

Math 623 (IOE 623), Fall 2012: Final exam

Name:

Student ID:

This is a closed book exam. You may bring up to ten one sided A4 pages of notes to the exam. You may also use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. When asked to compute a numerical value, it must be clear from your answer how you obtained it.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

- (1) Suppose I wish to compute the value of a *European put* option on a stock with strike price 28. The current value of the stock is 29 and it expires 9 months from today. The stock volatility is $\sigma = 0.18$ and the risk free interest rate is $r = 0.04$. The value of the option is to be obtained by numerically solving a terminal-boundary value problem for a PDE. The PDE is the Black-Scholes PDE transformed by the change of variable $S = e^x$, where S is the stock price. Hence the value $u(x, t)$ of the option as a function of the variables x and t with the units of t in years, is a solution of the differential equation

$$\frac{\partial u}{\partial t} + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - ru = 0,$$

on the interval $a < x < b$, $0 \leq t \leq T$, where $t = 0$ denotes today, with terminal condition $u(x, T) = \Psi(x)$, $a < x < b$, and boundary conditions $u(a, t) = 28e^{T-t} - e^a$ and $u(b, t) = 0$ for $0 \leq t \leq T$. I use the explicit Euler method to numerically solve the problem. This can be written in the form

$$u(x, t - \Delta t) = Au(x, t) + Bu(x + \Delta x, t) + Cu(x - \Delta x, t), \quad a < x < b,$$

where A, B, C are constant parameters. In the problem we take $a = 2.9, b = 3.7$ and $\Delta t = .015$.

- Give the formula for the function $\Psi(\cdot)$.
- Find a suitable choice for Δx and explain your answer.
- We wish to modify the numerical scheme to value the barrier put option with strike 28 which pays zero if the stock price rises above 35 between today and the expiration of the option. Explain how we modify the scheme to value the barrier option.
- Do you expect there to be a significant difference between the price of the barrier option and the price of the standard put option? Explain your answer.

(a) Soln: $\Psi(x) = \max[28 - e^x, 0]$.

(b) Soln: We need Δx to satisfy the inequality $\Delta t/(\Delta x)^2 < 1/\sigma^2$ due to the stability condition. Hence we need $\Delta x > \sigma\sqrt{\Delta t} = .18\sqrt{.015} = .022$, so $\Delta x = .025$ would be good.

(c) Soln: Observe that $\log 35 = 3.55 = b_1 < 3.7 = b$ so we need to replace the interval $a < x < b$ by $a < x < b_1$ with the same boundary condition at a as before, and zero boundary condition at b_1 i.e. $u(b_1, t) = 0$ for $0 \leq t \leq T$.

(d) Soln: We need to compute the number of standard deviations of b_1 from $\log K$, which is $(b_1 - \log K)/\sigma\sqrt{T}$. This is given by $(3.55 - \log 28)/(.18 * \sqrt{.75}) = 1.43$. If Z is the standard normal variable then $P(Z > 1.43) = .08$, so the barrier should not make much difference to the price of the option.

- (2) Suppose I wish to compute the value of a *European call* option on a stock using the Monte-Carlo (MC) method. The current value of the stock is 33 and it expires 8 months from today. The stock volatility is $\sigma = .26$ and the risk free interest rate is $r = 0.035$. The value V of the option is given as an expectation $V = E[\Phi(\xi)]$, where ξ is a standard normal variable.
- (a) Assuming the strike price of the option is 34, find an expression for the function $\Phi(\xi)$.
 - (b) Suppose that in 10^6 MC simulations of $\Phi(\xi)$ in (a) the sample mean is 2.694717 and the sample variance is 20.857. Find the 99% confidence interval for the value of the option.
 - (c) Suppose now that the strike price is 50. Explain why it makes sense to use importance sampling to estimate the value of the option.
 - (d) If we use importance sampling to value the call option with strike 50 then its value $V = E[\Psi(\xi)]$ where ξ is standard normal. Find a formula for $\Psi(\xi)$ and explain why doing MC simulations of $\Psi(\xi)$ will give higher accuracy than MC simulations of $\Phi(\xi)$.

(a) Soln:

$$\Phi(\xi) = e^{-rT} \max \left[S_0 \exp\{(r - \sigma^2/2)T + \sigma\sqrt{T}\xi\} - K, 0 \right] .$$

Since $e^{-rT} = \exp[-.035 * (2/3)] = 0.977$, $\sigma\sqrt{T} = .26\sqrt{2/3} = 0.212$, $S_0 \exp[(r - \sigma^2/2)T] = 33 \exp[(.035 - .26^2/2) * (2/3)] = 33.026$, we conclude that $\Phi(\xi) = 0.977 \max[33.026 \exp\{0.212\xi\} - 34, 0]$.

