Math 623, F 2005: Homework 2. Solutions.

(1) (a) The payoff $\Phi(S)$ is as in the picture below.

(b) The payoff is $\Phi(S) = (80 - S)^+ + (S - 120)^+$, i.e. a sum of a put and a call.

(c) When $S_t = 0$, we will have $S_u = 0$ for $t \le u \le T$. Exercising the option at time u will then yield 80, which corresponds to $80e^{-r(u-t)} < 80$ at time t. It is therefore optimal to exercise the option immediately at time t, so

$$V(0,t) = 80$$

Similarly, if $S_t = 300$, then we will most likely have $S_u \ge 120$ for $t \le u \le T$, so that the payoff if exercising at time u will be $S_u - 120$. At time t, this corresponds to $300e^{-D(u-t)} - 120e^{-r(u-t)}$. A direct computation (using $r = 0.03$, $D = 0.01$, $T = 0.5$) shows that the minimum of this occurs when $u = T = 0.5$. Thus

$$V(300,t) = 300e^{-0.01(0.5-t)} - 120e^{-0.03(0.5-t)}.$$

(d) The variational problem is

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV \le 0 \\ V(S,t) \ge (80 - S)^+ + (S - 120)^+ \\ \text{equality holds in one of the above.} \end{cases}$$

This holds in the region $0 < S < 300$, $0 < t < 0.5$. The terminal condition is

$$V(S, 0.5) = (80 - S)^+ + (S - 120)^+$$

and the boundary conditions are

$$\begin{cases} V(t,0) = 80 \\ V(t,300) = 300e^{-0.01(0.5-t)} - 120e^{-0.03(0.5-t)}. \end{cases}$$

(e) Use the grid

$$t = 0, \Delta t, \dots, M\Delta t = 0.5 \quad \text{and} \quad S = 0, \Delta S, \dots, N\Delta S = 300.$$

Write V_n^m for the approximate value of $V(n\Delta S, m\Delta t)$. The terminal condition translates into

$$V_n^M = (80 - n\Delta S)^+ + (n\Delta S - 120)^+, \quad 0 \le n \le N,$$

and the boundary conditions are

$$\begin{cases} V_0^m = 80 \\ V_N^m = 300e^{-0.01(0.5-m\Delta t)} - 120e^{-0.03(0.5-m\Delta t)}, \end{cases} \qquad m = 0, 1, \dots, M-1$$

Finally, the Crank-Nicholson discretization of the variational problem is given by the inequality

$$\begin{aligned}\frac{V_n^{m+1}-V_n^m}{\Delta t} &+ \frac{1}{2}\sigma^2(n\Delta S)^2\frac{1}{2}\left[\frac{V_{n+1}^{m+1}-2V_n^{m+1}+V_{n-1}^{m+1}}{(\Delta S)^2}+\frac{V_{n+1}^{m}-2V_n^{m}+V_{n-1}^{m}}{(\Delta S)^2}\right]\\ &+(r-D)(n\Delta S)\frac{1}{2}\left[\frac{V_{n+1}^{m+1}-V_{n-1}^{m+1}}{2\Delta S}+\frac{V_{n+1}^{m}-V_{n-1}^{m}}{2\Delta S}\right]\\ &-r\frac{1}{2}[V_n^{m+1}+V_n^m]\\ &\leq 0.\end{aligned}$$

together with

$$V_n^m \geq (80-n\Delta S)^+ + (n\Delta S-120)^+.$$

Both of these hold for $0 \leq m < M$ and $0 < n < N$ and for each such (m,n) we have equality in one of the inequalities.

