

Math 623 (IOE 623), Winter 2008: Final exam

Name:

Student ID:

This is a closed book exam. You may bring up to ten one sided A4 pages of notes to the exam. You may also use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. State explicitly any additional assumptions you make.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

- (1) Suppose I wish to compute the value of a *European Butterfly Spread* option on a stock. The option pays $\Phi(S)$ if exercised when the stock price is S , where

$$\Phi(S) = \begin{cases} 0 & \text{if } S \leq 40 \\ S - 40 & \text{if } 40 \leq S \leq 50 \\ 60 - S & \text{if } 50 \leq S \leq 60 \\ 0 & \text{if } S \geq 60. \end{cases}$$

The current value of the stock is 48 and it expires 6 months from today. Its volatility (in units of years^{-1/2}) is 0.34. Assume that the continuous rate of interest over the lifetime of the option is 3.75 percent. The value of the option is to be obtained by numerically solving a terminal-boundary value problem for a PDE. The PDE is the Black-Scholes PDE transformed by the change of variable $S = e^x$, where S is the stock price.

- Write down the PDE for the value of the option as a function of the variables x and t , where the units of t are in years.
- The PDE is to be numerically solved in the region $a < x < b$, $0 < t < T$. Find suitable *numerical values* for a, b, T and explain your reasoning for this choice..
- What should the terminal and boundary conditions for the PDE be? Explain your answer.
- Assume you decide to numerically solve the terminal-boundary value problem for the PDE by using the explicit Euler method. How large can you take $\Delta t/(\Delta x)^2$ and the numerical scheme remain stable? Justify your answer.

Solution (a):

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + (r - \frac{1}{2}\sigma^2) \frac{\partial u}{\partial x} - ru = 0.$$

Solution (b): Evidently $T = 0.5$ and $\sigma = 0.34$. Writing $X(t) = \log S(t)$, then $X(T)$ is a Gaussian variable with mean $X(0) + (r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. The option is in the money at the expiration date T if $\log 40 < X(T) < \log 60$. Hence if $X(0) < \log 40 - 3\sigma\sqrt{T}$ or $X(0) > \log 60 + 3\sigma\sqrt{T}$, then for the option to be in the money at time T requires $X(T)$ to be 3 standard deviations from its mean. Since the probability of this occurring is of order 10^{-3} , we may take $a = \log 40 - 3\sigma\sqrt{T} = 2.97$, and $b = \log 60 + 3\sigma\sqrt{T} = 4.82$.

Solution (c): The terminal condition on $u(x, t)$ is: $u(x, T) = 0$, $x < \log 40$, $u(x, T) = e^x - 40$, $\log 40 < x < \log 50$, $u(x, T) = 60 - e^x$, $\log 50 < x < \log 60$, $u(x, T) = 0$, $x > \log 60$. The boundary conditions are $u(a, t) = u(b, t) = 0$, $0 < t < T$.

Solution (d): The explicit Euler method for the PDE in (a) is given by

$$u(x, t - \Delta t) = \left[1 - r\Delta t - \sigma^2 \frac{\Delta t}{(\Delta x)^2}\right] u(x, t) + \left[\sigma^2 \frac{\Delta t}{2(\Delta x)^2} + (r - \frac{1}{2}\sigma^2) \frac{\Delta t}{2(\Delta x)}\right] u(x + \Delta x, t) \\ + \left[\sigma^2 \frac{\Delta t}{2(\Delta x)^2} - (r - \frac{1}{2}\sigma^2) \frac{\Delta t}{2(\Delta x)}\right] u(x - \Delta x, t).$$

The coefficients sum up to $1 - r\Delta t \leq 1$, so for stability all we need is that all the coefficients are positive. Thus we require $\Delta t/(\Delta x)^2 < 1/\sigma^2 = 8.65$.

- (2) We wish to find the value of an Asian option on a zero dividend stock which expires 9 months from today. The payoff on the option is the excess of the stock price at expiration over the continuous average of the stock price during the 9 month life span of the option. The current price of the stock is 35 and its volatility is 0.28 per annum. The risk free interest rate is 0.038 per annum.
- (a) Write down the two variable partial differential equation for a function w of (t, ξ) on intervals $0 < t < 3/4$, $0 < \xi < \xi_{\max}$, together with boundary and terminal conditions, which one needs to solve to compute the value of the Asian option. Find a suitable numerical value for $\xi_{\max}$.
 - (b) The solution $w(t, \xi)$ of the equation can be written as an expectation value $w(t, \xi) = E[\Phi(\xi(T)) | \xi(t) = \xi]$, where $\xi(t)$ is the solution of a stochastic differential equation. Write down a formula for the function $\Phi(\xi)$ and also the stochastic equation which $\xi(t)$ must satisfy.
 - (c) Suppose I wanted to find the the value of the option using the Monte-Carlo method. Explain carefully how I could do this, either using the representation in (b) or an alternative method.

