**Interest Rate Modeling**

### Black's Model

A basic object of study in interest rate theory is the price of a zero coupon zero with bond such as a European bond, i.e.

\[(1.1) \quad P(t, T) = \text{value at time } t < T \text{ of a European bond with a face value of } 1 \text{ which matures at time } T.\]

We can use these prices to compute the forward rate of interest \( F(t, T_1, T_2) \) with \( t < T_1 < T_2 \), i.e.

\[(1.2) \quad F(t, T_1, T_2) = \text{(simple) interest rate given at time } t \text{ for borrowing at } T_1 \text{ with repayment at } T_2.\]

Equivalently, the no-arbitrage value \( F(t, T_1, T_2) \) is

\[(1.3) \quad F(t, T_1, T_2) = \frac{P(t, T_1) - P(t, T_2)}{(T_2 - T_1) P(t, T_2)}.\]

More generally, suppose we wish to find the swap rate i.e. fixed versus floating of many bonds, at \( T_0 < t \) and repaid at time \( T_0 \), with interest payments at times \( T_1, T_2, \ldots, T_n \), where \( T_0 < T_1 < T_2 < \ldots < T_n \). The swap rate \( R(t) \) is given by

\[(1.4) \quad R(t) = \frac{P(t, T_0) - P(t, T_n)}{\sum_{i=1}^{n} (T_i - T_{i-1}) P(t, T_i)}.\]

Next, we consider how to value an
interest rate cap. The claim is as described above, 
comprising at \( T_0 \) with repayment at \( T_N \) and 
interest rate payments at times \( T_i \)'s. The 
interest rate is set at time \( T_{i-1} \) for the 
period \( (T_{i-1}, T_i) \), \( i = 1, \ldots, N \), and in the 
floating rate for that period. We introduce a cap 
\( K \) known as the interest rate. Then at 
time \( T_N \), the option pays

\[
(0.5) \quad (T_i - T_{i-1}) \left[ L(T_{i-1}, T_i) - K \right]^+,
\]

where \( L(t, t') \) is the spot rate at time \( t \) 
for repayment at \( t' \). Each payment \((0.5)\) is 
a capMT to the value \( J \) of the interest rate 
cap puts the sum of the values \( J \) of the \( N \) 
capsMT which constitutes the option.

The simplest way to value a capMT is by 
using Black's model. Suppose no interest payment is 
not a period \( \delta \). Then an option \( L(t) = 
L(t, t+\delta) \) Black proposes that \( L(t) \) is given by

\[
(0.6) \quad L(t) = F(0, t, t+\delta) S(t) \quad \text{for } t \geq 0,
\]

where

\[
(0.7) \quad \frac{dS(t)}{S(t)} = \delta \, dW(t), \quad S(0) = S_0,
\]

and \( W(t) \) is Brownian motion. Note that \( \delta = \infty 
\) then the interest rate \( \text{for the period } (t, t+\delta) \)
is just its drift forward rate. Thus \( \delta = 0 \) yields 
a deterministic situation. If \( \delta \neq 0 \) then the 
fluctuations in the floating rate at time \( t \)
goes with $t$. Using this model we can easily find the value of the caplet $(1.5)$ from the Black–Scholes formula. Thus

$$(1.8) \text{ value of caplet } = \text{ Black–Scholes price of a call option on a stock with strike price $K$, volatility } \sigma, \text{ expiration date } \tau, \text{ and $\tau$-days stock price given by } \mathbb{P}(0,\tau) \mathbb{I}(\tau)\mathbb{I}(\tau,\tau)$$

We may estimate $\sigma$ in $(1.8)$ from historical data or by using Black's formula to compute implied volatility from market data on interest rate caps or interest rate swaptions.

There is a corresponding Black model for swaptions. Suppose we obtain a bond at time $t_0$ with payments at time $T_0$, and interest rate payments at regular intervals. The bond has a payment option at time $T_0 < T_1$. The originally scheduled interest rate payments from $T_0$ to $T_1$ are at $T_1$, $T_2$, ..., $T_N$ and the bond is $K$. We wish to value the payment option. To do this let $R(t)$ be the swap rate for a swap starting at $T_0$ with payment at $T_1$ and interest rate payments at $T_1$, $T_2$.

The value of the payment option is then

$$(1.9) \text{ value of swaplet } = \left[ K - R(T_0) \right] \mathbb{I}(T_0) \mathbb{I}(T_1) + \left[ (T_1 - T_0) \mathbb{I}(T_0) \right] \mathbb{I}(T_1) \mathbb{I}(T_2) + \left[ (T_N - T_{N-1}) \mathbb{I}(T_{N-1}) \right] \mathbb{I}(T_N) \mathbb{I}(T_0)$$

Now $R(t)$ is deterministic and given in terms of $T_0$ days.
Formulas by (1.4). Black's model for valuing the European is then to set

\[
(1.10) \quad R(t) = R(0) S(t), \quad \text{with } S(t) \text{ as in (1.7)}.
\]

Thus we have

\[
(1.11) \quad \text{Value of a European} = \sum_{k=1}^{n} \left( T_k - T_{k-1} \right) R(0) S(T_k)^2.
\]

Which includes one of a host option with strike price \( K \), volatility \( \sigma \), expiration date \( T_0 \), and 2-step stock price given by \( R(0) S \).

Now one of the main uses volatility of interest rates is measured is by the use of Black implied volatilities for both interest rate caps and swaptions. Evidently, these volatilities are the same periods say \( T_0 \) to \( T_2 \) are related, but it is not so easy to sort out the relationship. We can therefore derive to consider a more basic approach, namely the model for evolution of the spot rate.

§ 2 \quad \text{Hull–White Model}

Let \( V(t) \) be the short rate at time \( t \geq 0 \), \( V(0) \) is known. In a model of the short rate desirable properties are: (i) \( V(t) \to 0 \) with prob 1

(ii) \( V(t) \) is large with small probability. In the H–W model we set the \( V(t) \) as the Markovian

(i.e. memory) and governed by the SDE

\[
(2.1) \quad dV(t) = \left[ \theta(t) - a \left( V(t) - \mu \right) \right] dt + \sigma \left( V(t) - \mu \right) dW(t),
\]

where the parameters \( \theta(t) \), \( a \), \( \sigma \rightarrow 0 \).
The advantage of (2.1) is that it is especially suitable once the parameter values are known. Then (2.1) implies

\[(2.2) \quad d [\varphi(k) x^t] = x^t e^{\lambda(k) x^t} + e^{\lambda(k) x^t} \cdot \omega(k).\]

Note (2.2) is obtained from (2.1) by the rules for ordinary calculus since the (the contribution of \(e^\varphi \cdot \omega\) is 0. Thus, we have

\[(2.3) \quad \varphi(k) = \lim_{e \to 0} e^{-\lambda(k) e} \cdot \int_k^t e^{-\lambda(k) e} \cdot e^{\lambda(k) x^t} \cdot \omega(k) \; dx.\]

Thus \(\varphi(k)\) is a linear function with mean and variance given by

\[(2.4) \quad \text{mean} = \lim_{e \to 0} e^{-\lambda(k) e} \cdot \int_k^t e^{-\lambda(k) e} \cdot e^{\lambda(k) x^t} \cdot \text{E}(\omega(k) x^t) \; dx,\]

\[(2.4) \quad \text{variance} = \lim_{e \to 0} e^{-\lambda(k) e} \cdot \int_k^t e^{-\lambda(k) e} \cdot \text{E}(\omega(k) x^t)^2 \; dx = \lim_{e \to 0} \int_k^t e^{-\lambda(k) e} \cdot \text{E}(\omega(k) x^t)^2 \; dx = \text{const.}\]

Thus (b) is satisfied and in fact we say that the process (2.1) is a mean reverting process because \(\varphi(k)\) needs to be pushed towards \(\text{E}(\omega(k) x^t)\). In the model bond prices are deterministic functions of the interest rate \(\omega(k)\). Thus let

\[(2.5) \quad f(k, t, T) = \text{bond price at time } t < T \]

for bond with maturity \(T\) when \(\varphi(k) = \text{const}\), then

\[(2.6) \quad P(k, t, T) = \text{E} \left[ e^{-\int_k^T \omega(k) x^t \; dt} \quad | \varphi(k) = \text{const} \right].\]
null
(2.9) \( \Delta s^*(t) = -\alpha \Delta s(t) \Delta t + \sigma \Delta W(t) \).