On a more detailed level, we solve the following system using SOR:

$$\begin{cases} V_n^m \geq p_n^- V_{n-1}^m + p_n^+ V_{n+1}^m + b_n^{m+1}, \\ V_n^m \geq (80-n\Delta S)^+ + (n\Delta S-120)^+ \\ \text{equality in one of these.}\end{cases} \tag{$\dagger$}$$

where

$$p_n^- = \frac{\frac{1}{4}(\sigma^2 n^2\Delta t-(r-D)n\Delta t)}{1+\frac{1}{2}\sigma^2n^2\Delta t+\frac{1}{2}r\Delta t} \qquad p_n^+ = \frac{\frac{1}{4}(\sigma^2 n^2\Delta t+(r-D)n\Delta t)}{1+\frac{1}{2}\sigma^2n^2\Delta t+\frac{1}{2}r\Delta t}$$

$$\begin{aligned} b_n^{m+1} = \frac{1}{1+\frac{1}{2}\sigma^2n^2\Delta t+\frac{1}{2}r\Delta t}\Bigg[&\frac{1}{4}\left(\sigma^2n^2\Delta t-(r-D)n\Delta t\right)V_{n-1}^{m+1}\\ &+\left(1-\frac{1}{2}\sigma^2n^2\Delta t-\frac{1}{2}r\Delta t\right)V_n^{m+1}\\ &+\frac{1}{4}\left(\sigma^2n^2\Delta t+(r-D)n\Delta t\right)V_{n+1}^{m+1}\Bigg]\end{aligned}$$

Thus, we can solve ($\dagger$) iteratively

1. Set $V_n^{m,0} = V_n^{m+1} \quad 1 \leq n \leq N-1$
2. For $k \geq 0$ compute $V_n^{m,k+1}$ for $n = 1, 2, \ldots, N-1$ using:

$$\begin{cases} \tilde{V}_n^{m,k+1} = p_n^- V_{n-1}^{m,k+1} + p_n^+ V_{n+1}^{m,k} + b_n^{m+1} \\ V_n^{m,k+1} = V_n^{m,k+1} + \omega(\tilde{V}_n^{m,k+1} - V_n^{m,k}) \end{cases}$$

where we set

$$\begin{cases} V_0^{m,k+1} = 80 \\ V_N^{m,k} = 300e^{-0.01(0.5-m\Delta t)} - 120e^{-0.03(0.5-m\Delta t)}. \end{cases}$$

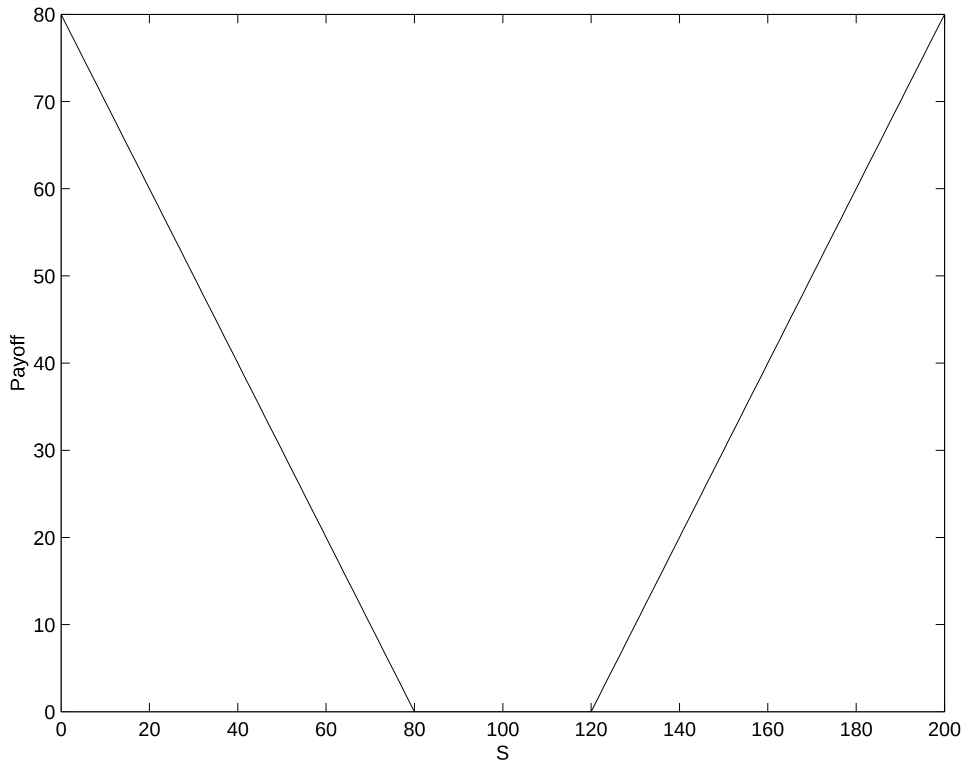
3. Stop loop in k when we have convergence, i.e. when

