

Math 623 (IOE 623), Fall 2005: Final exam

Name:

Student ID:

This is a closed book exam. You may bring up to ten one sided A4 pages of notes to the exam. You may also use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. State explicitly any additional assumptions you make.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

- (1) Consider the PDE $u_t = \frac{1}{2}(1 + 2x)u_{xx}$, $3 < x < 7$, $t > 0$, with the initial condition $u(x, 0) = x - 1$, $3 < x < 7$, and the boundary conditions $u(3, t) = 2$, $u(7, t) = 2t + 6$. The initial value problem is to be solved numerically by an explicit finite difference scheme, where the time step is denoted Δt and the space step Δx .
- (a) Write down the forward Euler finite difference scheme for the initial value problem. If $\Delta x = 0.5$ and $\Delta t = 0.01$, find the value the difference scheme gives for $u(3.5, 0.02)$.
- (b) If $\alpha = \Delta t/(\Delta x)^2$, find the maximum value α can be and the scheme still remain stable. Explain your answer.

- (2) We wish to find the value of an American put option on a zero dividend stock which expires 6 months from today. The strike price of the option is 50 and the current price of the stock is 45. The volatility of the stock is 0.32 per annum and the interest rate is 0.05 per annum. The problem is to be solved by using the Crank-Nicholson finite difference scheme to numerically solve the Black-Scholes equation.
- (a) Suppose we wish to estimate the value of the option correct to the nearest cent. Estimate the interval $S_{\min} < S < S_{\max}$ on which we should solve the Black-Scholes equation to guarantee this. Give also the boundary conditions at S_{\min} and S_{\max} . Explain your reasoning.
 - (b) Write down the Crank-Nicholson scheme for the problem and explain how it can be used to obtain an algorithm for pricing the American option.

- (3) The Heston stochastic volatility model for the evolution of the price S_t of a stock in the risk neutral world is given by the system of equations,

$$\begin{cases} dY_t = (0.11 - Y_t) dt + 0.3\sqrt{Y_t} dW_t, \\ \frac{dS_t}{S_t} = 0.03 dt + \sqrt{Y_t}[-0.6 dW_t + .8 dZ_t], \end{cases}$$

where $W_t, Z_t, t > 0$ are independent copies of Brownian motion. We wish to estimate the price of a European call option on the stock using the Monte Carlo method. The strike price of the option is 28, the expiration date is eight months from today, and the current stock price is 30.

- (a) We shall assume that $Y_0 = 0.11$ in implementing the Monte Carlo method for the problem. Why is this assumption reasonable?
- (b) Suppose you are given the following values of i.i.d. standard normal variables to implement the Monte Carlo method with $\Delta t = 1/3$: For W_t the values 0.1746, -0.1867 and for Z_t the values 0.7258, -0.5883 . Find the corresponding values for $Y_t, S_t, t = 1/3, 2/3$, given by solving the system of stochastic equations above using the Euler method.
- (c) Find the corresponding value of the option the path generated in (b) gives.

- (4) Suppose we did not know the Black-Scholes formula and wanted to estimate the price of a European call option on a stock using the Monte Carlo method. The strike price of the option is 40, the expiration date is nine months from today, and the current stock price is 43. The volatility of the stock is 0.24 per annum and the risk free rate is 5 percent per annum.
- (a) You are given the following three values of i.i.d. standard normal variables: 1.1909, -0.5376 , 0.8392. Estimate the value of the option the Monte Carlo method gives using these three samples of i.i.d. normal variables.
 - (b) Compute the standard error for the estimate in (a).
 - (c) Estimate the value of the option from the three values for the normal variable given in (a), but also using the method of antithetic variables.

- (5) In order to estimate the price of a European basket option on 2 stocks the risk neutral evolution of the 2 stocks is modelled as follows:

$$\begin{cases} \frac{dS_{1,t}}{S_{1,t}} = 0.03 dt + 0.15 dW_{1,t}, \\ \frac{dS_{2,t}}{S_{2,t}} = 0.03 dt + 0.28 dW_{2,t}, \end{cases}$$

where the $W_{1,t}$, $W_{2,t}$, $t > 0$, are correlated Brownian motions with $E[dW_{1,t} dW_{2,t}] = -0.37 dt$.

- (a) Obtain a representation for the evolution of the two stocks which has the form,

$$\begin{cases} \frac{dS_{1,t}}{S_{1,t}} = r_1 dt + a dZ_{1,t}, \\ \frac{dS_{2,t}}{S_{2,t}} = r_2 dt + b dZ_{1,t} + c dZ_{2,t}, \end{cases}$$

where the $Z_{1,t}$, $Z_{2,t}$, $t > 0$, are independent Brownian motions. Find the values of r_1 , r_2 , a , b , c .

- (b) Show that the system you obtained in (a) is equivalent to the original system.

- (6) The short rates at the first six nodes in a Black-Derman-Toy tree is given by the table below. The time step is $\Delta t = 0.25$.

| | | |
|--------|--------|--------|
| 0.0342 | 0.0325 | 0.0306 |
| | 0.0353 | 0.**** |
| | | 0.0378 |

- Find the price of a zero coupon bond which matures 6 months from today.
- Find the value of the number 0.**** in the tree.
- Find the yield volatility for bonds which mature in 9 months i.e if $y(t)$ is the yield at time $t \geq 0$ on a bond which matures 9 months from today ($t = 0$), find

$$\sqrt{\text{Var}\{\ln[y(0.25)]\}/.25}$$

