Math 623 (IOE 623), Winter 2009: Final exam

Name:

Student ID:

This is a closed book exam. You may bring up to ten one sided A4 pages of notes to the exam. You may also use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. State explicitly any additional assumptions you make.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

(1) Suppose I wish to compute the value of a *European Bear Spread* option on a stock. The option pays $\Phi(S)$ if exercised when the stock price is S, where

$$\Phi(S) = \begin{cases} 10 & \text{if } S \le 25 \\ 35 - S & \text{if } 25 \le S \le 35 \\ 0 & \text{if } S \ge 35. \end{cases}$$

The current value of the stock is 29 and it expires 8 months from today. The value of the option is to be obtained by numerically solving a terminal-boundary value problem for a PDE. The PDE is the Black-Scholes PDE transformed by the change of variable $S = e^x$, where S is the stock price. The stock volatility is 0.38 and risk free interest rate is 0.04. The value of the option as a function of the variables x and t with the units of t in years, is $u(x,t)$ where $u(x,t)$ is to be computed by solving the terminal value problem

$$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \omega u = 0,$$

on the interval $a < x < b,\ 0 \le t \le T$, where $t = 0$ denotes today, with terminal condition $u(x,T) = \Psi(x),\ a < x < b$, and boundary conditions $u(a,t) = f(t),\ u(b,t) = g(t),\ 0 \le t \le T$.

(a) Find the values of the coefficients $\lambda,\ \mu,\ \omega$ in the PDE.

(b) Find the value of $\Psi(3.40)$.

(c) Compute suitable values for $a,\ b$, and the corresponding boundary functions $f(t),\ g(t),\ 0 \le t \le T$. Explain your answers.

(d) In using the explicit Euler method to numerically solve the above terminal value problem, suppose one takes Δx to be given by $\Delta x = 0.05$. Is the corresponding error in the value of the option likely to be closer to 0.1 or 0.01? Explain your answer.

Solution (a): $\omega = -r = -0.04,\ \mu = \sigma^2/2 = (0.38)^2/2 = 0.0722,\ \lambda = r - \sigma^2/2 = -0.0322.$

Solution (b): $\Psi(3.40) = \Phi(S)$ where $S = \exp[3.40] = 29.96$. Hence $\Psi(3.40) = 35 - 29.96 = 5.04$.

Solution (c): Since the initial stock price of 29 is in the money we choose $a,\ b$ to be $3\sigma\sqrt{T}$ out of the money. Thus $a = \log(25) - 3\sigma\sqrt{T} = 2.881$ and $b = \log(35) + 3\sigma\sqrt{T} = 4.4862$. For boundary conditions we set $u(b,t) = 0$ and $u(a,t) = 10\exp[-r(T-t)] = 10\exp[-0.04(2/3 - t)]$.

Solution (d): The error is $O[(\Delta x)^2]$. Since $(\Delta x)^2 = 0.0025$ one would expect the error to be closer to 0.01.

(2) We wish to find the value of an Asian option on a zero dividend stock which expires 6 months from today. The payoff on the option is the excess of the stock price at expiration over the continuous average of the stock price during the 6 month life span of the option. The current price of the stock is 31. It has a continuous dividend payment rate of 0.015 per annum, and its volatility is 0.42 per annum. The risk free interest rate is 0.05 per annum.

(a) Write down the two variable partial differential equation for a function w of (t, ξ) on intervals $0 < t < 1/2,\ 0 < \xi < \xi_{\max}$, together with boundary and terminal conditions, which one needs to solve to compute the value of the Asian option.

(b) Find a suitable numerical value for $\xi_{\max}$. Explain your answer.

(c) Suppose you are given that the value of the option today i.e. $t = 0$ is 2.53. For what value of (ξ, t) does this determine $w(\xi, t)$? Find also the corresponding value of $w(\xi, t)$.

(d) Suppose that in numerically solving the terminal value problem for $w(\xi, t)$ by the explicit Euler method, we take $\Delta\xi = 0.025,\ \Delta t = 0.0038$. Do you expect the numerical solution to be stable? Explain your answer.