$$\sum_{n=0}^{N}(V_n^{m,k+1} - V_n^m)^2 < \epsilon.$$

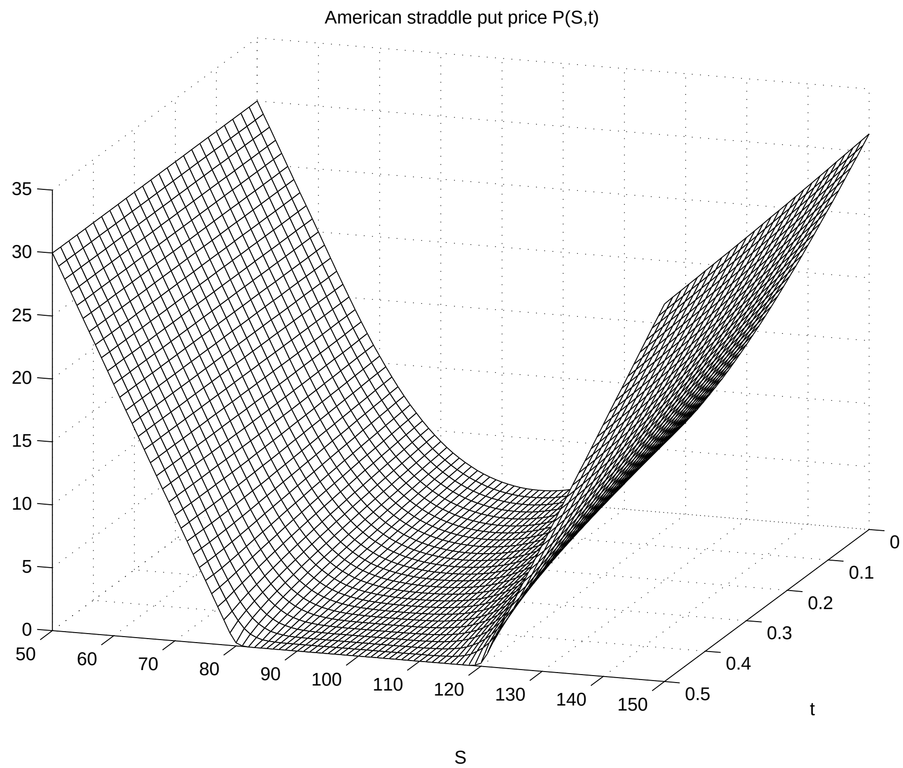
The code is given below.