(b) Soln: The 99% confidence interval is 2.6 standard deviations from the mean. Thus the confidence interval is $2.694717 \pm 2.6\sigma/\sqrt{N}$ where $\sigma^2 = 20.857$, $N = 10^6$. Hence the interval is $[2.683, 2.707]$.

(c) Soln: We need to measure how far the option is out of the money in terms of standard deviations i.e. $\log(K/S_0)/\sigma\sqrt{T} = \log(50/33)/(.26\sqrt{2/3}) = 1.96$. Since the option is about 2 standard deviations out of the money about 98% of MC simulations will yield 0 for the value of the option. It makes sense therefore to use importance sampling.

(d) Soln: We have that for any real number β ,

$$E[\Phi(\xi)] = E[\Psi_\beta(\xi)], \quad \text{where } \Psi_\beta(\xi) = \Phi(\xi - \beta)e^{\beta\xi - \beta^2/2}.$$

We choose now β so that $\Psi_\beta(\xi)$ is just in the money for $\xi = 0$ so that approximately 50% of MC simulations for $\Psi_\beta(\cdot)$ will be positive. To do this we choose β as in (c) so $\beta = 1.96$. Hence we are taking β to have the minimum value so that a significant percentage of simulations are in the money. That way we keep variance to a minimum. If we keep increasing β variance will start to grow again very rapidly (like e^{β^2}).

- (3) I wish to find the value of an Asian option on a stock by using the MC method. The price of the stock today is 25, its volatility is $\sigma = .24$ and the risk free rate of interest is $r = .03$. The expiration date for the option is 6 months. In order to do this I numerically solve the GBM stochastic differential equation

$$dS(t) = S(t)[r dt + \sigma dB(t)] .$$

- (a) Taking $\Delta t = .0125$ and using Euler's method to solve the SDE, write a code which will obtain one MC value for the option price.
- (b) Suppose we are able to generate at most 10^8 i.i.d. copies of the standard normal variable *in total* to run our simulation to estimate the value of the Asian option using Euler's method. What would be a good choice for Δt in this case? Explain your answer.
- (c) Write down an alternative numerical scheme for solving the SDE which is more accurate than Euler's method. Explain why this is the case.

(a) Soln: The code is as follows:

$V = 0, T = 0.5, \Delta t = 0.125, M = T/\Delta t. \quad \sigma = .24, r = 0.03, \quad S = \text{zeros}(M+1, 1), S(1) = 25, \xi = \text{randn}(M, 1).$

for $j = 1 : M, \quad S(j+1) = S(j)[1 + r\Delta t + \sigma\sqrt{\Delta t} \xi(j)], \quad \text{end.}$

$V = \max[S(M+1) - \text{mean}(S), 0].$

(b) Soln: The error for the stock price at time T in N MC simulations of the random path using Euler's method is $O(\sqrt{\Delta t}) + O(1/\sqrt{N})$. Hence we should take approximately $M = N$ to minimize error. Since $MN = 10^8$ we should take $M = 10^4$ so $\Delta t = 10^{-4} * .5 = .00005$.

(c) Soln: We can solve the GBM equation exactly and this yields

$$S(t + \Delta t) = S(t) \exp \left[(r - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} \xi \right] ,$$

where ξ is standard normal. Hence a more accurate simulation would yield in the code $S(j+1) = S(j) \exp \left[(r - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} \xi(j) \right]$ instead of Euler's formula. If we are seeking to find the value of the continuous Asian option the only error now is in approximating $\int_0^T S(t) dt$ by a Riemann sum.

- (4) The value V of a basket option on d stocks with expiration date T can be estimated using the MC method as $V = E[\Phi(\xi_1, \dots, \xi_d)]$, where $\xi_1, \dots, \xi_d$ are i.i.d. standard normal. Suppose the stocks have values $S_1(t), \dots, S_d(t)$ at time t , the stock volatilities are $\sigma_1, \dots, \sigma_d$ and the risk free rate of interest is r .
- (a) Write down the payoff function for the *arithmetically averaged* and *geometrically averaged* call options on the d stocks with strike price K .
- (b) Suppose the d stock prices are driven by d Brownian motions with covariance matrix given by a $d \times d$ matrix $\rho = [\rho_{i,j}]$, $1 \leq i, j \leq d$. What properties does ρ have that it can be written as $\rho = \mathbf{A}\mathbf{A}^T$ for some $d \times d$ matrix $\mathbf{A}$?
- (c) Assuming $\mathbf{A} = [a_{i,j}]$, $1 \leq i, j \leq d$, find a formula for the function $\Phi(\xi_1, \dots, \xi_d)$ in the case of the *arithmetically averaged* option.

