Math 623 (IOE 623), Fall 2003: Final exam

Name:

Student ID:

This is a closed book exam. The only notes you may bring must be written on a 3x5" card (both sides). You may use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. State explicitly any additional assumptions you make.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

(1) Explain briefly the difference and relation between the concepts *implied volatility surface* and *local volatility surface*. You do *not* need to derive any complicated equations relating the two.

(2) Explain how to price a basket option on two stocks using Monte Carlo simulations.

The option pays, six months from now, the difference between the arithmetic average of the two stocks and the strike price 50, but never less than zero.

The two stocks follow correlated Geometric Brownian Motion. To find the parameters in the model, you are given hourly (historical) observations of the two stocks during a four-month period.

For the Monte Carlo simulation, you have access to as many independent draws of an $N(0,1)$ as you need and a computer program that can do some useful matrix operations. No variance reduction techniques are needed.

(3) The solution to the following PDE is the price $V(S,I,t)$ of an Asian option on a dividend-paying stock. Today is $t=0$ and S, I denotes stock price and accumulated sum, respectively.

$$\begin{cases} \frac{\partial V}{\partial t} + 0.02S^2\frac{\partial^2 V}{\partial S^2} + 0.01S\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - 0.02V = 0, \quad 0 < S < \infty, 0 < I < \infty, 0 < t < 0.5 \\ V(S,I,0.5) = (S-2I)^+. \end{cases}$$

(a) What is the interest rate r and the continuous dividend yield D?

(b) Exploit the homogeneity in the problem and use a similarity reduction as follows. Write $\xi = I/S$ and define the function $W(\xi,t)$ by $V(S,I,t) = SW(I/S,t)$. Write down the PDE for W.

(c) What is the terminal condition for the PDE in (b)?

(d) Truncate the domain of W to $0 \le t \le 0.5$ and $0 \le \xi \le \xi_{\max}$. Why could $\xi_{\max} = 2$ be a reasonable value and what boundary condition for W at $\xi = \xi_{\max}$ could we use?

(e) Write down the (implicit) boundary condition at $\xi = 0$ resulting from the PDE with $\xi = 0$, there.

(f) Write down (carefully) the explicit finite difference scheme for the PDE in (b)-(e). Use a forward difference for the $\frac{\partial W}{\partial \xi}$ term.

(4) The short rates at the first six nodes in a Black-Derman-Toy tree is given by the table below. The time step is $\Delta t = 0.25$.

0.030000	0.027011	0.024321
	0.032991	0.029706
		0.036284

(a) What are the probabilities along the edges in the tree?
(b) Qualitatively, what kind of information is needed to compute the short rates at the nodes above? (No numbers needed.)
(c) What specific input would give the short rates above? (Numbers needed.)

(5) Consider a European put option on a stock in the Black-Scholes model, expiring one year from now and with strike price $K = 50$. The interest rate is $r = 2\%$, continuously compounded, and the volatility of the stock is $\sigma = 0.2$. The stock pays no dividends and is currently trading at $S_0 = 80$.

(a) Write the value of the stock price S_T at time T as a function of a standard normal variate ξ (under the risk-neutral measure).

(b) Describe how to use "vanilla" Monte Carlo to compute the price of the option by generating samples of the standard normal variate ξ in (a).

(c) Describe how you may use importance sampling to improve the accuracy of the Monte Carlo estimate.