

Math 623, F 2013: Homework 4.

For full credit, your solutions must be clearly presented and all code included.

- (1) In this problem we will be interested in valuing an out-of-the-money bear spread option on a stock S_t which evolves by geometric Brownian motion with volatility $\sigma = 0.20$. The risk free rate is $r = 0.046$ and the expiration date of the option is $T = 0.8$. The payoff on the option is $\Phi(S_T)$, where

$$\Phi(S) = \begin{cases} 10 & \text{if } S \leq 40 \\ 50 - S & \text{if } 40 \leq S \leq 50 \\ 0 & \text{if } S \geq 50. \end{cases}$$

The stock is currently trading (at time $t = 0$) at a value S_0 .

- (a) Show that the bear spread option is equivalent to the difference of two put options.
 - (b) Using the MATLAB function *blsprice* or alternative method of computing the Black-Scholes formula, graph the value of the bear spread option as a function of S_0 , for S_0 in the range $50 \leq S_0 \leq 80$. Compute the value of the option for $S_0 = 80$.
 - (c) Use the Monte Carlo method with $N = 10^5$ to compute the price of the option for the same range of values of S_0 as in part (b) and graph the value of the bear spread option as a function of S_0 . Compare with the graph in (b). Compute the value of the option for $S_0 = 80$.
 - (d) Graph the ratio of the Monte Carlo value of the option obtained in (c) to the Black-Scholes value obtained in (b) for S_0 in the range $50 \leq S_0 \leq 80$.
 - (e) Now redo the Monte Carlo method for (c) using importance sampling as outlined in class, with the zero value of the new normal variable corresponding to the strike price at time T being 50. As in (d) graph the ratio of the Monte Carlo value of the option to the Black-Scholes value for S_0 in the range $50 \leq S_0 \leq 80$.
 - (f) What conclusions can you draw on comparing the graphs in (d) and (e)? Explain your results.
- (2) Consider three stocks with the following dynamics under the risk-neutral measure Q :

$$\frac{dS_{i,t}}{S_{i,t}} = r dt + \sigma_i dW_{i,t}, \quad i = 1, 2, 3, \quad 0 \leq t \leq T = 1/2, \quad r = 0.045.$$

Here $W_{i,t}$ are *correlated* Brownian motions: $E[dW_{i,t} dW_{j,t}] = \rho_{ij} dt$, where the correlation matrix $\rho = (\rho_{ij})$ is

$$\rho = \begin{bmatrix} 1 & 3/5 & -1/4 \\ 3/5 & 1 & -1/3 \\ -1/4 & -1/3 & 1 \end{bmatrix}$$

Moreover, we have

$$\begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 0.27 & 0.21 & 0.31 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S_{1,0} \\ S_{2,0} \\ S_{3,0} \end{bmatrix} = \begin{bmatrix} 38 \\ 40 \\ 35 \end{bmatrix}$$

- (a) Use Cholesky factorization to write the stock price dynamics as

$$\frac{dS_{i,t}}{S_{i,t}} = r dt + \sum_{j=1}^3 \tilde{\sigma}_{ij} dZ_{j,t}, \quad i = 1, 2, 3.$$

where $Z_{i,t}$ are *independent* Brownian motions under Q and $\tilde{\sigma}_{ij}$ are constants. Find the matrix $\tilde{\sigma}$.

- (b) Let $S(T) = (S_{1,T}S_{2,T}S_{3,T})^{1/3}$ be the geometric average of the terminal stock prices. Suppose a call option dependent on the three stocks has payoff $\max(S(T) - K, 0)$. This can be computed from the Black-Scholes formula using suitable values of the volatility and initial stock price parameters in the formula, which we denote by σ_0 , S_0 , but keeping the same values of the interest rate r , strike price K and expiration time T as in the basket option. If $K = 39$, find the values of the parameters σ_0 , S_0 , and the corresponding Black-Scholes price for the option.
- (c) Using the Monte-Carlo method with antithetic variables and $N = 10^6$, estimate the values of the two options, one with arithmetic payoff $X(T) = \max([S_{1,T} + S_{2,T} + S_{3,T}]/3 - K, 0)$ at time T , and the other with geometric payoff $Y(T) = \max((S_{1,T}S_{2,T}S_{3,T})^{1/3} - K, 0)$ at time T .
- (d) Using the Monte-Carlo method with antithetic variables and $N = 10^4$, estimate the variance of the variable $Y(T)$, and the covariance of the variables $X(T)$, $Y(T)$. Estimate also the values of the coefficient of correlation ρ for $X(T)$, $Y(T)$, and β .
- (e) Estimate the value of the arithmetic option with $N = 10^4$ as in (c) and then its improved value using the control variate $Y(T)$. Compare your answers to what you obtained for (c).
- (3) In this problem we will find the value of an Asian option using the Monte Carlo method. The option is a *discretely sampled, arithmetically averaged, average strike Asian option*. The values of the parameters are as in problem 2 of homework II, i.e. $T = 0.8$, $\sigma = 0.35$, $r = 0.052$, $D = 0.02$, and today's stock price is 22. The stock price is sampled at equal time intervals T/M where M is an integer. The average stock price over the course of the option contract is then the average of the sampled values, including the initial value $S(0) = 22$ and the final value $S(T)$.
- (a) One can generate a single sample of the $M + 1$ stock values $S(mT/M)$, $m = 0, \dots, M$, by setting

$$S\left(\frac{mT}{M}\right) = S(0) \exp \left[\left(r - D - \frac{1}{2}\sigma^2 \right) \frac{mT}{M} + \sigma(\xi_1 + \dots + \xi_m) \sqrt{\frac{T}{M}} \right],$$

where the $\xi_1, \dots, \xi_M$ are independent standard normal variables. Explain why this is the case.

- (b) Applying the Monte Carlo method, use the representation of (a) to estimate the value of the Asian option when $M = 2, 4, 8, 16, 32, 64, 128$, with $N = 10^5$ simulations.
- (c) Now using the Brownian bridge method to construct path samples, estimate the value of the option when $M = 2, 4, 8, 16, 32, 64, 128$, with $N = 10^5$ simulations.
- (d) Compare the answers you obtain in (b) and (c) to the value of the *continuously* sampled Asian option you obtained in problem 2, homework II.