(a) Soln: The payoffs on the basket options are as follows:

$$\begin{aligned} \text{arithmetic payoff} &= \max \left[\frac{1}{d} \sum_{j=1}^d S_j(T) - K, 0 \right], \\ \text{geometric payoff} &= \max \left[\left(\prod_{j=1}^d S_j(T) \right)^{1/d} - K, 0 \right]. \end{aligned}$$

(b) Soln: The $d \times d$ matrix $\rho = [\rho_{i,j}]$ should have the following properties

$$\text{symmetry : } \rho_{i,j} = \rho_{j,i} \text{ for } 1 \leq i, j \leq d.$$

$$\text{non-negative definiteness : } \sum_{i,j=1}^d \rho_{i,j} \xi_i \xi_j \geq 0 \text{ for all vectors } \xi = [\xi_1, \xi_2, \dots, \xi_d].$$

Also $\rho_{i,i} = 1$ for $i = 1, \dots, d$, but this is not necessary for the representation $\rho = \mathbf{A}\mathbf{A}^T$.

(c) Soln:

$$S_i(T) = S_i(0) \exp[(r - \sigma_i^2/2)T + \sigma_i B_i(T)], \quad B_i(T) = \sum_{j=1}^d a_{i,j} Z_j(T),$$

where the $Z_j(\cdot)$ are independent BMs. We can therefore write $Z_j(T) = \sqrt{T} \xi_j$, $j = 1, \dots, d$, where the $\xi_1, \dots, \xi_d$ are i.i.d. standard normal. Hence the function $\Phi(\xi_1, \dots, \xi_d)$ is given by

$$\Phi(\xi_1, \dots, \xi_d) = e^{-rT} \max \left[\frac{1}{d} \sum_{i=1}^d S_i(0) \exp[(r - \sigma_i^2/2)T + \sigma_i \sqrt{T} \sum_{j=1}^d a_{i,j} \xi_j] - K, 0 \right].$$

- (5) We consider a discretization of the Hull-White model for the short rate $r(t)$ satisfying the SDE

$$dr(t) = [\theta(t) - 0.08 r(t)]dt + .025dB(t), \quad \text{where } B(\cdot) \text{ is Brownian motion.}$$

The lattice sites for the model are (m, j) , $m = 0, 1, 2, \dots$, $|j| \leq \min\{m, J\}$. A lattice site (m, j) corresponds to time $t = m\Delta t$ and interest rate r_j^m . In the model we take $\Delta r = .003$.

- (a) Find the corresponding value of Δt according to the Hull-White prescription so that third moments match.
- (b) Assuming that $r_{10}^{25} = 0.062$, find the value α^{25} of the expected short rate at time $t = 25\Delta t$.
- (c) The discount recurrence equation on internal vertices of the model is given by

$$V(m, j) = e^{-r_j^m \Delta t} [p_u(j)V(m+1, j+1) + p_s(j)V(m+1, j) + p_d(j)V(m+1, j-1)] .$$

Suppose we solve this recurrence equation with terminal data $V(M, j) = 1$ for all j , where $M\Delta t = 12$. What is the meaning of the values $V(m, j)$ when $m < M$?

(a) Soln: The HW prescription is $\Delta t/(\Delta r)^2 = 1/3\sigma^2$. From the SDE we see that $\sigma = .025$ so $\Delta t = (.003)^2/3(.025)^2 = .0048$.

(b) Soln: We use the formula $r_j^m = \alpha^m + j\Delta r$ with $m = 25, j = 10, \Delta r = .003$ to obtain $.062 = \alpha^{25} + 10(.003)$, whence $\alpha^{25} = .032$.

(c) Soln: The quantity $V(0, 0)$ is today's value of the zero coupon bond with face value 1 and maturity in 12 years. For $0 < m < M$ then $V(m, j) = P(j, m, M)$ is the bond price attached to the lattice site (m, j) for the bond with maturity in $T = 12$ years. This determines the distribution of the random variable $P(t, T)$ with $m\Delta t = t$, $M\Delta t = T$, where $P(t, T)$ is the bond price at time t of the bond with maturity T .

