Math 623 (IOE 623), Fall 2008: Final exam

Name:

Student ID:

This is a closed book exam. You may bring up to ten one sided A4 pages of notes to the exam. You may also use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. State explicitly any additional assumptions you make.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

(1) Suppose I wish to compute the value of a *European Straddle* option on a stock. The option pays $\Phi(S)$ if exercised when the stock price is S, where

$$\Phi(S) = \begin{cases} 52 - S & \text{if } S \leq 52 \\ S - 52 & \text{if } S \geq 52. \end{cases}$$

The current value of the stock is 53 and it expires 9 months from today. The value of the option is to be obtained by numerically solving a terminal-boundary value problem for a PDE. The PDE is the Black-Scholes PDE transformed by the change of variable $S = e^x$, where S is the stock price.

(a) The value of the option as a function of the variables x and t with the units of t in years, is $u(x,t)$ where $u(x,t)$ satisfies the PDE,

$$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} + .0392 \frac{\partial^2 u}{\partial x^2} - .035u = 0.$$

Find the value of the coefficient λ in the PDE.

(b) The PDE is to be numerically solved in some region $a < x < b,\ 0 < t < T$, using the explicit Euler method. Setting $u(n\Delta x, m\Delta t) = u_n^m$ then u_n^m satisfies the recurrence equation

$$u_n^{m-1} = Au_n^m + Bu_{n-1}^m + Cu_{n+1}^m, \quad m \leq M.$$

Taking $\Delta t = 0.006, \Delta x = 0.02$ find numerical values for M and u_{200}^M.

(c) Calculate the numerical value of the coefficient A in the scheme given in (b).

(d) Using the value of A you computed in (c), determine whether the scheme in (b) is stable. Explain your answer.

Solution (a):

$$\lambda = r - \sigma^2/2 = 0.035 - 0.0392 = -0.0042.$$

Solution (b): Since $M\Delta t = T$ we have $M = 0.75/0.006 = 125$. We also have that

$$u_{200}^M = \exp[200(.02)] - 52 = 54.60 - 52 = 2.60.$$

Solution (c):

$$A = 1 - \sigma^2 \Delta t/(\Delta x)^2 - r\Delta t = 1 - 2(0.0392)(0.006)/(0.02)^2 - (0.035)(0.006) = -0.17621.$$

Solution (d): The scheme is not stable since $A < -r\Delta t$ and $A+B+C = 1-r\Delta t$, so $|A|+|B|+|C| > 1$.

(2) We wish to find the value of an American put option which expires 8 months from today. The put option is on a stock which has current price 31 and volatility 0.34. The strike price for the option is 28 and the risk free rate of interest is 4.2%. The value of the option is to be obtained by numerically solving a terminal-boundary value problem for a PDE. The PDE is the Black-Scholes PDE transformed by the change of variable $S = e^x$, where S is the stock price. Let $u(x, t)$ be the value of the option corresponding to the stock price $x = \log(S)$ and time $t < T$, where T is the expiration date of the option.

(a) You have some numerical code which solves the PDE in $u(x, t)$ for x in the finite interval $1 < x < 8$ and $t < T$. Write down the boundary conditions for $u(x, t)$ when $x = 1$ and $x = 8$.

(b) A colleague suggests to you that the numerical code can be made more efficient without loss of accuracy by reducing the interval $1 < x < 8$ to a smaller interval $2.5 < x < 4$. Would you agree or disagree? Explain why.

(c) Suppose the numerical code stores the values of $u(0.01n, 0.004m)$ as u_n^m. Write an algorithm to determine and plot the location of the no-exercise boundary for the American option.