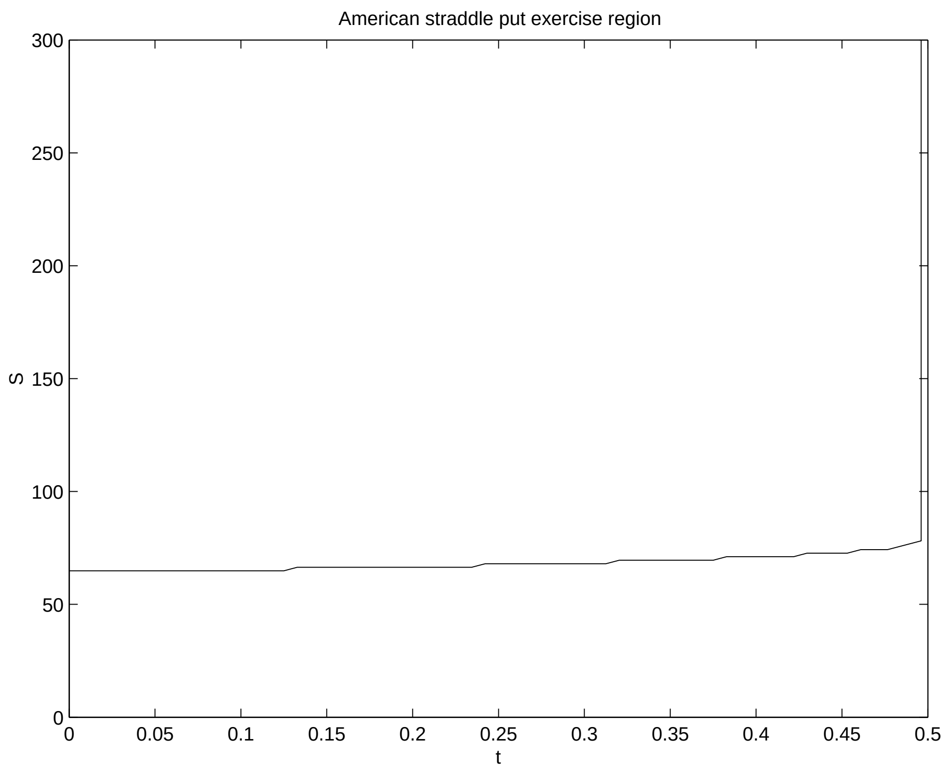
(f) The plot is shown below.

(g) The plot is shown below. The exercise region is the part below the curve (the exercise boundary). The apparent discontinuity at $t = 0.5$ is illusory.



Payoff
0
10
20
30
40
50
60
70
80
0
20
40
60
80
100
120
140
160
180
200
S


American straddle put price P(S,t)
35
30
25
20
15
10
5
0
50
60
70
80
90
100
110
120
130
140
150
S
0
0.1
0.2
0.3
0.4
0.5
t


American straddle put exercise region
300
250
200
150
100
50
0
S
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
t

```
% Code for hw2, problem 1e
clear;

r=0.03;
D=0.01;
sig=0.2;
T=0.5;

Smin=0;
Smax=300;
N=3*2^6;
dS=(Smax-Smin)/N;
S=[Smin:dS:Smax]';
M=2^6;
dt=T/M;
t=[0:dt:T];

P=zeros(N+1,M+1);
Pold=zeros(1,N+1);
Pnew=zeros(1,N+1);
Ptilde=zeros(1,N+1);

%boundary conditions
P(1,:)=80-Smin;
P(N+1,:)=max(Smax-120,Smax*exp(-D*(T-t))-120*exp(-r*(T-t)));

%terminal condition
P(:,M+1)=max(80-S,0)+max(S-120,0);

%for the SOR step
loops=zeros(1,M);
eps=1e-6;
omega=1.05;

for m=M:-1:1
  Pnew(2:N)=P(2:N,m+1);
  Pnew(1)=P(1,m);
  Pnew(N+1)=P(N+1,m);
  Pmisc=(Pnew(1:N-1).*(1/4*sig^2*[1:N-1].^2*dt-1/4*(r-D)*[1:N-1]*dt)+...
         Pnew(2:N).*(1-1/2*sig^2*[1:N-1].^2*dt-1/2*r*dt)+...
         Pnew(3:N+1).*(1/4*sig^2*[1:N-1].^2*dt+1/4*(r-D)*[1:N-1]*dt))./...
        (1+1/2*sig^2*[1:N-1].^2*dt+1/2*r*dt);
  error=Inf;
```

```
  loops(m)=0;
  while(error>eps)
    loops(m)=loops(m)+1;
    Pold=Pnew;
    for n=2:N
      Ptilde(n)=Pmisc(n-1)+...
                (Pnew(n-1).*(1/4*sig^2*(n-1).^2*dt-1/4*(r-D)*(n-1)*dt)+...
                 Pold(n+1).*(1/4*sig^2*(n-1)^2*dt+1/4*(r-D)*(n-1)*dt))/...
                 (1+1/2*sig^2*(n-1)^2*dt+1/2*r*dt);
      Pnew(n)=max(max(80-S(n),0)+max(S(n)-120,0),...
                   Pold(n)+omega*(Ptilde(n)-Pold(n)));
    end
    error=norm(Pnew-Pold);
  end
  P(:,m)=Pnew';
end

%interpolate a little
tvec=0:0.02:0.5;
Svec=50:1:150;
P0=interp2(t,S,P,tvec,Svec');

%3D plot of option value
mesh(tvec,Svec,P0);
title('American straddle put price P(S,t)');
xlabel('t');
ylabel('S');

%exercise region
Sgrid=repmat(S,1,M+1);
exval=max(80-Sgrid,0)+max(Sgrid-120,0);
contour(t,S,P>exval,1,'b');
title('American straddle put exercise region');
xlabel('t');
ylabel('S');
```

