

Math 623 (IOE 623), Fall 2011: Final exam

Name:

Student ID:

This is a closed book exam. You may bring up to ten one sided A4 pages of notes to the exam. You may also use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. When asked to compute a numerical value, it must be clear from your answer how you obtained it.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

- (1) Suppose I wish to compute the value of a *European Straddle* option on a stock. The option pays $\Phi(S)$ if exercised when the stock price is S , where

$$\Phi(S) = \begin{cases} 34 - S & \text{if } S \leq 34 \\ S - 34 & \text{if } S \geq 34. \end{cases}$$

The current value of the stock is 33 and it expires 8 months from today. The stock volatility is $\sigma = 0.31$ and the risk free interest rate is $r = 0.05$. The value of the option is to be obtained by numerically solving a terminal-boundary value problem for a PDE. The PDE is the Black-Scholes PDE transformed by the change of variable $S = e^x$, where S is the stock price. Hence the value $u(x, t)$ of the option as a function of the variables x and t with the units of t in years, is a solution of the differential equation

$$\frac{\partial u}{\partial t} + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - ru = 0,$$

on the interval $a < x < b$, $0 \leq t \leq T$, where $t = 0$ denotes today, with terminal condition $u(x, T) = \Psi(x)$, $a < x < b$, and boundary conditions $u(a, t) = f(t)$, $u(b, t) = g(t)$, $0 \leq t \leq T$. I use the explicit Euler method to numerically solve the problem. This can be written in the form

$$u(x, t - \Delta t) = Au(x, t) + Bu(x + \Delta x, t) + Cu(x - \Delta x, t), \quad a < x < b,$$

where A, B, C are constant parameters and $a = 2.7$, $b = 4.3$.

- Explain why the choices for a, b are suitable for this problem.
- Find the values of $f(0)$ and $g(T)$.
- In the numerical scheme I take $\Delta x = 0.04$, $\Delta t = 0.015$. Find the value of the parameter B .
- Explain why the choice $\Delta x = 0.04$, $\Delta t = 0.015$ in (c) is suitable for the numerical scheme but the choice $\Delta x = 0.04$, $\Delta t = 0.02$ is not suitable.

(a) Soln: $\sigma = .31$, $T = 2/3$, $K = 34$. Hence we should choose $b \geq \log K + 3\sigma\sqrt{T} = 4.28$ and $a \leq \log K - 3\sigma\sqrt{T} = 2.76$.

(b) Soln: We have $f(t) = Ke^{-r(T-t)} - e^a$ and $g(t) = e^b - Ke^{-r(T-t)}$. Hence $f(0) = 18.0056$ and $g(T) = 39.6998$.

(c) Soln: The explicit Euler numerical scheme for the PDE is

$$\begin{aligned} \frac{u(x, t) - u(x, t - \Delta t)}{\Delta t} + \left(r - \frac{\sigma^2}{2}\right) \frac{u(x + \Delta x, t) - u(x, t)}{2\Delta x} \\ + \frac{\sigma^2}{2} \left[\frac{u(x + \Delta x, t) + u(x - \Delta x, t) - 2u(x, t)}{(\Delta x)^2} \right] - ru(x, t) = 0. \end{aligned}$$

We conclude that

$$B = \frac{\sigma^2 \Delta t}{2(\Delta x)^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{\Delta t}{2\Delta x} = .4508.$$

(d) Soln: For $\Delta x = 0.04$, $\Delta t = 0.02$ we have $\sigma^2 \Delta t / (\Delta x)^2 = .901 < 1$, so this choice for Δx , Δt yields a stable numerical scheme. If $\Delta x = 0.04$, $\Delta t = 0.015$ we have $\sigma^2 \Delta t / (\Delta x)^2 = 1.2 > 1$, so this choice for Δx , Δt yields an unstable numerical scheme.