Solution (a):

$$u(1, t) = 28 - e^1 = 25.28, \quad u(8, t) = 0.$$

Solution (b): With $K = 28,\ \sigma = 0.34,\ T = 2/3$, we have $\log K - 3\sigma\sqrt{T} = 2.50,\ \log K + 3\sigma\sqrt{T} = 4.17$. The interval $2.50 < x < 4.17$ gives the value of the option with an error approximately $2 - 2P(Z < 3) = 2 - 2(0.99865) = 0.0027$ where Z is the standard normal variable. Hence I would agree with my colleague that the interval $2.5 < x < 4$ is sufficient.

Solution (c): Suppose $N_{\min}$ is the minimum value of n. Then we could have:

$S = \text{zeros}(1, M); \quad t = 0.004\ \text{ones}(1, M);$
for m=1:M
$n = N_{\min}$; while $u_n^m = 28 - \exp(0.01n), n = n + 1,$
end
$S(m) = \exp(0.01n);$
end
plot(t, S);

(3) I wish to use the Monte-Carlo method to find the value of an option on a stock S_t which evolves according to geometric Brownian motion with volatility $\sigma = 0.27$. The risk free rate of interest is $r = 0.041$ and the expiration date of the option is 6 months from today. The payoff $\Phi(S)$ of the option is given by the formula,

$$\Phi(S) = \begin{cases} 10 & \text{if } S \le 25 \\ 35 - S & \text{if } 25 \le S \le 35 \\ 0 & \text{if } S \ge 35. \end{cases}$$

(a) Suppose today's stock price is 29 and $\xi = -0.1814$ is a random value of the standard normal variable. Find the value of the option in the Monte Carlo method which corresponds to this random value of ξ.

(b) Explain why it is appropriate to use the variance reduction technique of antithetic variables to value the option.

(c) Find the value of the option corresponding to the random value of ξ in (a) when one uses the variance reduction method of (b).

Solution (a): With $S_0 = 29,\ r = 0.041,\ \sigma = 0.27,\ T = 1/2$, we have

$$S(T) = S_0 \exp[(r - 0.5\sigma^2)T + \sigma\xi\sqrt{T}].$$

Taking $\xi = -0.1814$ we get $S(T) = 28.0766$ yielding an option value of $[35 - S(T)]\exp(-rT) = 6.7829$. alternatively one could take $\xi = 0.1814$, which yields $S(T) = 30.0903$ and a corresponding option value of 4.8101.

Solution (b): The use of antithetic variables is justified because:
(1) the standard normal variable has a symmetric distribution i.e. ξ and $-\xi$ have the same density;
(2) the payoff function $[35 - S(T)]\exp(-rT)$ is a monotonic (in fact decreasing) function of ξ.

Solution (c): In the variance reduction method the value of the option is the average of the values of the payoff for ξ and $-\xi$, which is $(6.7829 + 4.8101)/2 = 5.7965$.

(4) The current value of a stock is 24. The Heston stochastic volatility model for the evolution of the price S_t of the stock is used to estimate the value of a stock option with expiration 9 months from today. The Heston model is governed by the system of equations:

$$\begin{cases} dY_t &= [0.31 - 2.8\ Y_t]dt + \beta\sqrt{Y_t}\,dW_t \\ \frac{dS_t}{S_t} &= 0.032\,dt + \sqrt{Y_t}\,(\rho\,dW_t + \sqrt{1-\rho^2}\,dZ_t)\ . \end{cases}$$

Here W_t and Z_t are independent (uncorrelated) Brownian motions under the risk neutral measure Q.

(a) In order to price the option the system of equations is to be solved by the explicit Euler method applied to stochastic differential equations. Setting $S^m = S_{m\Delta t},\ Y^m = Y_{m\Delta t},\ m = 0, 1, 2...$, write down appropriate values for $S^0,\ Y^0$. Explain your answer.

(b) Suppose $\rho = -0.3,\ \beta = 0.5,\ \Delta t = 0.01$, and you have computed $S^m,\ Y^m$ for $m = 0, 1, ...20$, with $S^{20} = 18,\ Y^{20} = 0.19$. You are given the two values $\xi = 0.2316,\ \eta = 0.3517$ of independently sampled standard normal variables. Find the corresponding values of $S^{21},\ Y^{21}$.

