Math 623, Fall 2013: Homework 1.

For full credit, your solutions must be clearly presented and all code included.

(1) Consider the following initial value problem for the function $u = u(x)$ defined for $0 \le x \le 1$.

$$u_{xx} + (1+x^2)u_x - (1+x)u = 0 \quad \text{and } u(0) = 1, u_x(0) = -0.5.$$

(a) Write down the finite difference scheme for the ODE above, using a forward difference for u_x and a symmetric difference for u_{xx}. Implement the corresponding numerical algorithm with $\Delta x = 2^{-10}$, and plot the graph of $u(x)$, $0 \le x \le 1$. Also give your computed value of $u(1)$ correct to six decimal places. (You should get $u(1) \simeq 1.138$).

(b) Same question as in (a) but use a backward difference for u_x.

(c) Same question as in (a) but use a central difference for u_x. Make sure that the initial conditions you use in the numerical algorithm are second order accurate.

(d) For $\Delta x = 2^{-N}$, $N = 1, 2.., 15$, let $u_{\Delta x}(1)$ be the computed value of the solution $u(x)$ at $x = 1$. Set $\epsilon(\Delta x) = u_{\Delta x}(1) - u_{2\Delta x}(1)$ to be an estimate of the error from the true solution when the distance between grid points is Δx. Plot $-\log|\epsilon(\Delta x)|$ as a function of $-\log \Delta x$ for $N = 2, .., 15$. What do you observe?

(2) The Black-Scholes PDE for determining the value V of an option is given by,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \quad S > 0,\ t < T, \tag{1}$$

where S is the stock price (in dollars) and t is time (in years). The expiration date of the contract is T, σ is stock volatility per annum, and r the annual continuous rate of interest.

(a) Show that for any constant K, the function $V(S,t) = S - Ke^{-r(T-t)}$ is a solution of the PDE (1).

(b) Show that with the transformation $S = e^x$, if $V(S,t)$ is a solution to the PDE (1) then $V(S,t) = u(x,t)$ where $u(x,t)$ is a solution of the PDE,

$$u_t + \frac{1}{2}\sigma^2 u_{xx} + (r - \frac{1}{2}\sigma^2)u_x - ru = 0, \quad -\infty < x < \infty,\ t < T. \tag{2}$$

(c) We will compute the value of a European call option with strike price K by numerically solving the PDE (2). With $t = 0$ corresponding to today, we will solve (2) for $0 \le t \le T$, with boundary and terminal conditions on the interval $|x - \ln K| < 3\sigma\sqrt{T}$ which we denote $a \le x \le b$, given by

$$\begin{cases} u(x,T) = [e^x - K]^+, \\ u(a,t) = 0, \quad u(b,t) = e^b - Ke^{-r(T-t)}, \quad 0 < t < T. \end{cases} \tag{3}$$

Explain why the boundary and terminal conditions are appropriate for computing the value of the option. Explain also why the interval $[a,b]$ is a good choice.

(d) Write down (carefully) the explicit finite difference algorithm you will use to numerically solve the problem in (c). For what values of $\alpha = \Delta t/(\Delta x)^2$ is the scheme stable? Explain your answer.

(e) Now take $K = 20$, $\sigma = 0.32$, $r = 0.052$, $T = 0.5$ in (c) . Implement the scheme with $\Delta x = (b - a)/2^5$ and for the two values of α, $\alpha = 9$ and $\alpha = 11$. In both cases plot the graph of the computed value of the option against stock price. Plot also the Black-Scholes price (using the MATLAB function *blsprice* for example) of the option against stock price and compare with the graphs for the two values of α.

(3) Consider the same situation as in problem (2) of computing the value of a call option..

(a) With $K = 20, \sigma = 0.32$, $T = 0.5$, implement the scheme with $\Delta x = (b - a)/2^7$, $\alpha = 8$ and r varying in the interval $0 \leq r \leq 0.1$, taking $\Delta r = 0.005$. Assuming the current value of the stock is 20, plot the graph of the computed value of the option against interest rate. Plot also the Black-Scholes price of the option against interest rate, and compare with the graph for the numerical solution of the PDE.

(b) With $K = 20, r = 0.052$, $T = 0.5$, implement the scheme with $\Delta x = (b - a)/2^7$, $\alpha = 0.09$ and σ varying in the interval $0 \leq \sigma \leq 3.0$, taking $\Delta\sigma = 3/20$. Assuming the current value of the stock is 20, plot the graph of the computed value of the option against volatility. Plot also the Black-Scholes price of the option against volatility, and compare with the graph for the numerical solution of the PDE.

(c) With $K = 20$, $r = 0.052$, $\sigma = 0.32$, $T = 0.5$, and the current value of the stock at 20, estimate the value of the Greek variables Delta Δ, Vega $\mathcal{V}$, and rho ρ, for the option. Give your answer correct to 2 decimal places.