- (2) We wish to find the value of an Asian option on a zero dividend stock which expires 9 months from today. The payoff on the option is the excess of the stock price at expiration over the continuous average of the stock price during the 9 month life span of the option. The current price of the stock is 28 and its volatility is 0.37 per annum. The risk free interest rate is 0.04 per annum. One can compute the value of the Asian option by solving a terminal-boundary value problem for a PDE. The PDE for the function $w(\xi, t)$ is given by

$$\frac{\partial w}{\partial t} + a(\xi) \frac{\partial w}{\partial \xi} + b(\xi) \frac{\partial^2 w}{\partial \xi^2} + c(\xi) w = 0,$$

where $a(\xi), b(\xi), c(\xi)$ are known functions of ξ . The equation is to be solved in the interval $0 < t < T$, $0 < \xi < \xi_{\max}$, with terminal condition $w(\xi, T) = \Phi(\xi)$, where $t = 0$ denotes today. We use the explicit Euler method with $\Delta \xi = 0.025$, $\Delta t = 0.001$ to numerically solve the problem. This can be written in the form

$$w(\xi, t - \Delta t) = A(\xi)w(\xi, t) + B(\xi)w(\xi + \Delta \xi, t) + C(\xi)w(\xi - \Delta \xi, t), \quad 0 \leq \xi < \xi_{\max}.$$

- (a) Find the value of $a(0.6)$ and $\Phi(0.6)$.
- (b) Find the values of $A(\xi), B(\xi), C(\xi)$ when $\xi = 0$.
- (c) Suppose that in the numerical scheme we approximate the derivative $\partial w(\xi, t)/\partial \xi$ by a symmetric difference for $\xi > \xi_0 > 0$ and a forward difference for $0 \leq \xi \leq \xi_0$. Explain why it makes sense to do this and obtain a suitable value for ξ_0 .

(a) Soln: We have that $a(\xi) = 1 - r\xi$ and $\Phi(\xi) = \max[1 - \xi/T, 0]$. Since $r = .04$, $T = 3/4$, we conclude that $a(0.6) = .976$ and $\Phi(0.6) = .200$.

(b) Soln: The boundary condition at $\xi = 0$ is given by

$$\frac{\partial w(\xi, t)}{\partial t} + \frac{\partial w(\xi, t)}{\partial \xi} = 0,$$

which in the discrete approximation yields

$$\frac{w(\xi, t) - w(\xi, t - \Delta t)}{\Delta t} + \frac{w(\xi + \Delta \xi, t) - w(\xi, t)}{\Delta \xi} = 0.$$

Hence we have that

$$w(\xi, t - \Delta t) = \frac{\Delta t}{\Delta \xi} w(\xi + \Delta \xi, t) + \left[1 - \frac{\Delta t}{\Delta \xi}\right] w(\xi, t).$$

We conclude that $B(0) = \Delta t/\Delta \xi = 0.04$ and $A(0) = 1 - B(0) = 0.96$. Evidently $C(0) = 0$.

(c) Soln: We should try to approximate the derivative $\partial w(\xi, t)/\partial \xi$ by the second order accurate approximation

$$\frac{\partial w(\xi, t)}{\partial \xi} \simeq \frac{w(\xi + \Delta \xi, t) - w(\xi - \Delta \xi, t)}{2\Delta \xi}$$

so that cumulative error is $O[(\Delta \xi)^2]$. This is not possible for all $\xi > 0$ as we can see that the boundary condition forces us to use the forward difference approximation

$$\frac{\partial w(\xi, t)}{\partial \xi} \simeq \frac{w(\xi + \Delta \xi, t) - w(\xi, t)}{\Delta \xi}$$

at $\xi = 0$, which is first order accurate. The numerical scheme with the second order difference approximation is only stable for $\xi > \sqrt{\Delta \xi}/\sigma$, but the scheme with the first order difference approximation is stable for all $\xi \geq 0$. Hence we should take $\xi_0 = \sqrt{\Delta \xi}/\sigma = .4273$.