(2) (a) The terminal condition is $V(S,I,T) = (S - I/T)^+$ (where $T = 0.5$).

(b) We get

$$\begin{cases} \frac{\partial V}{\partial t} = S\frac{\partial W}{\partial t} \\ \frac{\partial V}{\partial S} = W + S\frac{\partial W}{\partial \xi}(-\frac{I}{S^2}) = W - \xi\frac{\partial W}{\partial \xi} \\ \frac{\partial^2 V}{\partial S^2} = \frac{\partial W}{\partial \xi}(-\frac{I}{S^2}) + \frac{I}{S^2}\frac{\partial W}{\partial \xi} - \xi\frac{\partial^2 W}{\partial \xi^2}(-\frac{I}{S^2}) = \frac{1}{S}\xi^2\frac{\partial^2 W}{\partial \xi^2} \\ \frac{\partial V}{\partial I} = S\frac{\partial W}{\partial \xi}\frac{1}{S} = \frac{\partial W}{\partial \xi} \end{cases}$$

After some computations, this leads to the PDE

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\xi^2\frac{\partial^2 W}{\partial \xi^2} + (1 - (r - D)\xi)\frac{\partial W}{\partial \xi} - DW = 0$$

in the domain $0 < t < 0.5$ and $0 < \xi < \infty$.

(c) From (a) we get

$$V(S,I,T) = SW(I/S,T) = (S - I/T)^+ = S(T - I/S)^+/T,$$

which leads to

$$W(\xi, 0.5) = 2(0.5 - \xi)^+,$$

since $T = 0.5$.

(d) At the terminal time $t = 0.5$, the value of W is zero for $\xi > 0.5$. So we could expect that the value is very close to zero for $xi = \xi_{\max} = 2$ for any $t \leq 0.5$. Alternatively, $\xi = 2$ translates into $I = 2S$, and if $I_t = 2S_t$ at some t, then, with very high probability we will have $I_T > 0.5S_T$ so that the option ends up out of the money. Thus, $W(2,t) = 0$ is a good approximation.

(e) Just plug in $\xi = 0$ into the PDE:

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial \xi} - DW = 0.$$

(f) Use the grid

$$\begin{cases} t = 0, \Delta t, 2\Delta t, \ldots, M\Delta t = T = 0.5 \\ \xi = 0, \Delta \xi, 2\Delta \xi, \ldots, M\Delta \xi = \xi_{\max} = 2. \end{cases}$$

Write W_n^m for the approximate value of $W(n\Delta\xi, m\Delta t)$. The terminal condition translates into

$$W_n^M = 2(0.5 - n\Delta\xi)^+, \quad \text{for } 0 \leq n \neq N. \tag{1}$$

The boundary condition at $\xi = 2$ becomes

$$W_N^m = 0 \quad \text{for } 0 \leq m < M.$$

The implicit boundary condition at $\xi = 0$ can be discretized as

$$\frac{W_0^m - W_0^{m-1}}{\Delta t} - \frac{3W_0^m - 4W_1^m + W_2^m}{2\Delta\xi} - DW_0^m = 0 \quad \text{for } 1 \leq m \leq M.$$

Finally, the discretization of the PDE, using an explicit scheme, a symmetric difference for $\frac{\partial^2 W}{\partial \xi^2}$ and a central difference for $\frac{\partial W}{\partial \xi}$, becomes

$$\frac{W_n^m - W_n^{m-1}}{\Delta t} + \frac{1}{2}\sigma^2 (n\Delta\xi)^2 \frac{W_{n+1}^m - 2W_n^m + W_{n-1}^m}{(\Delta\xi)^2} + (1 - (r-D)n\Delta\xi)\frac{W_{n+1}^m - W_{n-1}^m}{2\Delta\xi} - DW_n^m = 0$$

for $0 < n < N$ and $0 < m \leq M$.

