Math 623 (IOE 623), Fall 2007: Final exam

Name:

Student ID:

This is a closed book exam. You may bring up to ten one sided A4 pages of notes to the exam. You may also use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. State explicitly any additional assumptions you make.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

(1) Suppose I wish to compute the value of a European *put* option on a stock. The option has strike price 23 and it expires 4 months from today. The current value of the stock is 24. Its volatility (in units of years$^{-1/2}$ is 0.29. Assume that the continuous rate of interest over the lifetime of the option is 4.34 percent. The value of the option is to be obtained by numerically solving a terminal-boundary value problem for a PDE. The PDE is the Black-Scholes PDE transformed by the change of variable $S = e^x$, where S is the stock price.

(a) Write down the PDE for the value of the option as a function of the variables x and t, where the units of t are in years.

(b) The PDE is to be numerically solved in the region $a < x < b,\ 0 < t < T$. What are suitable values for a, b, T? Explain your answer.

(c) What should the terminal and boundary conditions for the PDE be? Explain your answer.

(d) Assume you decide to numerically solve the terminal-boundary value problem for the PDE by using the explicit Euler method. For the x increment you take $\Delta x = 1/100$. Find an appropriate value for the t increment Δt. How accurate would you expect your computed value for the option to be i.e. how small the error from the exact Black-Scholes price? Explain your answers.

Note: In your answers to the above you should use actual *numerical values*, not letters denoting parameters (like T, σ etc).

(a) Soln: The PDE for $u(x,t)$ is

$$u_t + \frac{1}{2}\sigma^2 u_{xx} + (r - \frac{1}{2}\sigma^2)u_x - ru = 0, \quad t < T.$$

Taking $r = 0.0434,\ \sigma = 0.29,\ T = 1/3$, the equation becomes

$$u_t + 0.042u_{xx} + 0.0014u_x - 0.434u = 0, \quad t < 1/3.$$

(b) Soln: $a = log(K) + 3\sigma\sqrt{T},\ b = log(K) - 3\sigma\sqrt{T}$. With $K = 23$ we have $a = 3.638,\ b = 2.633$.

(c) Soln: The terminal condition is $u(x,T) = \max(K - e^x, 0)$, so $u(x, 1/3) = \max(23 - e^x, 0)$. The boundary conditions are $u(b,t) = 0$ and $u(a,t) = Ke^{-r(T-t)} - e^a$.

(d) Soln: The explicit Euler method gives

$$\frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} + \frac{1}{2}\sigma^2 \frac{u(x+\Delta x,t) + u(x-\Delta x,t) - 2u(x,t)}{(\Delta x)^2} + (r - \frac{1}{2}\sigma^2)\frac{u(x+\Delta x,t) - u(x-\Delta x,t)}{2\Delta x} - ru(x,t) = 0.$$

Rewriting this as an explicit expression for $u(x, t-\Delta t)$ in terms of quantities already computed we have

$$u(x,t-\Delta t) = au(x,t) + bu(x+\Delta x,t) + cu(x-\Delta x,t),$$

where $a = 1 - \sigma^2\alpha - r\Delta t,\ b = \sigma^2\alpha/2 + (r - \frac{1}{2}\sigma^2)\alpha\Delta x/2,\ c = \sigma^2\alpha/2 - (r - \frac{1}{2}\sigma^2)\alpha\Delta x/2$ and $\alpha = \Delta t/(\Delta x)^2$. Note that $a + b + c = 1 - r\Delta t = 1 + O(\Delta t)$ and hence the numerical scheme is stable provided all of a, b, c are positive. For Δx small this occurs provided $\sigma^2\alpha < 1$. Thus an appropriate value for Δt is $\Delta t \sim (\Delta x/\sigma)^2 = 0.001189$, using $\Delta x = 1/100$. If today's stock price corresponds to a grid point then the accuracy is $O[(\Delta x)^2]$. If it does not correspond to a grid point it is necessary to do spline interpolation of grid point values to obtain second order accuracy. With second order accuracy the error in the numerical solution should be around $(\Delta x)^2 = 10^{-4}$.