- (3) I wish to find the value of a stock option by using the Monte-Carlo method. The value of the stock today is 21, the risk free rate of interest is 5% and the expiration date of the option is 9 months from today. We write the value V of the option as an expectation value $V = e^{-rT} E[\Phi(\xi)]$ where ξ is a standard normal variable, r is the risk free rate and T the expiration date. The function $\Phi(\cdot)$ is given by the formula $\Phi(\xi) = \max [21.17e^{0.2425\xi} - 20, 0]$. We do N independent simulations ξ_1, \dots, ξ_N of the variable ξ to estimate the value of the option.

- (a) Write down formulas for the sample mean and sample variance of $\Phi(\xi)$.
 (b) Suppose $N = 10^5$, the sample mean is 2.1547, and the sample variance is 9.3298. Estimate the value of the option and the confidence error in this estimation.
 (c) Find the strike price of the option and the volatility of the stock.

(a) Soln:

$$\begin{aligned} \text{Sample Mean} &= \frac{\Phi(\xi_1) + \dots + \Phi(\xi_N)}{N} = \hat{\Phi}_N . \\ \text{Sample Variance} &= \frac{1}{N} \sum_{j=1}^N \{\Phi(\xi_j) - \hat{\Phi}_N\}^2 = \frac{1}{N} \sum_{j=1}^N \Phi(\xi_j)^2 - \hat{\Phi}_N^2 . \end{aligned}$$

(b) Soln:

$$\text{Option value} = 2.1547e^{-rT} = 2.1547e^{-.05 \cdot .75} = 2.0754.$$

$$\text{Error} \simeq \left[\frac{\text{Variance}}{N} \right]^{1/2} = \left[\frac{9.3298}{10^5} \right]^{1/2} \simeq .01.$$

Hence the confidence error is 2 decimal places.

(c) Soln: $K = 20$ and $\sigma\sqrt{T} = .2425$. Hence $\sigma = .2425/\sqrt{.75} = .2800$.

- (4) The current value of a stock is 31. The Heston stochastic volatility model for the evolution of the price S_t of the stock is used to estimate the value of a stock option with expiration 8 months from today. The Heston model is governed by the system of equations:

$$\begin{cases} dY_t &= [\theta - 0.3 Y_t]dt + 0.4\sqrt{Y_t} dW_t \\ \frac{dS_t}{S_t} &= 0.023 dt + \sqrt{Y_t} (\rho dW_t + \sqrt{1 - \rho^2} dZ_t) . \end{cases}$$

Here W_t and Z_t are independent (uncorrelated) Brownian motions under the risk neutral measure Q . In order to price the option the system of equations is to be solved by the explicit Euler method applied to stochastic differential equations, where we set $S^m = S_{m\Delta t}$, $Y^m = Y_{m\Delta t}$, $m = 0, 1, 2, \dots$

- (a) Suppose $\rho = -0.5$, $\theta = 0.12$, $\Delta t = 0.0156$, and you have computed S^m , Y^m for $m = 0, 1, \dots, 38$, with $S^{38} = 29.75$, $Y^{38} = 0.48$. You are given the two values $\xi = 0.2137$, $\eta = -0.1549$ of independently sampled standard normal variables. Find the corresponding values of S^{39} , Y^{39} .
- (b) Explain why the choice of parameter $\theta = 0.12$ in (a) is appropriate for this model but the choice $\theta = 0.06$ is not.
- (c) Sketch a graph of the implied volatility curve you would expect to get from this model with parameters ρ, θ as in (a). Explain why the general shape of the graph is determined by the value of ρ .