Note that (2.9) is symmetric about 0, i.e. \( \Delta s^*(t) \) and \( -\Delta s^*(t) \) have the same distribution. The
radius of the form

(2.10) \((m, j) : 0 \leq m \leq M, \; |j| \leq \min \{m, J\}\).

Thus the lattice looks like:

\[
\begin{array}{cccc}
& x & x & x \\
& x & x & x \\
& x & x & x \\
& x & x & x \\
& x & x & x \\
& x & x & x \\
& x & x & x \\
& x & x & x
\end{array}
\]

\((3 \Delta t, 2 \Delta s)\) \(\Delta s = 5 \) month \(\Delta t = 1 \) month

Evidently the maximum

\[(m, j) \Rightarrow (m+1, j+1) \text{ with prob } \mu \]
\[(m, j) \Rightarrow (m+1, j) \text{ with prob } \nu \]
\[(m, j) \Rightarrow (m+1, j-1) \text{ with prob } \lambda \]

Evidently we must have \(\mu + \nu + \lambda = 1\). If

\( j = J \) then we need to modify the growth

rule since we cannot go up. We do this as follows:

\[(m, J) \Rightarrow (m+1, J) \text{ with prob } \mu \]
\[(m, J) \Rightarrow (m+1, J-1) \text{ with prob } \nu \]
\[(m, J) \Rightarrow (m+1, J-2) \text{ with prob } \lambda \]

Similarly at \( j = -J \), we have
\[(m, -J) \Rightarrow (m+1, -J) \text{ with prob } p_d,
(m, -J) \Rightarrow (m+1, -J+1) \text{ with prob } p_s,
(m, -J) \Rightarrow (m+1, -J+2) \text{ with prob } p_u.
\]

We need to match constants to determine \( p_d, p_s, p_u \). As previously we determine \( A \) above by equating 1st and 2nd moments for the I.D.F (2.9) at the corresponding values for the discrete model. Integrating (2.7) we have

\[
E \left[ \sum_{n=0}^{\infty} x^n \right] = e^t \Rightarrow e^{t+\Delta t}.
\]

\[
x(t+\Delta t) - x(t) = \int_t^{t+\Delta t} \sigma x(t) \, dW(t),
\]

\[
E \left[ \sum_{n=0}^{\infty} x^n \right] = e^{t+\sqrt{\Delta t} \xi}, \text{ where } \xi \sim N(0,1) \text{ to leading order, where (2.12) gives}
\]

\[
x(t+\Delta t) = x(t) + (\sigma x(t)) \Delta t + \sigma x(t) \sqrt{\Delta t} \xi
\]

Thus to highest order we have

\[
E \left[ \sum_{n=0}^{\infty} x^n \right] = -a \Delta t \, x(t) + \sigma x(t) \sqrt{\Delta t} \xi
\]

(2.15)

\[
E \left[ \sum_{n=0}^{\infty} x^n \right] = -a \Delta t \, x(t)
\]

(2.16)

\[
E \left[ \sum_{n=0}^{\infty} x^n \right] = \sigma^2 \Delta t + a^2 x^2(t) \Delta t^2
\]

For the discrete model, if we are at state \((m, j)\)

Then \( x(t) = \Delta x \), where

\[
E \left[ \sum_{n=0}^{\infty} x^n \right] = p_u \Delta x + p_s (x + p_d (x-\Delta x))
\]

and similarly...
\[
\begin{align*}
\psi &= -\frac{1}{2} + \frac{2}{3}a + \frac{4}{5}a^2 - \frac{6}{7}a^3 \\
\eta &= \frac{1}{2} + \frac{3}{4}a - \frac{5}{6}a^2 + \frac{7}{8}a^3 \\
\phi &= 2a + 3a^2 - 4a^3 \\
\end{align*}
\]

The second order equation (2.19) is given by

\[
\frac{d^2 \eta}{dx^2} + \frac{1}{x} \frac{d \eta}{dx} + \left( \frac{1}{x^2} - \frac{1}{x} \right) \eta = 0
\]

Write the general solution of the equation

\[
\eta(x) = A \sin \left( \frac{1}{x} \right) + B \cos \left( \frac{1}{x} \right)
\]

where A and B are constants.

Now, substitute the values of A and B into the equation and

\[
\begin{align*}
-\frac{1}{x} \frac{d \eta}{dx} &= -\frac{1}{x} \left( A \cos \left( \frac{1}{x} \right) - B \sin \left( \frac{1}{x} \right) \right) \\
\frac{d^2 \eta}{dx^2} &= \frac{1}{x^2} \left( A \cos \left( \frac{1}{x} \right) - B \sin \left( \frac{1}{x} \right) \right) - \frac{1}{x} \left( -\frac{1}{x} A \sin \left( \frac{1}{x} \right) - B \cos \left( \frac{1}{x} \right) \right)
\end{align*}
\]

Substituting these values into the equation, we get

\[
\frac{1}{x^2} \left( A \cos \left( \frac{1}{x} \right) - B \sin \left( \frac{1}{x} \right) \right) - \frac{1}{x} \left( -\frac{1}{x} A \sin \left( \frac{1}{x} \right) - B \cos \left( \frac{1}{x} \right) \right) + \frac{1}{x} \left( A \cos \left( \frac{1}{x} \right) - B \sin \left( \frac{1}{x} \right) \right) = 0
\]

Simplifying, we get

\[
A \cos \left( \frac{1}{x} \right) - B \sin \left( \frac{1}{x} \right) + A \sin \left( \frac{1}{x} \right) + B \cos \left( \frac{1}{x} \right) = 0
\]

Which simplifies to

\[
A \cos \left( \frac{1}{x} \right) + B \sin \left( \frac{1}{x} \right) = 0
\]

Thus, the solution is

\[
\eta(x) = A \sin \left( \frac{1}{x} \right) + B \cos \left( \frac{1}{x} \right)
\]

For the solution of the differential equation (2.18),

\[
\int \left( \frac{1}{x^2} - \frac{1}{x} \right) \frac{dx}{\eta(x)} = \frac{1}{2} \log \left( \frac{1}{x} \right) - \log x + C
\]

where C is a constant. Therefore, the solution is

\[
\int \frac{dx}{\eta(x)} = \frac{1}{2} \log \left( \frac{1}{x} \right) - \log x + C
\]
Note that the formulas (2.25) put a limit on the value of $J$. To see this observe first that since $\Delta t \ll \Delta s$ because $\Delta t \sim (\Delta s)^2$ we have $J \Delta t < \Delta s = O(\Delta)$ and $\Delta t \ll \Delta s$ is also small. Hence from (2.25) we must have

$$a J \Delta t < 1/3.$$  

We need to assign Hamilton probabilities at the boundary lattice points $(n, J)$, $n \geq J$. We do this by equating moments as before. Now we have

$$E[ \{ \psi(t + \Delta t) - \psi(t) \}^2 ] = p\psi(0) + f_3(-2\Delta s) + f_4( - 2 \Delta s)$$

and

$$E[ \{ \psi(t + \Delta t) - \psi(t) \}^2 ] = p\psi(0) + f_3(2\Delta s)^2 + f_4(2 \Delta s)^2.$$  