The algorithm thus becomes

1. Compute V_n^M for $0 \leq n \leq N$ using (1)
2. Suppose we have computed $V^M, \ldots, V^m$. Compute V^{m-1} as follows:

$$V_N^{m-1} = 0,$$

$$V_0^{m-1} = \left(1 - \frac{3}{2}\frac{\Delta t}{\Delta\xi} - D\Delta t\right) V_0^m + 2\frac{\Delta t}{\Delta\xi}V_1^m - \frac{1}{2}\frac{\Delta t}{\Delta\xi}V_2^m,$$

and, for $1 \leq n \leq N-1$:

$$V_n^{m-1} = p_n^- V_{n-1}^m + p_n^0 V_n^m + p_n^+ V_{n+1}^m.$$

where the coefficients p_n^* are given by

$$\begin{cases} p_n^- = \frac{1}{2}\sigma^2 n^2 \Delta t - \frac{1}{2}(1-(r-D)n\Delta\xi)\frac{\Delta t}{\Delta\xi} \\ p_n^0 = 1 - (\sigma^2 n^2 + D)\Delta t \\ p_n^+ = \frac{1}{2}\sigma^2 n^2 \Delta t + \frac{1}{2}(1-(r-D)n\Delta\xi)\frac{\Delta t}{\Delta\xi} \end{cases}$$

With this scheme, it is not really possible to guarantee that all three of p_n^+, p_n^0 and p_n^- are always positive. But at least we should make sure that $p_n^0 \geq 0$ for all n. This can be done by picking

$$\Delta t \leq \frac{1}{\sigma^2 N^2 + D} = \frac{(\Delta\xi)^2}{4\sigma^2 + D(\Delta\xi)^2},$$

since $N = 2/\Delta\xi$. To be safe, we may pick

$$\Delta t = \frac{(\Delta\xi)^2}{4(\sigma^2 + D)},$$

since $\Delta\xi \leq \xi_{\max} = 2$.

Note: by using a forward difference for $\frac{\partial W}{\partial \xi}$ it would be possible to have all three "probabilities" positive above.

(g) The code and plot are given below.

(h) The code gives the numerical value $W(0,0) = 0.034322$ so that

$$V(20, 0, 0) = 20W(0,0) = 0.686.$$

This also gives the Delta

$$\Delta(20, 0, 0) = \frac{\partial V}{\partial S}(20, 0, 0) = W(0,0) - 0\frac{\partial W}{\partial \xi}(0,0) = 0.0343.$$