Solution (a): The PDE is

$$\frac{\partial w(\xi,t)}{\partial t} + \frac{1}{2}\sigma^2\xi^2\frac{\partial w(\xi,t)}{\partial \xi^2} + [1-(r-D)\xi]\frac{\partial w(\xi,t)}{\partial \xi} - Dw(\xi,t) = 0,$$

where $\sigma = 0.42,\ D = 0.015,\ r = 0.05$. The terminal condition is $w(\xi, T) = \max[1-\xi/T, 0]$ where $T = 1/2$. The boundary conditions are

$$w(\xi,t) = 0,\ \text{at}\ \xi = \xi_{\max}; \qquad \frac{\partial w(\xi,t)}{\partial t} + \frac{\partial w(\xi,t)}{\partial \xi} - Dw(\xi,t) = 0\ \text{at}\ \xi = 0.$$

Solution (b): We can take $\xi_{\max} = T(1+3\sigma\sqrt{T}) = 0.9455$. We could also take $\xi_{\max} = T\exp(3\sigma\sqrt{T}) = 1.219$. This is justified because then the option with initial value $\xi = \xi_{\max}$ is 3 standard deviations out of the money.

Solution (c): The value of the option today is $S_0 w(0,0) = 2.53$. Since $S_0 = 31$ we conclude that at $(\xi, t) = (0, 0)$ one has $w(\xi, t) = 2.53/31 = 0.0816$.

Solution (d): For stability we need $\Delta t/(\Delta\xi)^2 < 1/\sigma^2\xi_{\max}^2 = 6.3413$ if $\xi_{\max} = 0.9455$. We have $\Delta t/(\Delta\xi)^2 = 0.0038/0.025^2 = 6.08$, so the numerical scheme is (barely) stable. For the other value $\xi_{\max} = 1.219$ it is unstable.

(3) We wish to use the Monte-Carlo (MC) method to estimate the value of a basket option on three stocks. The payoff on the option depends only on the *arithmetic* average of the three stocks at the expiration of the option, which is 6 months from today. The risk neutral evolution of the three stocks is modelled as follows:

$$\begin{cases} \frac{dS_{1,t}}{S_{1,t}} = r\ dt + \sigma_1\ dW_{1,t}, \\ \frac{dS_{2,t}}{S_{2,t}} = r\ dt + 0.31\ dW_{2,t}, \\ \frac{dS_{3,t}}{S_{3,t}} = r\ dt + 0.27\ dW_{3,t}, \end{cases}$$

where the $W_{1,t},\ W_{2,t},\ W_{3,t},\ t > 0$, are correlated Brownian motions with correlation matrix $\rho = [\rho_{i,j}]$, and $\rho_{i,j} dt = E[dW_{i,t}\ dW_{j,t}]$. The units of time t are in years. The values of the three stocks today are $S_{1,0} = 24,\ S_{2,0} = 29,\ S_{3,0} = 21$.

(a) In the MC simulation the stock $S_{1,T}$ at the expiration time is modeled as the exponential of a Gaussian variable which has mean 3.162 and variance 0.0722. Using this information, find the values of r and σ_1 above.

(b) The matrix ρ is given by the formula:

$$\rho = \begin{bmatrix} 1 & 0.27 & -0.45 \\ 0.27 & 1 & 0.32 \\ -0.45 & 0.32 & 1 \end{bmatrix}$$

The matrix A defined by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0.27 & 0.9629 & 0 \\ -0.45 & 0.4585 & 0.7663 \end{bmatrix}$$

satisfies the equation $AA^T = \rho$. In a MC simulation of $S_{1,T},\ S_{2,T},\ S_{3,T}$ we use samples of three independent standard normal variables $\xi_1,\ \xi_2,\ \xi_3$. The stock price $S_{3,T}$ is then given by the formula $S_{3,T} = \exp[b + b_1\xi_1 + b_2\xi_2 + b_3\xi_3]$. Calculate the values of $b,\ b_1,\ b_2,\ b_3$.

(c) We can use a control variate to reduce variance in the Monte-Carlo simulation for estimating the price of the basket option. Describe how one does this, paying particular attention to the criteria which need to be satisfied to justify the use of the control variate method.

Solution (a): We use the identity $\log(S_{1,T}) = \log(S_{1,0}) + (r - \sigma_1^2/2)T + \sigma_1\sqrt{T}\xi$, where ξ is a standard normal variable. Hence to find r, σ_1 we solve

$$3.162 = \log(24) + (r - \sigma_1^2/2)/2; \qquad 0.0722 = \sigma_1^2/2.$$

This yields $\sigma_1 = 0.38,\ r = 0.0401$.

