Solution:** For (a) we have,

$$E[S] = 1(140)(.2) + 2(90)(.15) + 3(70)(.1) = 76 \ ,$$

$$Var[S] = 1^2(140)(.2)(1-.2) + 2^2(90)(.15)(1-.15) + 3^2(70)(.1)(1-.1) = 125 \ .$$

For (b) we observe by the central limit theorem that

$$\frac{S - E[S]}{\sqrt{Var[S]}} = \frac{S - 76}{5\sqrt{5}} \sim Z,$$

where $Z$ is the standard normal variable. The security loading $\theta$ is defined by $P(S < (1+\theta)E[S]) = .95$ whence,

$$P\left(\frac{S-76}{5\sqrt{5}} < \frac{76\theta}{5\sqrt{5}}\right) = .95 \ .$$

Thus we have $76\theta/5\sqrt{5} = 1.645$ and so $\theta = .2419$. The security loading is 24.2%.

(3) A compound Poisson claims variable $S$ has moment generating function $\Phi_S(t)$ given by the formula,

$$\Phi_S(t) = \exp\left[10e^t + 40e^{2t} - 50\right] \ .$$

(a) Find $E[S]$ and $Var[S]$.
(b) If $X$ is the severity variable find $E[X]$ and $Var[X]$.

**Solution:** To solve (a) we use the fact that,

$$E[S] = \frac{d}{dt}\ln[\Phi_S(t)], \quad Var[S] = \frac{d^2}{dt^2}\ln[\Phi_S(t)]$$

evaluated at $t = 0$. We have now $\ln[\Phi_S(t)] = 10e^t + 40e^{2t} - 50$, whence

$$\begin{aligned} \frac{d}{dt}\ln[\Phi_S(t)] &= 10e^t + 80e^{2t}, \\ \frac{d^2}{dt^2}\ln[\Phi_S(t)] &= 10e^t + 160e^{2t} \ . \end{aligned}$$

Thus $E[S] = 90, \ Var[S] = 170$.

For (b) we use the fact that since $S$ is compound Poisson one has

$$\Phi_S(t) = exp\left[\lambda\{\Phi_X(t) - 1\}\right] \ ,$$

where $\lambda$ is the expected frequency. Comparing with the given formula for $\Phi_S(t)$ we see that $\lambda = 50$. Hence $E[N] = Var[N] = 50$. Using $E[S] = E[N]E[X]$ we conclude $E[X] = 90/50 = 9/5$. Using $Var[S] = E[N]Var[X] + Var[N]E[X]^2$ we have $170 = 50[Var[X] + (9/5)^2]$ and so $Var[X] = 4/25$.

(4) An aggregate loss distribution $S$ is compound Poisson with the expected number of claims being 4. Individual claim amounts can take only the values 1 and 3 with 1

having probability 0.7.
(a) Find $P(S \leq 3)$.
(b) Find the pure premium for stop-loss insurance on the risk $S$ with deductible 3.5.

**Solution:** For (a) we use the recurrence relation for the compound Poisson variable with severity variable which takes integer values. Thus we have $\lambda = 4,\ \lambda_1 = 0.7(4) = 2.8,\ \lambda_2 = 0,\ \lambda_3 = 0.3(4) = 1.2$. Thus we have,

$$P(S=0) = e^{-\lambda} = e^{-4}, \quad P(S=1) = \lambda_1 P(S=0) = 2.8e^{-4} \ ,$$

$$P(S=2) = \frac{1}{2}[\lambda_1 P(S=1) + 2\lambda_2 P(S=0)] = 1.4(2.8)e^{-4} = 3.92e^{-4},$$

$$P(S=3) = \frac{1}{3}[\lambda_1 P(S=2) + 2\lambda_2 P(S=1) + 3\lambda_3 P(S=0)] = \frac{1}{3}[(2.8)(3.92)e^{-4} + 3(1.2)e^{-4}] = 4.86e^{-4} \ .$$

Hence $P(S \leq 3) = [1 + 2.8 + 3.92 + 4.86]e^{-4} = 12.58e^{-4} = 0.23$.

For (b) we need to compute $E[(S-3.5)^+]$. We use the equation,

$$E[(S-3.5)] = E[(S-3.5)^+] - 0.5P(S=3) - 1.5P(S=2) - 2.5P(S=1) - 3.5P(S=0) \ .$$

Now $E[X] = 0.7(1) + 0.3(3) = 1.6$ and $E[N] = 4$ so $E[S] = 4(1.6) = 6.4$. We conclude that $E[(S-3.5)^+] = 6.4 - 3.5 + [0.5(4.86) + 1.5(3.92) + 2.5(2.8) + 3.5(1)]e^{-4} = 3.24$.

(5) An insurance company assumes that the aggregate claims variable for a corporate policy is compound Poisson. It has a full credibility standard of 3000 claims for claim frequency. Suppose the severity variable is uniform on the interval $[0, 3]$.
(a) Find the number of claims needed for full credibility of pure premium.
(b) Find the credibility of 500 claims for frequency.

**Solution:** Since $X$ is uniform on $[0, 3]$ we have that $E[X] = 3/2,\ Var[X] = 3^2/12 = 3/4$. For (a) we have then,

$$\text{number of claims} = 3000\left\{1 + \frac{Var[X]}{E[X]^2}\right\} = 3000[1 + \frac{1}{3}] = 4000 \ .$$

For (b) we use the formula $n_Z = Z^2 n_F$, whence $500 = Z^2(3000)$ so $Z = 1/\sqrt{6} = 0.41$.

(6) An auto insurance company divides its customers into 2 types, A and B. Type A customers have a probability 1/4 of making a claim in a given year. Type B customers have a probability 3/5 of making a claim in a given year. The probability of a random customer being Type A is 40%. Suppose a customer does not make a claim in 2004. Find
(a) The probability the customer is Type A,
(b) The probability the customer will make a claim in 2005.

Assume at most one claim is made in a given calendar year and independence from year to year. Use a Bayesian analysis.

**Solution:** We have by Bayes formula that the probability the customer does not make a claim in 2004 is given by,

$$P(A)P(A \text{ makes no claim})+P(B)P(B \text{ makes no claim}) = 0.4(1-1/4)+0.6(1-3/5) = 0.54\,.$$

Similarly we have,

$$P(\text{customer is A and makes no claim}) = P(A)P(A \text{ makes no claim}) = 0.4(1-1/4) = 0.3\,.$$

Hence $P(\text{customer is A given no claim}) = 0.3/0.54 = 0.556$.

For (b) we have the probability the customer makes a claim in 2005 is given by,

$$\begin{aligned} P(\text{customer is A given no claim in 2004})\ P(A \text{ makes claim}) &\quad + \\ P(\text{customer is B given no claim in 2004})\ P(B \text{ makes claim}) &= (.556)(1/4) + (1-.556)(3/5) = .41\,. \end{aligned}$$