Then the equations for $f_3, f_4$ become

$$f_3 + 2 f_4 \Delta s = 0 \Delta t + J \Delta s,$$

$$(f_3 + 2 f_4) \Delta s = \sigma^2 \Delta t + a^2 J^2 (\Delta s)^2 (\Delta t)^2.$$  

This gives us

$$f_4 = \frac{1}{6} + \frac{1}{2} [ a^2 J^2 (\Delta t)^2 - a J \Delta t ],$$

$$f_3 = -\frac{1}{3} - a^2 J^2 (\Delta t)^2 + 2a J \Delta t,$$

$$p\psi = \frac{1}{6} + \frac{1}{2} [ a^2 J^2 (\Delta t)^2 - 3a J \Delta t ].$$  

We need to rewrite $p\psi$ in the above formula. We rewrite it as
\( (2.36) \quad \rho_s = \frac{2}{3} - \left[ a \sqrt{\Delta t} - 1 \right]^2, \) where we note
\( \frac{\Delta t}{\Delta \theta} \) is given by the formula.

\( (2.31) \quad 1 - \frac{\sqrt{2}}{3} < a \sqrt{\Delta t} < 2 + \frac{\sqrt{2}}{3}. \)

Since we want to ensure (2.26) also holds, we take \( \Delta t \) to be given by the formula
\( (2.32) \quad a \sqrt{\Delta t} = 1 - \frac{\sqrt{2}}{3} = 0.184. \)

This is the Hall-White value for \( \Delta t \). Observe that
\[ \frac{\Delta t}{\Delta \theta} = 1 \]
we have the same formulas as (2.29) for the Hamilton probabilities, but with \( \rho_s \) and \( \rho_d \) interchanged.

Next we wish to argue that we can take a much smaller value for \( \Delta t \) than in (2.31) without any particular loss in accuracy. To see this, we go back to the formula (2.3), recalling that the \( \rho \) for \( \rho(\Delta t) \) does not exceed \( \Delta t / \sqrt{2\pi} \). Since \( r(\tau) = 0 \)
we really only need a bistable model which crosses \( 3 \Delta \theta \) from 0, where we can take \( \Delta t \) by
\( (2.33) \quad \Delta t = 3 \Delta \theta / \sqrt{2\pi} \Rightarrow \Delta t = \left[ \frac{3}{2} \left( 2a \Delta \theta \right)^2 \right]^{1/2} \).

Evidently for a \( \Delta t \ll 1 \) the value of \( \Delta t \) in (2.33) is much smaller than the \( \Delta t \) or \( \Delta \theta \) value (2.32). With \( \Delta t \) given by (2.33) we cannot make use of the formulas (2.29) for the Hamilton probabilities. Instead we use simply a reflecting boundary condition. That is, \( \rho_s = 0 \), \( \rho_d = 0 \).

The reason for this is that \( \rho(\Delta t) = 3 \Delta \theta \) for any \( \Delta t \) with very small probability. If \( \rho(\Delta t) = 3 \Delta \theta \) the condition (2.34) is met. In particular it will
not introduce an instability, something which occurs with negative probabilities.

Finally we need to make some comment about the value of $\Delta t$. Our chosen value for $\Delta t$ is $\Delta t = 2.5$ corresponding to 2 month bonds. If we take $\Delta t$ to be 0.015 then (2.22) yields $\Delta s = \sqrt{2} \Delta t = 0.13$. Thus for Abus lattice model the possible interest rate jumps exceed $\Delta s$ which is uncritically too large. One way of getting around this problem is to make $\Delta t$ smaller and evaluate the local prices needed to calibrate the model i.e. find $\theta(t)$ from bond prices via (2.11) by using some type of interpolation technique.

Next we wish to show how to determine the function $\theta(t)$ from 4-day's yield curve by using (2.1). First we observe that a lattice model for the SDE (2.1) can be obtained from the lattice model for the SDE (2.9) in a natural way. To see this we set $\Delta(t) = \Delta(t) - \Delta(t')$ and note that (2.1) and (2.9) imply

$$\Delta(t) = \int [\theta(t) - a(t)] dt, \quad \theta(t) = \nu.$$

Thus $\theta(t)$ is determined in terms of $\theta(t)$. Put another way, the lattice we have already constructed for the evolution of $e^{i(t)}$ also models the evolution of $\theta(t)$ provided we equate with a lattice point $(m, j)$ the interest rate

$$\tau = m \Delta s + \Delta s \Delta t$$

we should write (in $\Delta(t) = \Delta(t)$ with $\Delta(t)$ evolving as in (2.35)).
Consider now how to determine the values \( d^y \) for \( y = 0, 1, \ldots \) from the 350 yield curve. First we approximate

\[
(2.36) \quad \int_0^T r(t) dt = r(0) \Delta t + r(\Delta t) \Delta t + \ldots + r((M-1)\Delta t) \Delta t
\]

where \( \Delta t = T \). Then from (2.27) we have that

if \( \rho^2 \) is the price of a 350 year bond maturing at time \( \Delta t \),

\[
(2.37) \quad \rho^2 = \exp \left[ -2^0 \Delta t \right] \text{ yielding}
\]

\[
(2.38) \quad \lambda = -\frac{1}{\Delta t} \log \left( \rho^2 \right).
\]

We are looking for \( \lambda \), where \( r(0) = 2^0 \). There are 3 possible values:

\( \rho^2 = 2^0 + r^*(\Delta t) \) since \( r^*(\Delta t) = 0, \pm \Delta r \). Writing \( \lambda \)

\[
(2.39) \quad \int_0^T r(t) \Delta t = x_0 \Delta t + \left[ 2^0 + r^*(\Delta t) \right] \Delta t
\]

and using the fact that \( r^*(\Delta t) = 0 \) with prob \( p_x \),

\( r^*(\Delta t) = \pm \Delta r \) with prob \( p_y \), and \( r^*(\Delta t) = \mp \Delta r \) with prob. \( p_z \), we conclude that

\[
(2.40) \quad \rho = \exp \left[ -\left\{ 2^0 + 2^2 + 2 \Delta r \right\} \right] p_x + \exp \left[ -\left\{ 2^0 + 2^2 - 2 \Delta r \right\} \right] p_y + \exp \left[ -\left\{ 2^0 + 2^2 \Delta r \right\} \right] p_z
\]

Here we are using the notation \( \rho^m \) in the price of a 350 year bond maturing at time \( m \Delta t \). Note that \( \rho_x \) (2.40) we have from (2.27) that \( p_x = 1/6 \), \( p_y = 1/6 \), \( p_z = 2/3 \). Evidently (2.40) determines the value of \( d^y \).
We need to come up with an algorithm to compute
the values of $d^m$ for all $m$. To achieve this,
we introduce the notion of an AAD (Analytical
Algorithm) security. For $m = 1, 2, \ldots, M \leq \min \{m, J\}$ we
define the AAD security at node $(m, j)$ by

$$Q^m_{j, \Delta} = E \left[ \exp \left\{ - \int_0^\Delta \left( \mu dt \right)_j \right\} ; \tau^j \left( m \Delta t \right) = \Delta \right] .$$

Thus, we have

$$Q^m_\Delta = e^{-\Delta \kappa}$$
$$Q^0_\Delta = e^{-\Delta \kappa}$$

Further, we have that

$$Q^m_{j+1} = \sum_{j=-\min\{m, J\}}^{\min\{m, J\}} Q^m_{j, \Delta} \exp \left\{ - (k^m + \Delta \kappa) \Delta t \right\}.$$  

The basic point now is that $Q^m_{j, \Delta}$ are
computable in terms of $d^m$, $d^{m-1}$. Thus, (2.43)
determines the value of $d^m$.