(2) We have written some computer code to find the value of a European put option with strike price 37, and wish to alter the code to find the value of the corresponding American option. If $u(x,t)$ denotes the value of the European option at time t (in years) when the stock price is $S = e^x$, then the numerically computed value $u_{n,m}$ approximates the value of $u(0.02n, 0.004m)$. The explicit Euler method yields the algorithm:

$$u_{n,m-1} = 0.37482\ u_{n,m} + 0.313875\ u_{n+1,m} + 0.311125\ u_{n-1,m}\ .$$

(a) Find the values of the continuous interest rate and the stock volatility (with the time unit being years) which have been used to price the option. Explain your answer.
(b) Show how to alter the algorithm above to price the American option. Explain your answer.
(c) Suppose we have stored the values of $u_{n,m}$. Write an algorithm to determine the location of the no-exercise boundary for the American option. Explain your answer.

(a) Soln: We see that $\Delta x = 0.02,\ \Delta t = 0.004$, whence $\alpha = 10$. Now using the formula $a + b + c = 1 - r\Delta t$, we conclude that $0.004r = 1.8 * 10^{-4}$, so $r = 0.045$. We can find the value of σ now from the value of a, whence $1 - 10\sigma^2 - r\Delta t = 0.37482$. Hence $\sigma^2 = 0.0625$ and so $\sigma = 0.25$.

(b) Soln: The modified algorithm is:

$$\begin{aligned} u_{n,m-1} &= 0.37482\ u_{n,m} + 0.313875\ u_{n+1,m} + 0.311125\ u_{n-1,m}\ , \\ u_{n,m-1} &= \max[u_{n,m-1}, 37 - \exp(0.02n)]. \end{aligned}$$

(c) Soln: Setting $S(n) = \exp(0.02n)$ then the following would work:

for $m = 1 : M$
counter= 0;
for $n = 2 : N$
if $u_{n,m} == 37 - S(n)$
counter=counter+1;
end
end
Exercise(m) =counter;
end

Now plot $t = m\Delta t$ against $S(Exercise(m))$.

(3) We have been given some data from a million i.e. 10^6 simulations of the Monte-Carlo method to compute the value of an option. The sum of the million values we obtained is 1340389, and the sum of the squares of the million values we obtained is 7660517.

(a) Find the estimated value of the option from the million simulations.
(b) Estimate the *percentage* error for the option value given in (a).
(c) Does it seem reasonable that the data given above came from a million Monte Carlo simulations? Explain your answer.

(a) Soln: The value of the option is the sample mean $V_1 = 1340389/10^6 = 1.340389$.

(b) Soln: We have now that the sample mean of the square of the option value is $V_2 = 7660517/10^6 = 7.660517$. Hence if ε_N is the standard error for the $N = 10^6$ simulations, then $\varepsilon_N = \sqrt{(V_2 - V_1^2)/N} = 0.00242$, whence the percentage error is $0.00242/1.340389 = 0.0018$.

(c) Soln: The standard error in Monte Carlo should be $O((1/\sqrt{N})$ for N simulations. Since the standard error in this case is of order 10^{-3} it seems reasonable that $N = 10^6$.

(4) We wish to estimate the value of a basket option on three stocks. The payoff on the option depends only on the *arithmetic* average of the three stocks at the expiration of the option, which is 6 months from today. The risk neutral evolution of the three stocks is modelled as follows:

$$\begin{cases} \frac{dS_{1,t}}{S_{1,t}} = 0.047\ dt + 0.29\ dW_{1,t}, \\ \frac{dS_{2,t}}{S_{2,t}} = 0.047\ dt + 0.34\ dW_{2,t}, \\ \frac{dS_{3,t}}{S_{3,t}} = 0.047\ dt + 0.28\ dW_{3,t}, \end{cases}$$

where the $W_{1,t},\ W_{2,t},\ W_{3,t},\ t > 0$, are correlated Brownian motions with correlation matrix $\rho = [\rho_{i,j}]$, and $\rho_{i,j}dt = E[dW_{i,t}\ dW_{j,t}]$. The units of time t are in years. The values of the three stocks today are $S_{1,0} = 24,\ S_{2,0} = 29,\ S_{3,0} = 21$.

(a) The matrix ρ is given by the formula:

$$\rho = \begin{bmatrix} * & -0.54 & * \\ * & * & 0.28 \\ 0.37 & * & * \end{bmatrix}$$

Find the numerical values of the entries in the matrix which are denoted by '$*$'. Explain your answer.

(b) We use a control variate for a variance reduction method for the Monte-Carlo simulation to estimate the price of the basket option. The control variate is the *geometric* average of the three stocks at the expiration of the option. Show that the control variate is the exponential of a Gaussian variable and compute the mean and standard deviation of this Gaussian variable.