(a) Soln: The explicit Euler scheme for the stochastic differential equation yields

$$\begin{aligned} Y(t + \Delta t) &= Y(t) + [\theta - 0.3 Y(t)]\Delta t + 0.4\sqrt{Y(t)}\sqrt{\Delta t} \xi , \\ S(t + \Delta t) &= S(t) + S(t) \left[0.023\Delta t + \sqrt{Y(t)}(\rho\xi + \sqrt{1 - \rho^2} \eta)\sqrt{\Delta t} \right] , \end{aligned}$$

where ξ, η are samples of independent standard normal variables. Taking the given values with $m = 38$ we find that $Y^{39} = .4870$, $S^{39} = 29.1403$.

(b) Soln: For the model to make sense we need to have $\theta > \beta^2/2$, where $\beta = 0.4$ for this model. If $\theta < \beta^2/2$ then $Y(t)$ becomes zero after some finite random time τ and $Y(t) = 0$ for $t > \tau$. Since we expect volatility always to be positive, we require $\theta > \beta^2/2$ in our model, so $\theta > (.4)^2/2 = .08$. Hence the choice of $\theta = .06$ is not appropriate for the model.

(c) Soln: Since $\rho < 0$ the stock price and stock volatility are negatively correlated. Thus as stock price increases the volatility tends to decrease. Now the price of the option is largely determined by stock values close to the strike price of the option. Hence we expect implied volatility to be a decreasing function of strike price, so the graph of implied volatility as a function of strike price should slope downwards.

- (5) We consider a discretization of the Hull-White model for the short rate $r(t)$ satisfying the SDE

$$dr(t) = [\theta(t) - 0.13 r(t)]dt + \sigma dB(t), \quad \text{where } B(\cdot) \text{ is Brownian motion.}$$

In the discretization we take $\Delta r = 0.003$ and $\Delta t = 0.01$. The lattice sites for the model are (m, j) , $m = 0, 1, 2, \dots$, $|j| \leq \min\{m, J\}$. A lattice site (m, j) corresponds to time $t = m\Delta t$ and interest rate r_j^m .

- (a) Find the value of the volatility σ which is used in the model, as prescribed by Hull-White so that third moments match.
- (b) Assuming that $r_{12}^{20} = 0.058$, find the value of r_7^{20} .
- (c) Suppose the model has been completely calibrated and we wish to compute the value of an interest rate cap where the cap is 3%. Let $V(m, j)$ be the value of the cap corresponding to a lattice point (m, j) . You are given that $V(31, 6) = 7.24653$, $V(31, 5) = 7.24645$, $V(31, 4) = 7.24638$, and also that $\alpha^{30} = 0.022$. Find the value of $V(30, 5)$ correct to 5 decimal places.

(a) Soln: We use the equation $\Delta t/(\Delta r)^2 = 1/3\sigma^2$ so $\sigma = \Delta r/\sqrt{3\Delta t} = 0.0173$.

(b) Soln: Since $r_j^m = r_0^m + j\Delta r$, we have that $r_7^{20} = r_{12}^{20} - 5\Delta r = 0.0430$.

(c) Soln: We use the formulas for the probabilities

$$p_u(j) = \frac{1}{6} + \frac{1}{2} [(aj\Delta t)^2 - aj\Delta t], \quad p_d(j) = \frac{1}{6} + \frac{1}{2} [(aj\Delta t)^2 + aj\Delta t], \quad p_s(j) = \frac{2}{3} - (aj\Delta t)^2.$$

Since from the stochastic differential equation we see that $a = .13$ we conclude that $p_u(5) = 0.1634$, $p_d(5) = 0.1699$, $p_s(5) = 0.6666$. To compute $V(30, 5)$ we use the recurrence equation

$$V(m, j) = \exp[-r_j^m \Delta t] \{ (r_j^m - .03)^+ + p_u(j)V(m+1, j+1) + p_s(j)V(m+1, j) + p_d(j)V(m+1, j-1) \},$$

together with the formula $r_j^m = \alpha^m + j\Delta r$. Now taking $m = 30, j = 5$ we see that $r_5^{30} = 0.0370$ and $V(30, 5) = 7.2500$.