To complete the algorithm for determining $d^m$ we obtain a recurrence rela
for $Q^m_{j, \Delta}$. Thus, (2.43) becomes

$$Q^m_{j, \Delta} = \sum_{j=-\min\{m, J\}}^{\min\{m, J\}} Q^m_{j, \Delta} \exp \left\{ - (k^m + \Delta \kappa) \Delta t \right\}.$$  

Next, we can write

$$Q^m_{j+1} = \sum_{j=-\min\{m, J\}}^{\min\{m, J\}} Q^m_{j, \Delta} \exp \left\{ - (k^m + \Delta \kappa) \Delta t \right\}.$$  

Thus, we can express $Q^m_{j, \Delta}$ as a

$$Q^m_{j+1} = \sum_{j=-\min\{m, J\}}^{\min\{m, J\}} Q^m_{j, \Delta} \exp \left\{ - (k^m + \Delta \kappa) \Delta t \right\}.$$  

$$Q^m_{j+1}$$ as follows:

$$Q^m_{j+1} = \sum_{j=-\min\{m, J\}}^{\min\{m, J\}} Q^m_{j, \Delta} \exp \left\{ - (k^m + \Delta \kappa) \Delta t \right\}.$$  

Thus, we can express $Q^m_{j, \Delta}$ as a

$$Q^m_{j+1} = \sum_{j=-\min\{m, J\}}^{\min\{m, J\}} Q^m_{j, \Delta} \exp \left\{ - (k^m + \Delta \kappa) \Delta t \right\}.$$  

Thus, we can express $Q^m_{j, \Delta}$ as a

$$Q^m_{j+1} = \sum_{j=-\min\{m, J\}}^{\min\{m, J\}} Q^m_{j, \Delta} \exp \left\{ - (k^m + \Delta \kappa) \Delta t \right\}.$$  

Thus, we can express $Q^m_{j, \Delta}$ as a

$$Q^m_{j+1} = \sum_{j=-\min\{m, J\}}^{\min\{m, J\}} Q^m_{j, \Delta} \exp \left\{ - (k^m + \Delta \kappa) \Delta t \right\}.$$  

Thus, we can express $Q^m_{j, \Delta}$ as a
\( Q^m_j = \exp \left[ - \left( \frac{2^m + 2j + 2mT}{2} \right) DT \right] p_0(j) Q^m_j + \exp \left[ - \left( \frac{2^m + 2j + 2mT}{2} \right) DT \right] p_1(j+1) Q^m_{j+1} + \exp \left[ - \left( \frac{2^m + 2j - 2mT}{2} \right) DT \right] p_1(j-1) Q^m_{j-1}. \)

Evidently by combining (2.45) and (2.43) we can compute all the \( Q^m_j \) and \( p_0(j) \) recursively. Note that (2.45) needs to be modified when \( j \) is close to the boundary of the lattice. To give an example consider the HW assignment of probability at the boundary given by (2.21). If \( m = \frac{1}{2} \), then \( Q^m_{j-2} \) is a sum of \( Q^m_j \) with \( j = \frac{1}{2}, \frac{3}{2}, 1, \frac{5}{2} \).

We can think of the AP sensitivity as an artificial priced instrument. Thus

\[ Q^m_j = \text{value today of a security which at time } mDT \text{ pays } 1 \text{ if } \text{at a rate at time } mDT \text{ corresponds to node } j \text{ and pays } 0 \text{ if not}. \]

We can also model to use the HW model to value interest rate derivatives. We are assuming now that the model is fully calibrated.

Suppose first we have an interest rate cap with cap \( K \), and suppose the domain extends to time \( T \) with \( MDT = T \). Let \( X(m, j) \geq m = 0, \ldots, M, \)

\[ |j| \leq \min \{m, J \} \] the value of the interest rate cap in node \( X(0, c) \). We compute this by
working backwards in the plane. Thus \( V(M, 3) = 0 \) and \( V(m, j) \) for \( m < M \) is given by the recurrence

\[
V(m, j) = \begin{cases} 
0 & \text{if } m = 0 \\
\Delta \exp \left[ \frac{m}{\Delta} - \frac{J}{\Delta^2} \right] + p_u V(m+1, j) + p_d V(m+1, j-1) \exp \left[ -\frac{m}{\Delta} \right] \Delta J ,
\end{cases}
\]

where \( \Delta = \frac{m + 2 \Delta^2}{\Delta} \). Explicitly the formula

\[
(2.49) \quad V(M, j) = \int_0^\infty \exp \left[ -\frac{m}{\Delta} \right] \Delta J 
\]

We may also find the value of the integral

\[
(2.47) \quad \int_0^\infty \exp \left[ -\frac{m}{\Delta} \right] \Delta J = \int_0^\infty \exp \left[ -\frac{m}{\Delta} \right] \Delta J 
\]

We may also find the value of the integral

\[
(2.48) \quad \int_0^\infty \exp \left[ -\frac{m}{\Delta} \right] \Delta J = \int_0^\infty \exp \left[ -\frac{m}{\Delta} \right] \Delta J 
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\[
(2.49) \quad \int_0^\infty \exp \left[ -\frac{m}{\Delta} \right] \Delta J = \int_0^\infty \exp \left[ -\frac{m}{\Delta} \right] \Delta J 
\]
We can similarly find the value of a swap in a loan, where the swap is one of a perpetual option on a loan. The loan is one of a fixed 0, 1, 2, ... T years with a perpetual option at year T. The interest is paid annually. Thus, the swap is a put. If the fixed rate on the loan is \( r \), then the value of the option at time \( T_0 \) at rate \( j \) is given by

\[
V(T_0, j) = \left[ X - \text{swap} (T_0, j) \right] + \left[ P(1, T_0, T_0+1) + P(1, T_0, T_0+2) + \ldots + P(1, T_0, T) \right] j
\]

where \( P(1, T_0, T) \) is the price at rate \( j \) of a bond maturing at time \( T \) > \( T_0 \). The swap rate swap \( (T_0, j) \) is given by the usual formula

\[
\text{swap} (T_0, j) = \left[ (1 - P(1, T_0, T)) j \right] / \left( P(1, T_0, T_0+1) + \ldots + P(1, T_0, T) \right)
\]

Thus, to compute the value of \( V(T_0, j) \) we must find the values of the various bonds. This can be done by the usual discounting procedure.

\[
V(m, j) = 1 \left[ \text{swap} \left( m, j \right) + \text{swap} \left( m+1, j \right) + \ldots + \text{swap} \left( m+i, j \right) \right]
\]

For a bond with maturity \( T \), we set \( V(m, j) \) = \( 2 \) with \( m+i = T \). Similarly, one at time \( m+i-j-1 \) day exp \( \left[ - \frac{T}{j} \right] \), for a bond with maturity \( T \).
§3 Black-Derman-Toy (B-D-T) Model

The B-D-T model is a model for the evolution of the short rate which has some advantages and also some disadvantages over the HW model. First we define a discrete model and then take its continuum limit. An index here is parameterized by \( m \in \mathbb{N}^* \) with \( m = 0, 1, 2, \ldots \). The lattice consists of \( (m,j) \) with \( 0 \leq j \leq m \). To attach interest rates to the lattice points \((m,j)\) \( 0 \leq j \leq m \), at fixed \( m \) we need a parameter \( r_0^m \) and \( \beta^m \).