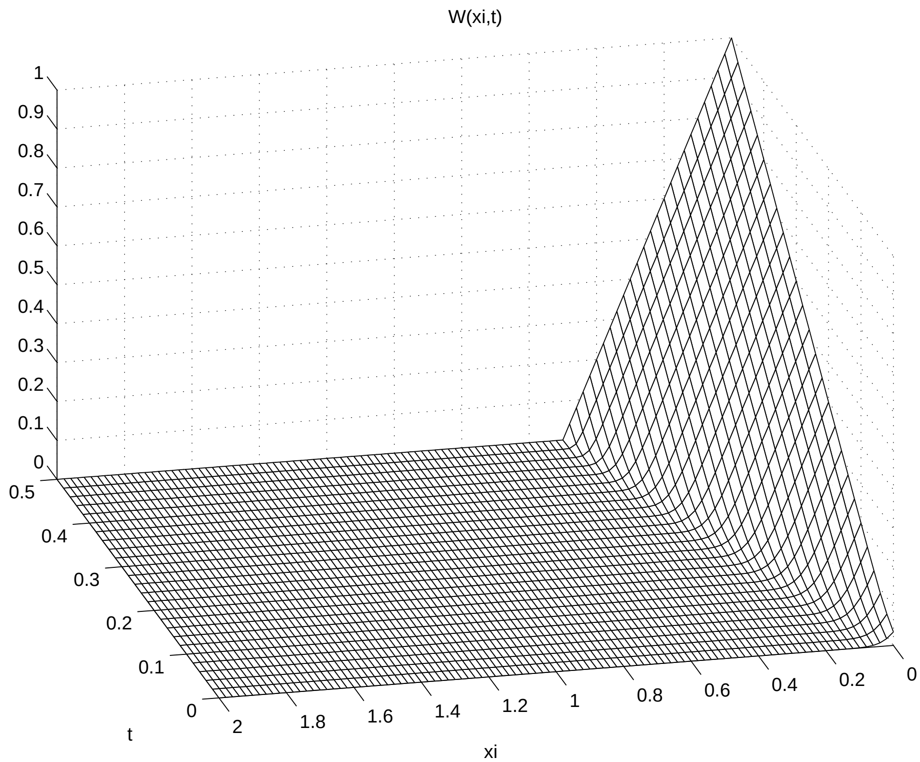
To get the Vega we run the code with several values of $\sigma \approx 0.2$. We get

$$
\begin{aligned}
V(20,0,0;0.190) &= 0.65281 \\
V(20,0,0;0.195) &= 0.66966 \\
V(20,0,0;0.199) &= 0.68309 \\
V(20,0,0;0.201) &= 0.68979 \\
V(20,0,0;0.205) &= 0.70317 \\
V(20,0,0;0.210) &= 0.71984
\end{aligned}
$$

This leads to the three approximations

$$
\begin{aligned}
\mathcal{V}(20,0,0) &\approx \frac{0.71984 - 0.65281}{0.02} = 3.35 \\
\mathcal{V}(20,0,0) &\approx \frac{0.70317 - 0.66966}{0.01} = 3.35 \\
\mathcal{V}(20,0,0) &\approx \frac{0.68979 - 0.68309}{0.002} = 3.35
\end{aligned}
$$

so the options vega is $\mathcal{V}(20,0,0) = 3.35$.


W(xi,t)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.5
0.4
0.3
0.2
0.1
0
t
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
xi

```
% Code for hw2, problem 2f
clear;

sig=input('sigma= ');
%sig=0.2;
r=0.03;
D=0.01;
T=0.5;
ximin=0;
ximax=2;
N=2^9;
dxi=(ximax-ximin)/N;
dt=1/(sig^2*N^2+D);   % to ensure convergence/stability
M=ceil(T/dt);
dt=T/M;

W=zeros(N+1,M+1);
xi=[0:dxi:ximax]';
t=[0:dt:T];
nvec=[0:N]';

% terminal condition
W(:,M+1)=2*max(0.5-xi,0);

% first boundary condition
W(N+1,:)=0;

% coefficients
pm=0.5*sig^2*nvec.^2*dt-0.5*dt/dxi+0.5*(r-D)*dt*nvec;
p0=1-dt*(sig^2*nvec.^2+D);
pp=0.5*sig^2*nvec.^2*dt+0.5*dt/dxi-0.5*(r-D)*dt*nvec;

%coefficients for implicit boundary condition
q0=1-1.5*dt/dxi-D*dt;
q1=2*dt/dxi;
q2=-0.5*dt/dxi;

% go backwards in time
for m=M:-1:1
  % implicit boundary condition
  W(1,m)=q0*W(1,m+1)+q1*W(2,m+1)+q2*W(3,m+1);

  % main part
```

```
  W(2:N,m)=pm(2:N).*W(1:N-1,m+1)+...
           p0(2:N).*W(2:N,m+1)+...
           pp(2:N).*W(3:N+1,m+1);
end

%interpolate a little
tvec=0:0.02:0.5;
xivec=0:0.02:2;
W0=interp2(t,xi,W,tvec,xivec');

%3D plot of option value
mesh(tvec,xivec,W0);
title('W(xi,t)');
xlabel('t');
ylabel('xi');
```