(c) For what range of parameter values β is this Heston model appropriate to use to value stock options? Explain your answer.

Solution (a): We take $S_0 = 24,\ Y_0 = 0.31/2.8 = 0.1107$. Note the initial value for Y_0 makes the drift $[0.31 - 2.8Y_t]$ in the SDE for Y_t equal to 0. Hence because of mean reversion Y_t will have this value on average, and so we choose it as our initial value Y_0.

Solution (b): We have

$$Y^{21} = Y^{20} + (0.31 - 2.8\ Y^{20})\Delta t + \beta\xi\sqrt{Y^{20}}\sqrt{\Delta t} = 0.1928,$$

$$S^{21} = S^{20}[1 + 0.032\Delta t + \sqrt{Y^{20}}\{\xi\rho + \eta\sqrt{1-\rho^2}\}\sqrt{\Delta t}] = 18.2145.$$

Solution (c): The model is appropriate if $\theta > \beta^2/2$, where $\theta = 0.31$ is the constant term in the drift of the SDE for Y_t . Thus we need $\beta < \sqrt{0.62} = 0.7874$. The reason for this is that when β violates this condition, then there is with probability 1 a (random) time τ such that $Y_t = 0$ for all $t > \tau$. Thus volatility is eventually zero with probability 1 which is not realistic.

(5) The Hull-White model with $\sigma = 0.0087$, $a = 0.14$, $\Delta t = 1/16$, and J the largest integer smaller than $\sqrt{3/(2a\Delta t)}$, is to be used to find the value of some interest rate derivatives. The lattice sites for the model are (m, j), $m = 0, 1, 2..$, $|j| \le \min\{m, J\}$. A lattice site (m, j) corresponds to time $t = m\Delta t$ and interest rate r_j^m.

(a) Suppose we know that $r_5^{24} = 0.023$. Find the value of r_2^{24}.

(b) Write down formulas for the transition probabilities $p_u(j)$, $p_s(j)$, $p_d(j)$ when $j = 13$. Give an explanation of why the formulas have the given values.

(c) The HW model is to be used to find the value of a swaption. The swaption consists of a prepayment option after 6 years on a 15 year loan with a notional principle of 100, where the rate is 4.5% payable annually. Let $V(m, j)$ denote the value of the swaption corresponding to the lattice site (m, j), so today's value of the prepayment option is $V(0, 0)$. Taking r_5^{24} to have the value in part (a), find $V(24, 5)$ given that $V(25, 5) = 1.5473$, $V(25, 6) = 1.5362$, $V(25, 4) = 1.5624$.

Solution (a): We have that $r_2^{24} = r_5^{24} - 3\Delta r$. Taking $\Delta r = \sigma\sqrt{3\Delta t} = 0.0038$, we obtain $r_2^{24} = 0.023 - 3(0.0038) = 0.0116$.

Solution (b): Observe that $\sqrt{3/(2a\Delta t)} = 13.09$, so $J = 13$. Hence we have the exceptional HW formulas for the transition probabilities. Thus $j = 13 = J$, and $p_u(J) = 0$, $p_s(J) = 1$, $p_d(J) = 0$.

Solution (c): We have by the discount formula that

$$V(24, 5) = \exp(-r_5^{24}\Delta t)[p_u(5)V(25, 6) + p_s(5)V(25, 5) + p_d(5)V(25, 4)].$$

Now $p_u(5)$ etc are given by the standard HW formulas, $p_u(j) = \frac{1}{6} + \frac{1}{2}[a^2 j^2 (\Delta t)^2 - aj\Delta t]$ etc. This yields $p_u(5) = 0.1457$, $p_s(5) = 0.1895$, $p_d(5) = 0.6648$, and consequently $V(24, 5) = 1.5463$.