Then the interest rate associated with the lattice point \((m,j)\) is given by

\[
(3.1) \quad r_{m,j} = r_0^m \exp \left[ -2j \beta^m \Delta t \right], \quad 0 \leq j \leq m
\]

Now \( r_0^m \beta^m \to 0 \) as \( m \to \infty \) while in HW all interest rates are positive. Note that \( r_0^m \) is the maximum possible interest rate allowed at time \( m \Delta t \). For transition probabilities we have

\[
(3.2) \quad (m,j) \Rightarrow (m+1,j) \quad \text{with prob } \frac{1}{2}
\]

(\( m,j \) \( \Rightarrow \) \( (m+1), j+1 \) \( \text{with prob } \frac{1}{2} \))
We show that $\beta^m$ is a volatility. To see this, let

\[
(3.3) \quad \log \frac{c(m \Delta t)}{c((m-1) \Delta t)} = \frac{1}{2} \log \left( \frac{c(m \Delta t)}{c((m-1) \Delta t)} \right) = \frac{1}{2} \log \left( \frac{c(m \Delta t)}{c((m-1) \Delta t)} \right)
\]

(3.4) $\log \left( \frac{c(m \Delta t)}{c((m-1) \Delta t)} \right) = \left( \frac{\log c}{\Delta t} \right)^2 \Delta t$.

Consider now how to calibrate the parameters $c_0$, $\beta^m$ of the model to market data. Proceeding as for the HW model, let $Q_0 \Delta t$ be the AD

\[
(3.5) \quad Q_0 \Delta t = \exp \left[ - \int_0^t c(t) \Delta t \right] ; \quad c(t) = \frac{c_0}{\sqrt{\Delta t}}
\]

Then just as in (2.43) we have that

\[
(3.6) \quad \beta^{m+1} = \sum_{j=0}^{m} Q_j \Delta t \exp \left[ - \frac{c_j}{\sqrt{\Delta t}} \Delta t \right]
\]

As before, the $Q_j$ are determined by a recursive equation.

\[
(3.7) \quad Q_j \Delta t = \frac{1}{2} \exp \left[ - \frac{c_j}{\sqrt{\Delta t}} \Delta t \right] Q_j \Delta t + \\
\frac{1}{2} \exp \left[ - \frac{c_{j-1}}{\sqrt{\Delta t}} \Delta t \right] Q_{j-1} \Delta t, \quad 0 \leq j \leq m
\]

As before, but now the AD is

\[
(3.8) \quad Q_j \Delta t = \frac{1}{2} \exp \left[ - \frac{c_j}{\sqrt{\Delta t}} \Delta t \right] Q_j \Delta t + \\
Q_{m+1} = \frac{1}{2} \exp \left[ - \frac{c_0}{\sqrt{\Delta t}} \Delta t \right] Q_0 \Delta t.
\]
Evidently the $Q^m_{ij}$ are computable in terms of $r^{ij}_m, \beta^m$, $m'=e, \ldots, m-1$. We need 2 equations when $A$ determines $r_m^m, \beta_m^m$. One of these equations is (3.6) but to obtain a second one is not so evident. We deduce from information about the volatility of the yield curve.

We can approximately do this by using Black implied volatility for each $t$. To see this let us observe that

(3.1) \[
\log \tau(m \Delta t) = \log r_m^m + 2 \beta_m^m \sqrt{\Delta t} X_m,
\]

where $X_m$ is the binomial variable for the sum of $m$ independent random variables with prob. $\frac{1}{2}$. It follows that

\[
\mathbb{E}[X_m] = m \mathbb{E}[X] = \frac{m}{2},
\]

where we conclude

(3.10) \[
\mathbb{E}[\log \tau(m \Delta t)] = (\beta_m^m)^2 m \Delta t = (\beta^m)^2 t.
\]

Thus we have

(3.11) \[
\tau(m \Delta t) \sim \text{exponential of a Gaussian variable with variance } (\beta^m)^2 t.
\]

We should compare this to Black formula for estimating the value of a call $c$ at the time interval $[m \Delta t, (m+1) \Delta t]$. Thus

(3.12) \[
\mathbb{E}[c(t)] = \mathbb{E}[\text{exponential of a Gaussian variable with variance } \sigma^2 t, \text{ with constant } \sigma \text{ at time } t].
\]

Here we make the approximation

(3.13) \[
\beta^m = \text{Black implied volatility at a call in the interval } [m \Delta t, (m+1) \Delta t].
\]
since we have approximately

\[(3.14) \quad E\left[x(t)\right] = E\left(x, t\right), \quad \text{for small volatility}.
\]

Noting that we can come up with a formula

for the RHS of (3.14). Thus

\[(3.15) \quad E\left[\nu^2(t, \Delta t)\right] = \int_0^\infty \int_0^\infty E\left[\exp\left(2 \beta \sqrt{\Delta t} x\right)\right] \nu^2(t, \Delta t) dx = \int_0^\infty \int_0^\infty \exp\left(2 \beta \sqrt{\Delta t} x\right) \nu^2(t, \Delta t) dx.
\]

It is clear now that if we determine \(\beta^m\) from

(3.13) then we can use (3.6) and the AD formula to determine \(\nu^2\). Thus the model

is completely calibrated. We can determine the

value of interest rate. Anomalies in the one way

as we did for \(\nu^2\).

Next we wish to find the continuum limit of

the \(\Delta t\) model. Going back to (3.3) we see that

\[(3.16) \quad k = [\log c_m + 2\theta \beta^{m-1} \sqrt{\Delta t}] - [\log c_m + 2\beta \beta^{m-1} \Delta t]
\]

\[= \frac{\beta^{m-1}}{\beta^{m-1}} \left[\log c_{m-1} + 2\theta \beta^{m-1} \sqrt{\Delta t}\right] + \text{Function of only}
\]

Thus defining \(\beta^m = \sigma(\nu^2)\Delta t\) then we have

\[(3.17) \quad \log \sigma(t + \Delta t) - \log \sigma(t) = \left[\frac{\sigma(t + \Delta t) - \sigma(t)}{\sigma(t) \Delta t}\right] \frac{\beta^{m-1}}{\beta^{m-1}} \left[\log c_{m-1} + 2\theta \beta^{m-1} \sqrt{\Delta t}\right] + \text{Function of only}
\]

\[
\sigma(t) \Delta t + \sigma(t) \Delta W(t).
\]

Setting \(\Delta t \to 0\), we conclude that

\[(3.10) \quad \Delta \left[\log c(t)\right] = \left[\theta(t) + \frac{\sigma(t)}{\sigma(t)} \log c(t) + \frac{\sigma(t)}{\sigma(t)} \Delta W(t)\right] \Delta t.
\]
We can see from (3.18) that $r(t)$ is the expected value of a Gaussian variable. In fact, setting $X(0) = \ln Y_0$, we have from (3.18) that

$$\frac{dX(t)}{dt} = \frac{\theta(X(t))}{\sigma(X(t))} X(t) + \frac{\lambda(t)}{\sigma(X(t))}.$$  

Using Ito's formula we get then

$$d\left[ \frac{X(t)}{\sigma(t)} \right] = \frac{\theta(X(t))}{\sigma(t)} dt + \frac{\lambda(t)}{\sigma(t)} dW(t),$$

where

$$\frac{X(T)}{\sigma(T)} = X(0) \frac{\sigma(T)}{\sigma(0)} + \int_0^T \frac{\theta(Y(t))}{\sigma(Y(t))} dt + \sigma(T) W(T).$$

Thus $\log r(t)$ is Gaussian with variance $\sigma^2(t)^2$.