(c) Suppose that for the standard Monte-Carlo simulation the value of the arithmetic option is 1.3718, the value of the geometric option is 1.1197, and $\beta = 1.0984$. Find the improved value of the arithmetic option if the *exact* value of the geometric option is 1.1232.

(d) If in addition you are given that the variance of the arithmetic payoff is 5.7139 and the variance of the geometric payoff is 4.5155, find the coefficient of correlation for the two random variables corresponding to the arithmetic payoff and the geometric payoff.

(a) Soln: The matrix is ρ is symmetric with diagonal entries 1. Hence $\rho_{1,1} = 1,\ \rho_{3,2} = 0.28$, etc.

(b) Soln: If $S(t)$ is the control variate then

$$S(t) = S(0) \exp\left\{(r - [\sigma_1^2 + \sigma_2^2 + \sigma_3^2]/6)t + \xi(t)\right\},$$

where $S(0)^3 = S_{1,0}S_{2,0}S_{3,0} = 24 * 29 * 21$, and $\xi(t) = [\sigma_1 W_1(t) + \sigma_2 W_2(t) + \sigma_3 W_3(t)]/3$. Evidently $\xi(t)$ is Gaussian with zero mean and variance $\text{Var}[\xi(t)] = t{*}\sigma{*}\rho{*}\sigma^T/9$ where $\sigma = [\sigma_1, \sigma_2, \sigma_3] = [0.29, 0.34, 0.28]$. At $t = 1/2$ the mean of $\log S(t)$ is 3.1969 and the variance is 0.0158.

(c) Soln: Improved value is: $1.3718 + 1.0984(1.1232 - 1.1197) = 1.3756$.

(d) Soln: Let ρ be the coefficient of correlation. Then $\rho = \beta\sqrt{4.5155/5.7139} = 0.9765$.

(5) Today's bond prices with face value 100 and maturities from 1 up to 10 years is given by the following table:

Maturity	1	2	3	4	5	6	7	8	9	10
Price	95.24	91.31	87.31	82.92	78.89	75.35	71.83	68.27	64.71	61.51

The values of interest rate derivatives are to be computed using the Hull-White tree with $\sigma = 0.014$, $a = 0.11$ and the bond data above. We use linear interpolation of the bond prices above to estimate the values of bonds with maturity which is not an integer number of years.

(a) Find the value of Δr corresponding to $\Delta t = 1/8$.

(b) Find all the possible values for the interest rate r on the Hull-White tree corresponding to the times $t = 0$ and $t = 1/8$.

(c) Suppose that we wish to find the value of an interest rate cap of 7 percent on a 10 year loan with principal 100. Let $\alpha^m = 0.049$ correspond to time $t = m\Delta t$ where $\Delta t = 1/8$ and $m = 16$. Find all nodes (m, j), $|j| \leq m$, $m = 16$, for which the caplet corresponding to interest over the time interval from 2 years to 2 and 1/8 years is nonzero.

(a) Soln: Use $(\Delta t)/(\Delta r)^2 = 1/3\sigma^2$, whence $\Delta r = 0.0086$.

(b) Soln: Using $P(0,0) = 1$, $P(0,1) = 0.9524$, we have by linear interpolation $P(0,1/8) = 0.9940$, $P(0,1/4) = 0.9881$. We have with $\Delta t = 1/8$, then

$$\exp[-\alpha(0)\Delta t] = 0.9940, \quad \Rightarrow \alpha(0) = 0.0481.$$

To get $\alpha(1)$ we need to solve

$$0.9940 \exp\{-\alpha(1)\Delta t\}[2/3 + \exp(\Delta r \Delta t)/6 + \exp(-\Delta r \Delta t)/6] = 0.9881,$$

whence $\alpha(1) = 0.0476$. For $t = 0$ the value of r is $r = \alpha(0) = 0.0481$. For $t = 1/8$ there are three possible values of r, given by $r = \alpha(1)$, $r = \alpha(1) \pm \Delta r$, which is $r = 0.0476$, $r = 0.0562$, $r = 0.0390$.

(c) Soln: In Hull-White we take $J = [1 - \sqrt{2/3}]/a\Delta t = 13$. For the caplet to have a positive value we need $\alpha^m + j\Delta r > 0.07$, which gives $j > 2.44$. With $\Delta t = 1/8$ then $t = 2$ corresponds to $m = 16$ so $m \geq J$. Hence the nodes are $j = 3, 4, ..., 13$.