

Math 623 (IOE 623), Winter 2008: Final exam

Name:

Student ID:

This is a closed book exam. You may bring up to ten one sided A4 pages of notes to the exam. You may also use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. State explicitly any additional assumptions you make.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

- (1) Suppose I wish to compute the value of a *European Butterfly Spread* option on a stock. The option pays $\Phi(S)$ if exercised when the stock price is S , where

$$\Phi(S) = \begin{cases} 0 & \text{if } S \leq 40 \\ S - 40 & \text{if } 40 \leq S \leq 50 \\ 60 - S & \text{if } 50 \leq S \leq 60 \\ 0 & \text{if } S \geq 60. \end{cases}$$

The current value of the stock is 48 and it expires 6 months from today. Its volatility (in units of $\text{years}^{-1/2}$) is 0.34. Assume that the continuous rate of interest over the lifetime of the option is 3.75 percent. The value of the option is to be obtained by numerically solving a terminal-boundary value problem for a PDE. The PDE is the Black-Scholes PDE transformed by the change of variable $S = e^x$, where S is the stock price.

- (a) Write down the PDE for the value of the option as a function of the variables x and t , where the units of t are in years.
- (b) The PDE is to be numerically solved in the region $a < x < b$, $0 < t < T$. Find suitable *numerical values* for a, b, T and explain your reasoning for this choice..
- (c) What should the terminal and boundary conditions for the PDE be? Explain your answer.
- (d) Assume you decide to numerically solve the terminal-boundary value problem for the PDE by using the explicit Euler method. How large can you take $\Delta t/(\Delta x)^2$ and the numerical scheme remain stable? Justify your answer.

- (2) We wish to find the value of an Asian option on a zero dividend stock which expires 9 months from today. The payoff on the option is the excess of the stock price at expiration over the continuous average of the stock price during the 9 month life span of the option. The current price of the stock is 35 and its volatility is 0.28 per annum. The risk free interest rate is 0.038 per annum.
- (a) Write down the two variable partial differential equation for a function w of (t, ξ) on intervals $0 < t < 3/4$, $0 < \xi < \xi_{\max}$, together with boundary and terminal conditions, which one needs to solve to compute the value of the Asian option. Find a suitable numerical value for ξ_{\max} .
 - (b) The solution $w(t, \xi)$ of the equation can be written as an expectation value $w(t, \xi) = E[\Phi(\xi(T)) \mid \xi(t) = \xi]$, where $\xi(t)$ is the solution of a stochastic differential equation. Write down a formula for the function $\Phi(\xi)$ and also the stochastic equation which $\xi(t)$ must satisfy.
 - (c) Suppose I wanted to find the the value of the option using the Monte-Carlo method. Explain carefully how I could do this, either using the representation in (b) or an alternative method.

- (3) I wish to find the value of an option on a stock S_t which evolves according to geometric Brownian motion with volatility $\sigma = 0.26$. The risk free rate of interest is $r = 0.038$ and the expiration date of the option is 6 months from today. The payoff $\Phi(S)$ of the option is given by the formula,

$$\Phi(S) = \begin{cases} 15 & \text{if } S \leq 30 \\ 45 - S & \text{if } 30 \leq S \leq 45 \\ 0 & \text{if } S \geq 45. \end{cases}$$

- (a) Suppose today's stock price is 40. We wish to find the value of the option by implementing the Monte Carlo method. Thus we write the value V of the option as $V = E[\Psi(\xi)]$ where ξ is a standard normal variable. Find a formula for the function $\Psi(\xi)$.
- (b) Suppose that today's stock price is 70. Why does it make sense to look for a variance reduction method to find the value of the option?
- (c) If we use *importance sampling* as a variance reduction technique then $V = E[\Psi_1(\xi)]$ for some suitable function Ψ_1 different from Ψ . Obtain a formula for Ψ_1 .

- (4) We wish to use the Hull-White model to find the value of an interest rate cap on a 15 year loan with a notional principle of 100, where the cap is 5.5%. The parameters in the model are $\sigma = 0.015$, $a = 0.2$. The model has been calibrated to today's yield curve by taking $\Delta t = 1/32$ and using interpolation. The resulting α values corresponding to the time $m\Delta t$ are denoted α^m . The value of the cap is given by $V(0, 0)$, where $V(m, j)$ satisfies the recurrence

$$V(m, j) = \exp[-r(m, j)]\{\text{Cap}(m, j) + p_u(j)V(m+1, j+1) + p_s(j)V(m+1, j) + p_d(j)V(m+1, j-1)\},$$

if $|j| \leq \min(m, J-1)$, and $V(M, j) = 0$, $|j| \leq \min(M, J)$.

- (a) Find the value of M and the sum $p_u(j) + p_s(j) + p_d(j)$.
- (b) Obtain a formula for $r(m, j)$ which can be explicitly computed once we know m, j, α^m .
- (c) Obtain a formula for $\text{Cap}(m, j)$ which can be explicitly computed once we know m, j, α^m .
- (d) Suppose that today's yield curve varies with a minimum of 2.5% and a maximum of 6% over the 15 year period. Estimate how likely it is that the spot rate in the Hull-White model ever exceeds 10%.

- (5) We wish to find the value of a bond call option using the BDT interest rate model. The expiration date of the option is 6 years from today. The payoff is the excess over 55 of the value then of the bond with face value 100 and maturity 15 years from today. We take $\Delta t = 1/16$ in the model and assume the model has been calibrated to today's yield curve and volatilities, giving parameter values r_0^m , β^m , $m = 0, 1, \dots$ corresponding to the time $m\Delta t$.
- (a) Write down an algorithm which we can use to compute the value of the bond call option.
 - (b) Suppose the parameter values corresponding to $m = 80$ are $r_0^m = 0.0025$ $\beta^m = 0.1605$. Find from this the *mean* of the random variable $r(t)$ for some suitable t .
 - (c) With the same data as in part (b) find the *variance* of $\log r(t)$.