Solution (a): The PDE is

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2\xi^2\frac{\partial^2 w}{\partial \xi^2} + (1 - r\xi)\frac{\partial w}{\partial \xi} = 0.$$

The terminal condition is $w(\xi, T) = \max(1 - \xi/T, 0)$. The boundary conditions are

$$w(\xi_{\max}, t) = 0; \quad \frac{\partial w}{\partial t} + \frac{\partial w}{\partial \xi} = 0, \quad \xi = 0.$$

We may take $\xi_{\max} = T \exp(3\sigma\sqrt{T}) = 1.55$.

Solution (b): The function Φ is given by $\Phi(\xi) = \max(1 - \xi/T, 0)$. The stochastic equation which $\xi(t)$ satisfies is $d\xi(t) = (1 - r\xi)dt + \sigma\xi dW(t)$, where $W(t)$ is Brownian motion.

Solution (c): We will use the representation in (b) for the MC method. First solve the stochastic differential equation by using the explicit Euler method. Thus choose Δt small with $T = M\Delta t$ for some integer M . Then with M independent $N(0, 1)$ variables $\eta_m, m = 0, \dots, M-1$, we set

$$\xi_{m+1} = \max \left[0, \xi_m + (1 - r\xi_m)\Delta t + \sigma\xi_m\eta_m\sqrt{\Delta t} \right], \quad m = 0, 1, \dots, M-1,$$

where $\xi_0 = 0$, $\sigma = 0.28$, $r = 0.038$. The option value corresponding to this sampling is $V = S(0)\Phi(\xi_M) = 35\Phi(\xi_M)$. Now do this independently N times and take the average option value to obtain the value of the option correct to $O(1/\sqrt{N})$.

- (3) I wish to find the value of an option on a stock S_t which evolves according to geometric Brownian motion with volatility $\sigma = 0.26$. The risk free rate of interest is $r = 0.038$ and the expiration date of the option is 6 months from today. The payoff $\Phi(S)$ of the option is given by the formula,

$$\Phi(S) = \begin{cases} 15 & \text{if } S \leq 30 \\ 45 - S & \text{if } 30 \leq S \leq 45 \\ 0 & \text{if } S \geq 45. \end{cases}$$

- (a) Suppose today's stock price is 40. We wish to find the value of the option by implementing the Monte Carlo method. Thus we write the value V of the option as $V = E[\Psi(\xi)]$ where ξ is a standard normal variable. Find a formula for the function $\Psi(\xi)$.
- (b) Suppose that today's stock price is 70. Why does it make sense to look for a variance reduction method to find the value of the option?
- (c) If we use *importance sampling* as a variance reduction technique then $V = E[\Psi_1(\xi)]$ for some suitable function Ψ_1 different from Ψ . Obtain a formula for Ψ_1 .

Solution (a): We have that $\Psi(\xi) = \Phi(S(T))$, where

$$S(T) = S_0 \exp \left[\left(r - \frac{1}{2} \sigma^2 \right) T + \sigma \xi \sqrt{T} \right], \quad S_0 = 40, \quad r = 0.038, \quad \sigma = 0.26, \quad T = 0.5.$$

Solution (b): For the option to be in the money we need $S(T) < 45$, whence

$$\left(r - \frac{1}{2} \sigma^2 \right) T + \sigma \xi \sqrt{T} < -\log(70/45).$$

Thus $\xi < -2.41$, whence the probability of getting a non-zero value on a trial is around 10^{-2} . This is the situation where the use of a variance reduction technique is justified.

Solution (c): For importance sampling we have $\Psi_1(\xi) = \exp[\xi\alpha - \alpha^2/2]\Psi(\xi - \alpha)$. The parameter α is chosen so that when $\xi = 0$ the option is at the threshold of being in the money. Thus α is determined by the equation

$$70 \exp \left[\left(r - \frac{1}{2} \sigma^2 \right) T - \sigma \alpha \sqrt{T} \right] = 45,$$

which as in (b) yields $\alpha = 2.41$.