Solution (b): We have

$$b = \log(S_{3,0}) + (r - \sigma_3^2/2)T = 3.0463; \qquad b_1 = \sigma_3\sqrt{T}A_{3,1} = -0.0859,\ b_2 = \sigma_3\sqrt{T}A_{3,2} = 0.0875,\ b_3 = \sigma_3\sqrt{T}A_{3,3} = 0.1463.$$

Solution (c): The control variate is the corresponding *geometric* average option. Suppose $\Phi(x)$ is the payoff function for the option. Then we need that the coefficient of correlation between the two random variables

$$\Phi\left(\frac{S_{1,T} + S_{2,T} + S_{3,T}}{3}\right), \qquad \Phi\left([S_{1,T}S_{2,T}S_{3,T}]^{1/3}\right)$$

be very close in absolute value to 1.

(4) The current value of a stock is 24. We wish to use a binomial tree model to find the value of a European option on the stock. The volatility of the stock is 0.29, the risk free interest rate is 0.035 and the expiration date for the option is 9 months from today. In the tree model the stock can increase from the value S to the value uS or decrease to the value dS, each with probability 1/2 in a time step $\Delta t = 1/100$.

(a) Calculate the values of u and d.

(b) Suppose the lattice is defined by integer pairs (m, n), corresponding to time $t = m\Delta t$ and stock price $S = S_0 u^n d^{(m-n)}$, where S_0 is today's stock price. Find the maximum and minimum allowable values of the stock price S in the model for time $t = 1/4$.

(c) Let V(m,n) denote the value of the option at time $t = m\Delta t$ and stock price $S = S_0 u^n d^{(m-n)}$. You are given that $V(41, 10) = 3.47$ and $V(41, 11) = 3.52$. Find for what value of n this information determines $V(40, n)$ and calculate its value.

Solution (a): $u,\ d$ are given by the formulas,

$$u = e^{r\Delta t}\left\{1 + \sqrt{\exp(\sigma^2 \Delta t) - 1}\right\}, \qquad d = e^{r\Delta t}\left\{1 - \sqrt{\exp(\sigma^2 \Delta t) - 1}\right\} .$$

With $\sigma = 0.29,\ r = 0.035,\ \Delta t = 0.01$, this yields $u = 1.0294,\ d = 0.9713$.

Solution (b): The time $t = 1/4$ corresponds to $m = t/\Delta t = 25$. The maximum allowable value of the stock at $t = 1/4$ is therefore $S_0 u^{25} = 24 * 2.0635 = 49.52$ and the minimum value is $S_0 d^{25} = 24 * 0.4829 = 11.59$.

Solution (c): We have that

$$V(40, 10) = \exp(-r\Delta t)[0.5V(41, 10) + 0.5V(41, 11)].$$

Hence $n = 10$ and $V(40, 10) = 3.4938$.

(5) Consider the Hull-White model with $\sigma = 0.023$ and $\Delta r = 0.002$. The lattice sites for the model are (m, j), $m = 0, 1, 2..$, $|j| \leq \min\{m, J\}$. A lattice site (m, j) corresponds to time $t = m\Delta t$ and interest rate r_j^m.

(a) Suppose the model has been completely calibrated and we wish to compute the value of a bond at all lattice points. Let $V(m, j)$ be the value of the bond for a lattice point (m, j). You are given that $V(86, 1) = 0.7458$, $V(86, 0) = 0.7451$, $V(86, -1) = 0.7446$, and also that $\alpha^{85} = 0.045$. Find the value of $V(85, 0)$ correct to 5 significant digits, assuming (for the purposes of this calculation only) that $\Delta t = 0.002$.

(b) Suppose the standard deviation of the short rate $r(t)$ at large time is 0.063 i.e. 6.3%. Find the value of the parameter a in the model.

(c) Find the value of Δt to be used in the model, as prescribed by Hull-White so that third moments match.

(d) Suppose you are given that the mean of $r(t)$ when $t = 50\Delta t$ is 0.037. Find the value of r_{20}^{50}.

Solution (a): We have

$$V(85, 0) = \exp[-\alpha^{85}\Delta t] \left\{ \frac{2}{3}V(86, 0) + \frac{1}{6}V(86, 1) + \frac{1}{6}V(86, -1) \right\}.$$

This yields $V(85, 0) = 0.74507$.

Solution (b): The standard deviation of $r(t)$ at large time is $\sigma/\sqrt{2a}$. Hence $a = 0.5 * (0.023/0.063)^2 = 0.0666$.

Solution (c): The HW value for Δt is determined by the equation $\Delta t/(\Delta r)^2 = 1/3\sigma^2$. Hence $\Delta t = [0.002/0.023]^2/3 = 0.0025$.

Solution (d): We have $\alpha^{50} = 0.037$ since it is the mean of $r(t)$ when $t = 50\Delta t$. Then $r_{20}^{50} = \alpha^{50} + 20\Delta r = 0.077$.