We also easily see from (3.22) that

$$E[r(T)] = \exp \left[ \frac{\sigma^2(T)}{\sigma(0)} \log r(0) + \sigma(T) \int_0^T \frac{\theta(Y(t))}{\sigma(Y(t))} dt \right] + \frac{\sigma^2(T)}{2}.$$  

Note: This expression involves the volatility so it is not as easy to see from it that for small volatility the RHS is approximately the forward rate.

There is a significant disadvantage of PDT from HW: The HW model is more realistic, where interest rates cannot get large. We have seen that in choosing $\sigma > 0.2$, an $0.2 \%$ interest rate fluctuates by less than $0.02\%$.

Now from (3.18) we see that for the PDT
model to be mean reverting we need $\sigma^2(t) \lambda(t) < \theta$, in fact $\sigma^2(t) \lambda(t) < -\theta$ for some $\lambda(t)$ to make sure $\sigma(t)$ remains within some finite interval for all $t$ with high prob. Thus of course implies that $\sigma(t) < \sigma(0) e^{\lambda(t)}$, so volatility must go rapidly to 0, which certainly does not hold in practice.

Forward Rate Models

For the short rate models we have considered the short rate evolves according to an SDE

$$\frac{d\tilde{r}(t)}{\tilde{r}(t)} = \lambda(t, \tilde{r}) \, dt + \sigma(t, \tilde{r}) \, dW(t),$$

and

Next the value of a bond with maturity $T$ at time $t \leq T$ is given by

$$P(t, \tilde{r}, T) = E \left[ \exp \left( \int_t^T \lambda(s, \tilde{r}(s)) \, ds \right) \left| \tilde{r}(t) = \tilde{r} \right. \right].$$

One sees from this that $P(t, \tilde{r}, T)$ satisfies

the SDE

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2(t, \tilde{r}) \frac{\partial^2 P}{\partial \tilde{r}^2} + \lambda(t, \tilde{r}) \frac{\partial P}{\partial \tilde{r}} - \frac{r}{2} P = 0, \quad t < T,$$

let us next consider the time evolution of the bond price as a function of $t < T$, which we will as $P(t, \tilde{r}, T)$. Thus $P(t, \tilde{r}, T) = P(t + \delta t, \tilde{r}, T)$ where $\tilde{r}(T)$ satisfies (4.1). Thus

$$\frac{dP(t, \tilde{r}, T)}{dt} = \lambda(t, \tilde{r}) P(t, \tilde{r}, T) + \frac{1}{2} \sigma^2(t, \tilde{r}) \frac{\partial^2 P}{\partial \tilde{r}^2} \Delta t = \left[ \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2(t, \tilde{r}) \frac{\partial^2 P}{\partial \tilde{r}^2} + \lambda(t, \tilde{r}) \frac{\partial P}{\partial \tilde{r}} - \frac{r}{2} P \right] \Delta t + \frac{\partial P}{\partial \tilde{r}} \sigma(t, \tilde{r}) \Delta W(t) \Delta t.$$
Thus we have that
\[
\frac{d\theta(t, T)}{\theta(t, T)} = \frac{\sigma(t)\vartheta + \frac{1}{\rho} \frac{\partial}{\partial T} \sigma(t, T) \, d\mathcal{W}(T)}{\theta(t, T)},
\]
for the HW model we can explicitly find
\[\theta(t, T), \quad \text{Thus we have}
\]
\[
\theta(t) = \begin{cases} \theta(0) e^{-at} + \int_0^t e^{-a(t-u)} \theta(u) \, du & \text{if } t \leq T, \\ 0 & \text{if } t > T. \end{cases}
\]
Hence
\[
\int_0^T \theta(t) \, dt = \theta(0) \left[1 - e^{-aT}\right] + \int_0^T \frac{1 - e^{-a(T-u)}}{a} \theta(u) \, du + \sigma \int_0^T \frac{1 - e^{-a(T-u)}}{a} \, d\mathcal{W}(u).
\]
Thus
\[
\theta(t, T) = \mathbb{E} \left[ \exp \left[ \frac{\theta(0) \left[1 - e^{-aT}\right]}{a} + a(T-T) \right] \right],
\]
we conclude that
\[
\frac{1}{\rho} \frac{\partial}{\partial T} \theta(t, T) = \frac{1 - e^{-aT}}{a}.
\]
Here we evaluate of the bond price in the case of the HW model is given by
\[
\frac{\partial \theta(t, T)}{\theta(t, T)} = \sigma(t) \vartheta + \chi(t, T) \, d\mathcal{W}(T), \quad \text{where}
\]
\[
\chi(t, T) = \sigma \left[1 - 2 - a(T-t)\right] / a.
\]
Note that \( \lim_{t \to T} \gamma(t, T) = 0 \). This is natural since bond prices \( t \to T \) volatility should go to zero as \( t \) approaches the maturity \( T \).

The HJM model consists of equation (4.11) and three evolution equations and also puts the volatility \( \gamma(t, T) \) as a perfect delta. The basic object in the HJM model is the instantaneous forward rate \( f(t, T) \). To obtain an evolution equation for it, first derive from (4.11) that

\[
(4.13) \quad \frac{d}{dt} \log p(t, T) = \left[ \gamma(t) - \gamma(t, T)^2 \right] \frac{\gamma(t)}{2} + \gamma(t, T) \frac{\partial \gamma(t, T)}{\partial t},
\]

by Ito’s formula. Define now the forward rate \( f(t, T_1, T_2) \) with \( t < T_1 < T_2 \) by continuity as

\[
(4.14) \quad p(t, T_2) = p(t, T_1) \exp \left[ -\int_{T_1}^{T_2} f(t, T_1, T_2) dt \right],
\]

which is the same as

\[
(4.15) \quad f(t, T_1, T_2) = \frac{\log p(t, T_1) - \log p(t, T_2)}{T_2 - T_1}.
\]

Then (4.13) gives for the evolution of \( f(t, T_1, T_2) \),

\[
(4.16) \quad df(t, T_1, T_2) = \gamma(t, T_2)^2 - \gamma(t, T_1)^2 \frac{df(t, T_1, T_2)}{2(T_2 - T_1)} dt + \gamma(t, T_1) - \gamma(t, T_2) \frac{df(t, T_1, T_2)}{T_2 - T_1} dW(t).
\]

Next we let \( T_2 \to T_1 \) to get the evolution of the instantaneous forward rate.

\[
(4.17) \quad df(t, T) = \left[ \frac{1}{2} \frac{2}{\gamma^2(t)} \gamma(t, T)^2 \right] dt + \frac{\gamma(t, T)}{\gamma(t)} dW(t).
\]
Oliver (4.19) gives a relation between the drift and volatility of the instantaneous forward rate. Using the fact that \( \tau(t, T) = 0 \) we have
\[
(4.18) \quad \tau(t, T) = \int_t^T \frac{\partial}{\partial t} \log \sigma(\tau(t, u)) \, du.
\]
Here if we set \( \sigma(t, T) = -\sigma(x(t, T))/\sigma(t) \) then we have
\[
(4.19) \quad \rho(t, T) = \left[ \sigma(t, T) \int_t^T \sigma(x(t, u)) \, du \right] \, dt + \sigma(t, T) \, dW(t).
\]

In the case of the HJM model we have
\[
(4.20) \quad \sigma(t, T) = \sigma(t, T) \quad \text{where}
\]
the forward rate volatility depends only on time \( t \) and maturity. Moreover, the forward rate volatility increases with \( T \) with a maximum of \( \sigma(t, T) \).

If we take (4.19) as our basic equation in the case of some compound bond prices \( f(t, T) \) then instantaneous forward rates by means of the equation,
\[
(4.21) \quad f(t, T) = -\frac{2}{\sigma(t, T)} \log \, f(t, T).
\]
Thus
\[
(4.22) \quad f(t, T) = \exp \left[ -\int_t^T f(t, u) \, du \right].
\]

It is easy to prove from (4.19) and (4.20) that the equation (4.11) for the evolution of \( f(t, T) \), subject to \( f(t, t) = f(t, T) \).

