Math 623, F 2013: Homework 3.

For full credit, your solutions must be clearly presented and all code included.

(1) In the Black-Scholes method for pricing of options it is assumed that the stock price evolves according to geometric Brownian motion:

$$\frac{dS_t}{S_t} = r\ dt + \sigma\ dW_t, \tag{1}$$

where $W(t)$ is Brownian motion, r is the risk free rate of interest and σ is the volatility.

(a) Suppose $t = 0$ is today and the current stock price is S_0. The Black-Scholes price of the option is the expectation value of a function of $S(T)$, where $T > 0$ is the expiration date of the option. The random variable $\log\ S(T)$ is known to be Gaussian. Write down formulas for its mean and variance.

(b) Suppose $S_0 = 20,\ r = 0.045,\ \sigma = 0.28,\ T = 0.5$. Draw histograms for the distribution of $S(T)$ with numbers of simulations given by the values $N = 10^4,\ 10^5,\ 10^6$. You can use the MATLAB function *hist* to do this.

(c) With the numerical values given in (b) use the Monte-Carlo method to compute the value of a European call option with strike price $K = 21$. If V_N is the value of the option based on N simulations and ϵ_N is the standard error for the N simulations, plot the graphs of N against V_N (convergence diagram), and N against ϵ_N for $1 \le N \le 10^4$. Report the values of V_N and ϵ_N/V_N for $N = 10^6$. What is the significance of the reported value of ϵ_N/V_N ?

(d) With $K = 21,\ r = 0.045,\ \sigma = 0.28,\ T = 0.5$, plot the Monte-Carlo price of the European call option against stock price for $15 < S_0 < 25$ corresponding to $N = 10^4$ simulations. Plot also the Black-Scholes price (using the MATLAB function *blsprice* for example) of the option against stock price and compare it with the graph obtained from the Monte Carlo simulation.

(e) With $S_0 = 20,\ K = 21,\ r = 0.045,\ T = 0.5$, plot the Monte-Carlo price of the European call option against volatility for $0 < \sigma < 3$, corresponding to $N = 10^4$ simulations. Plot also the Black-Scholes price (using the MATLAB function *blsprice* for example) of the option against volatility and compare it with the graph obtained from the Monte Carlo simulation.

(2) We shall find the value of a European call option with a knock-out using the Monte-Carlo method and also the finite difference method. S is the stock price (in dollars) and t is time (in years). The expiration date of the contract is T, σ is stock volatility per annum, and r the annual continuous rate of interest. The strike price of the call option is K and the knockout barrier is $X < K$. For the option held at time $t < T$ the payoff is $\max(S(T) - K, 0)$ provided $S(t') \ge X,\ t \le t' \le T$. Otherwise the payoff is zero. Suppose $t = 0$ is today and the current stock price is S_0.

(a) The Monte-Carlo method may be used to find the value of the option by numerically solving the stochastic differential equation (1). Suppose $S_0 = 20,\ K = 20,\ X = 18,\ \sigma =$

0.31, $r = 0.048$, $T = 0.8$. With $\Delta t = T/100$ and number of simulations $N = 10^4$, find the corresponding value V_N of the barrier option. Report also the value of ϵ_N/V_N.

(b) With the other parameters fixed as in (a), plot the graph of the value of the option against the barrier value X for $10 < X < 20$. Explain the shape of the graph.

(c) The option can also be valued by solving the transformed Black-Scholes PDE, equation (2) of Homework I. Write down the terminal and boundary conditions for the PDE problem.

(d) Write down the explicit Euler method for solving the PDE problem (c), and use it to find the corresponding value of the option for the values of $S_0, ..., T$ given in part (a). Take $\Delta x = (b-a)/2^6$ and report the value of α you use.

(e) Write down the Crank-Nicholson algorithm for solving the PDE problem (c), and use it to find the corresponding value of the option for the values of $S_0, ..., T$ given in part (a). Again take $\Delta x = (b-a)/2^6$ but now take $\alpha = 100$ and use the Gauss-Seidel algorithm to solve the numerical linear algebra problem. Give your answers when the number of Gauss-Seidel iterations used is: 5,10,20, 30.

(f) Investigate improving the Monte-Carlo estimate of part (a) for the value of the option by making Δt smaller and N larger. What conclusions can you come to about the relevant importance of Δt and N?

(3) In this problem we consider the (Heston) stochastic volatility model for the price S_t of a stock

$$\begin{cases} dY_t &= [\theta - \kappa Y_t]dt + \beta\sqrt{Y_t}\,dW_t \\ \frac{dS_t}{S_t} &= r\,dt + \sqrt{Y_t}\,(\rho\,dW_t + \sqrt{1-\rho^2}\,dZ_t)\ . \end{cases} \tag{2}$$

Here W_t and Z_t are independent (uncorrelated) Brownian motions under the risk neutral measure Q.

(a) The values of the parameters θ, β must satisfy $\theta > \beta^2/2$ in order that the volatility process $Y(t)$ remain positive. Taking $\theta = .02$, $\beta = 0.4$, $\kappa = 2$ with $Y(0) = \theta/\kappa$, use Monte Carlo simulations to estimate the probability that $Y(t) = 0$ for some t, $0 < t < T$, where $T = 0.5$. Plot the graph of the exit probability against θ for $0 < \theta < 0.25$.

(b) The values of the parameters in the stochastic equation are taken to be $\theta = 0.25$, $\kappa = 2$, $\beta = 0.4$ and $r = 0.04$, $\rho = -0.7$. With $T = 0.5$ use the Monte Carlo method to estimate the mean and standard deviation of $S(T)$. Estimate also the mean and standard deviation of the stock volatility $\sqrt{Y(T)}$ at time T. The initial stock price is 25 and $Y(0) = \theta/\kappa$.

(c) With the parameter values given in part (b), estimate the value of a put option with expiration date $T = 0.5$ if the strike price is $K = 23$.

(d) Take the parameter values except for ρ and K as in part (c). Now for two fixed values of ρ, plot the graph of implied volatility against K with K varying in the interval $24 < K < 26$. The two values of ρ are $\rho = -0.5$ and $\rho = 0.5$. Explain the different shapes of the graphs. Finally find a value of ρ for which the graph of implied volatility against K in the region $24 < K < 26$ has a minimum and plot its graph.