- (4) We wish to use the Hull-White model to find the value of an interest rate cap on a 15 year loan with a notional principle of 100, where the cap is 5.5%. The parameters in the model are $\sigma = 0.015$, $a = 0.2$. The model has been calibrated to today's yield curve by taking $\Delta t = 1/32$ and using interpolation. The resulting α values corresponding to the time $m\Delta t$ are denoted α^m . The value of the cap is given by $V(0, 0)$, where $V(m, j)$ satisfies the recurrence

$$V(m, j) = \exp[-r(m, j)]\{\text{Cap}(m, j) + p_u(j)V(m+1, j+1) + p_s(j)V(m+1, j) + p_d(j)V(m+1, j-1)\},$$

if $|j| \leq \min(m, J-1)$, and $V(M, j) = 0$, $|j| \leq \min(M, J)$.

- (a) Find the value of M and the sum $p_u(j) + p_s(j) + p_d(j)$.
- (b) Obtain a formula for $r(m, j)$ which can be explicitly computed once we know m, j, α^m .
- (c) Obtain a formula for $\text{Cap}(m, j)$ which can be explicitly computed once we know m, j, α^m .
- (d) Suppose that today's yield curve varies with a minimum of 2.5% and a maximum of 6% over the 15 year period. Estimate how likely it is that the spot rate in the Hull-White model ever exceeds 10%.

Solution (a): We have $M = 15/\Delta t = 15 * 32 = 480$. Conservation of probability yields $p_u(j) + p_s(j) + p_d(j) = 1$.

Solution (b): We have $r(m, j) = (\alpha^m + j\Delta r)\Delta t = (\alpha^m + j\sigma\sqrt{3\Delta t})\Delta t$. The numerical value of this is $\alpha^m/32 + 10^{-4} * 1.435j$.

Solution (c): We have $\text{Cap}(m, j) = 100\Delta t \max[0, \alpha^m + j\sigma\sqrt{3\Delta t} - 0.055]$. From (b) we see the numerical value of this is $100 \max[0, \alpha^m/32 + 10^{-4} * 1.435j - 0.0017]$.

Solution (d): For the spot rate to exceed 10% we need $r^*(t) > 0.10 - 0.06 = 0.04$ for some t . The standard deviation of $r^*(t)$ is $\sigma/\sqrt{2a} = 0.024$. Thus $r^*(t) > 0.04$ requires approximately 1.67 standard deviations from its mean of 0. The probability of this is 5%.

- (5) We wish to find the value of a bond call option using the BDT interest rate model. The expiration date of the option is 6 years from today. The payoff is the excess over 55 of the value then of the bond with face value 100 and maturity 15 years from today. We take $\Delta t = 1/16$ in the model and assume the model has been calibrated to today's yield curve and volatilities, giving parameter values r_0^m , β^m , $m = 0, 1, \dots$ corresponding to the time $m\Delta t$.
- Write down an algorithm which we can use to compute the value of the bond call option.
 - Suppose the parameter values corresponding to $m = 80$ are $r_0^m = 0.0025$ $\beta^m = 0.1605$. Find from this the *mean* of the random variable $r(t)$ for some suitable t .
 - With the same data as in part (b) find the *variance* of $\log r(t)$.

Solution (a): Define M by $M\Delta t = 15$ so $M = 240$. Then the nodes of the BDT lattice are (m, j) , $0 \leq j \leq m$, $m = 0, \dots, M$. The time corresponding to the lattice point (m, j) is $m\Delta t$, and the spot rate is $r(m, j) = r_0^m \exp[2\beta^m j \sqrt{\Delta t}]$. First we compute the value of the bond with maturity 15 years. Let $P(m, j)$ be the value of the bond at the lattice point (m, j) . Then $P(M, j) = 1$, $0 \leq j \leq M$. For $m < M$ we compute $P(m, j)$ by the recurrence

$$P(m, j) = \exp[-r(m, j)\Delta t] \{0.5P(m+1, j) + 0.5P(m+1, j+1)\}, \quad 0 \leq j \leq m.$$

Now let $V(m, j)$, $0 \leq j \leq m$, $0 \leq m \leq M_0$, with $M_0\Delta t = 6$ be the value of the bond call option at node (m, j) . Then $V(M_0, j) = \max[0, P(M_0, j) - 55]$, $0 \leq j \leq M_0$. We compute the value of the option on the other nodes $m < M_0$ by the recurrence,

$$V(m, j) = \exp[-r(m, j)\Delta t] \{0.5V(m+1, j) + 0.5V(m+1, j+1)\}, \quad 0 \leq j \leq m.$$

The value of the option today is $V(0, 0)$.

Solution (b): There is the formula,

$$E[r(m\Delta t)] = r_0^m \{0.5[1 + \exp(2\beta^m \sqrt{\Delta t})]^m\}.$$

With $m = 80$ the numerical value of this is 0.0661.

Solution (c): The variance is given by the formula,

$$\text{Var}[\log r(t)] = (\beta^m)^2 t.$$

The numerical value for this when $m = 80$ is 0.129.