In the HJM world the basic equation is (4.19), modeling the evolution of forward rates as an SDE. i.e. Martingale with no mean. We have seen that for HW both evolution of
short rates and evolution of forward rates at both Maturities. Thus in an exceptional situation, one can see once generally that for both the evolution of \( S(t) \) and \( d(t, t) \) to be Markovian we need the volatility \( \sigma(t, T) \) to satisfy

\[
\sigma(t, T) = \phi(t, \psi(T)) \tag{4.23}
\]

Next we turn to a discrete version of the HJM model, the so-called Libor Forward Market model (LFM). Here we set discrete times \( T_0 < T_1 < T_2 < \ldots \) where typically \( T_n - T_{n-1} = \frac{1}{4} \) is a quarterly next time. The basic object is \( F_n(t) = F(t, T_n, T_{n-1}) \), \( n = 1, 2, \ldots, T < T_{n-1} \), so

\[
F_n(t) = \frac{\phi(t, T_{n-1}) - \phi(t, T_n)}{(T_{n-1} - T_{n-1}) \phi(t, T_n)} \tag{4.24}
\]

Here we have

\[
\log \left[ 1 + (T_n - T_{n-1}) F_n(t) \right] = \int_{T_{n-1}}^{T_n} \frac{\phi(t, T_n) - \phi(t, T_{n-1})}{(T_n - T_{n-1}) \phi(t, T_{n-1})} \, dt + \int_{T_{n-1}}^{T_n} [\phi(t, T_{n-1}) - \phi(t, T_n)] \, dW(t) \tag{4.25}
\]

Using the fact that

\[
1 + (T_n - T_{n-1}) F_n(t) = \exp \left\{ \log \left[ 1 + (T_n - T_{n-1}) F_n(t) \right] \right\} \tag{4.26}
\]

we have from (4.25) that

\[
\frac{(T_n - T_{n-1}) \, dF_n(t)}{1 + (T_{n-1} - T_n) F_n(t)} = \int_{T_{n-1}}^{T_n} \frac{\phi(t, T_{n-1}) - \phi(t, T_n)}{2} \, dt \tag{4.27}
\]
We can rewrite (4.22) as

\[ \frac{dF_x(t)}{F_x(t)} = m_x(t) dt + \sigma_x(t) \, dW(t), \]

where

\[ \sigma_x(t) = \frac{[v(T_{t+1}) - v(T_t)] [1 + (T_t - T_{t-1}) F_x(t)]}{(T_t - T_{t-1}) F_x(t)}, \]

\[ m_x(t) = \frac{[v(T_{t+1}) - v(T_t)] v(T_t) [1 + (T_t - T_{t-1}) F_x(t)]}{(T_t - T_{t-1}) F_x(t)} \]

We can rewrite (4.30) as

\[ \sigma_x(t) = -v(T_t) \, \sigma_x(t) \]

\[ v(T_t) = v(T_{m(t)}) + \sum_{k=m(t)+1}^{t} v(T_k) - v(T_{k-1}), \]

where \( m(t) \) is defined by

\[ m(t) = \text{smallest integer such that } t \leq T_{m(t)}. \]

Since \( v(t, t) = 0 \) we approximate \( v(T_{m(t)}) = 0 \),

where (4.31) becomes from (4.29)

\[ v(T_t) = -\sum_{k=m(t)+1}^{t} \frac{(T_k - T_{k-1}) F_k(t)}{1 + (T_k - T_{k-1}) F_k(t)} \sigma_x(t) \]

Then (4.23) becomes

\[ \sigma_x(t) = \frac{\sum_{k=m(t)+1}^{t} \frac{(T_k - T_{k-1}) F_k(t)}{1 + (T_k - T_{k-1}) F_k(t)} s_k(t) \, dW(t) + \sigma_x(t) \, dW(t)}{F_x(t)}. \]
Observe that the formula (4.35) is equivalent to

\[
\frac{\delta F_\lambda(t)}{F_\lambda(t)} = \frac{(T_\lambda - T_{x-1}) F_\lambda(t)}{1 + (T_\lambda - T_{x-1}) F_\lambda(t)} \frac{\delta x(t)^2 \Delta t}{\delta x_{x-1}(t)} \frac{d F_{x-1}(t)}{F_{x-1}(t)}, \quad t < T_{x-2}
\]

We need to think more carefully about what (4.36) means. First let us recall what our basic assumption was. We are no longer assuming that the evolution of the spot rate \( r_s(t) \) is Markovian, meaning \( \Delta r_s(t) \) is a Gaussian variable with mean and variance depending only on \( r(t) \) and \( \sigma(r) \). Instead we assume \( \Delta r_s(t) \) is a Gaussian variable with mean and variance which can depend on the history \( r(t) \), \( t < t \), \( t \in T \). We call this a Markov process for the interest \( r(t) \), \( t \in T \). The value today is given by

\[
E \left[ \exp \left( - \int_0^t r(t) \, d\tau \right) \Delta F_x(T_{x-1}) - \frac{1}{2} \right],
\]

where \( r(t) \) is the forward rate at time \( t \) for the interval \( (T_{x-1}, T_x) \) and \( K \) is the cap. The function \( F_x(t) \) is known by re-scaling bond prices. We can rewrite (4.38) as

\[
P(x, T) \cdot \mathbb{E} \left[ \left( \frac{F_x(T_{x-1})}{K} \right)^2 \right],
\]
where $E^Q_{t_i}$ is known as the risk neutral measure $P$

the general case $F_i(t)$, $t < T_{n-1}$. We shall show that $F_i(t)$, $t < T_{n-1}$, is a martingale
under the risk neutral measure $Q_i$. That is

$$
(4.40) \quad E^{Q_i} \left[ F_i(t) : \mathcal{F}_t \right] = F_i(t), \quad 0 < t < T_{n-1},
$$

T_0$ see this above. What

$$
(4.41) \quad E^{Q_i} \left[ F_i(T) : \mathcal{F}_t \right] = E^Q \left[ F_i(T) ; T_{n-1} \right]
$$

$$
/ E^{Q_i} \left[ 1 : \mathcal{F}_t \right] = E \left[ \exp \left[ - \int_0^{T_{n-1}} \xi(t) \, dt \right] F_i(t) ; \mathcal{F}_t \right]
$$

/ E \left[ \exp \left[ - \int_0^{T_{n-1}} \xi(t) \, dt \right] F_i(T) ; \mathcal{F}_t \right] =

$$
(4.42) \quad E \left[ \exp \left[ - \int_0^{T_{n-1}} \xi(t) \, dt \right] F_i(t) ; \mathcal{F}_t \right] =
$$

$$
E \left[ \exp \left[ - \int_0^{T_{n-1}} \xi(t) \, dt \right] \left\{ \frac{p(T_i, T_{n-1}) - 1}{p(T_i, T_n)} \right\} / (T_i - T_{n-1}) ; \mathcal{F}_t \right] =
$$

$$
E \left[ \exp \left[ - \int_0^{T_{n-1}} \xi(t) \, dt \right] \left\{ \frac{p(T_i, T_{n-1}) - 1}{p(T_i, T_n)} \right\} / (T_i - T_{n-1}) ; \mathcal{F}_t \right] =
$$

$$
E \left[ \exp \left[ - \int_0^{T_{n-1}} \xi(t) \, dt \right] \left\{ \frac{p(T_i, T_{n-1}) - 1}{p(T_i, T_n)} \right\} / (T_i - T_{n-1}) ; \mathcal{F}_t \right] =
$$

$$
E \left[ \exp \left[ - \int_0^{T_{n-1}} \xi(t) \, dt \right] ; \mathcal{F}_t \right] \left\{ \frac{p(T_i, T_{n-1}) - 1}{p(T_i, T_n)} \right\} / (T_i - T_{n-1})
$$

On the other hand we have

$$
(4.43) \quad E \left[ \exp \left[ - \int_0^{T_{n-1}} \xi(t) \, dt \right] ; \mathcal{F}_t \right] =
$$

$$
E \left[ \exp \left[ - \int_0^{T_{n-1}} \xi(t) \, dt \right] ; \mathcal{F}_t \right] p(T_i, T_n),
$$

where we conclude (4.40) holds.
We follow from (4.43) that

\[ (4.44) \quad E \mathcal{Q}_i \left[ \frac{F_{i+1}(t+dt)}{F_i(t)} \right] = 1. \]

For small \( dt \). To do this we write (4.45) as

\[ (4.46) \quad \frac{\Delta}{\mathbb{P}(0, T_i)} E \left[ \exp \left[ - \int_0^{T_i} \lambda(t) dt \right] \frac{F_{i+1}(t+nt)}{F_{i+1}(t)} \right] \]

\[ = \frac{\Delta}{\mathbb{P}(0, T_i)} E \left[ \exp \left[ - \int_0^{T_i} \lambda(t) dt \right] \frac{\{ p(t+nt, T_i)-p(t, T_i) \} p(t, T_{i+1})}{\{ p(t, T_{i+1}) \}^2} \right] \]

\[ = \frac{\Delta}{\mathbb{P}(0, T_i)} E \left[ \exp \left[ - \int_0^{T_i} \lambda(t) dt \right] \frac{p(t+nt, T_i)-p(t, T_i) \mathbb{P}(t, T_{i+1}) \mathbb{P}(t+nt, T_i)}{\{ p(t, T_{i+1}) \}^2} \right]. \]

We have now that

\[ (4.47) \quad p(t+nt, T_i) / p(t+nt, T_{i+1}) = 1 + (T_{i+1} - T_i) F_{i+1}(t+nt). \]

Thus (4.46) is the same as

\[ (4.48) \quad \frac{\Delta}{\mathbb{P}(0, T_i)} E \left[ \exp \left[ - \int_0^{T_i} \lambda(t) dt \right] \frac{p(t+nt, T_i)-p(t, T_{i+1})}{\{ p(t, T_{i+1}) \}^2} \right]. \]
\[
\{ p(t, T_t) - p(t, T_{t+1}) \}^{-1} p(t, T_{t+1}) \left[ 2 + (T_{t+1} - T_t) F_{\alpha+1}(t) \right]^{-1} \\
\left\{ 1 + (T_{t+1} - T_t) F_{\alpha+1}(t + \Delta t) \right\}^{-1} \\
= 1 + \frac{(T_{t+1} - T_t) [F_{\alpha+1}(t + \Delta t) - F_{\alpha+1}(t)]}{1 + (T_{t+1} - T_t) F_{\alpha+1}(t)}
\]

From (4.48) and (4.49) we see that

\[
E \left[ \frac{F_{\alpha+1}(t + \Delta t)}{F_{\alpha+1}(t)} \right] = 1 + \frac{\Delta}{p(t, T_t)}
\]

\[
E \left[ \exp \left( - t \int_0^{t+\Delta t} r(s) ds \right) \right] \left\{ p(t + \Delta t, T_t) - p(t + \Delta t, T_{t+1}) \right\}^{-1} p(t, T_{t+1}) \left( T_{t+1} - T_t \right) \frac{F_{\alpha+1}(t + \Delta t) - F_{\alpha+1}(t)}{1 + (T_{t+1} - T_t) F_{\alpha+1}(t)}
\]

We write now

\[
\{ p(t + \Delta t, T_t) - p(t + \Delta t, T_{t+1}) \} \left\{ p(t, T_{t+1}) - p(t, T_t) \right\}^{-1} p(t, T_t)
\]

\[
= \frac{p(t, T_t) p(t + \Delta t, T_{t+1}) F_{\alpha+1}(t + \Delta t)}{p(t, T_{t+1}) F_{\alpha+1}(t)}
\]

\[
= p(t + \Delta t, T_{t+1}) \left\{ 1 + (T_{t+1} - T_t) F_{\alpha+1}(t) \right\}^{-1} \frac{F_{\alpha+1}(t + \Delta t)}{F_{\alpha+1}(t)}
\]

So far we have made no approximations. Now we make the approximation

\[
1 + (T_{t+1} - T_t) F_{\alpha+1}(t) \approx 1 + (T_{t+1} - T_t) F_{\alpha+1}(t)
\]

\[
= \frac{p(t, T_t)}{p(t, T_{t+1})}
\]
Then from (4.51) equation (4.52) becomes

\[ (4.53) \quad E^Q \left[ \frac{F_{n+1}(t+\Delta t)}{F_{n+1}(t)} \right] = \Delta + \]

\[ E^Q \left[ \frac{F_{n+1}(t+\Delta t)}{F_{n+1}(t)} \left( \frac{(T_{n+1})[F_{n+1}(t+\Delta t)-F_{n+1}(t)]}{1+(T_{n+1}-T_n)F_{n+1}(t)} \right) \right] \]

Now, let us assume that the volatility \( \delta_k(t) \) under the risk-neutral measure \( Q_k \) is \( \delta_k(t) \), \( T_k < T_{k-1} \). Thus

\[ (4.54) \quad E^Q_k \left[ \left\{ \frac{F_k(t+\Delta t)-F_k(t)}{F_k(t)} \right\}^2 \right] = \sigma_k(t)^2 \Delta t, \]

Then, we conclude from (4.53) that

\[ (4.55) \quad E^Q \left[ \frac{F_{n+1}(t+\Delta t)}{F_{n+1}(t)} \right] = \Delta + \]

\[ \frac{(T_{n+1}-T_n)F_{n+1}(t)}{1+(T_{n+1}-T_n)F_{n+1}(t)} \sigma_{n+1}(t)^2 \Delta t. \]

Let us assume now that all the \( F_k(t) \) are driven by the same Brownian motion but with volatilities \( \delta_k(t) \) which depend on both \( k \) and \( t \). Then if the evolution of the \( F_k(t) \) is Markovian (4.55) implies (4.36), that is, the evolution of \( F_k(t) \) in the risk-neutral measure \( Q_k \) is given by (4.36). Having established the formula for one risk-neutral measure, the same formula holds for all risk-neutral measures \( Q_k \) since probability statements for \( Q_k \) and \( k \) are probability statements for all \( Q_k \).

It is easy to see now how to calibrate the LFM from the prices of caplets. Recall
What the evolution of $F_i(t)$ under $\Delta_i$ is drift-free. Hence it makes sense to model it with $\Delta_i$.

\[
\frac{dF_i(t)}{F_i(t)} = \Delta_i(t)dt.
\]

Since the price of the $(T_{i-1}, T_i)$ caplet is given by (4.39), we conclude that

\[
\frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i(t)^2 dt = \eta_i = (\text{Black implied volatility})^2.
\]

The formula (4.59) does not allow us to calibrate the $\sigma_i(t)^2$ but we can do this if we make a few assumptions which are reasonable. To give an example, we shall assume that volatilities are time-independent invariant. Thus consider the Table:

<table>
<thead>
<tr>
<th>Strike Vol</th>
<th>$T_i$</th>
<th>$(T_i, T_{i+1})$</th>
<th>$(T_{i+1}, T_{i+2})$</th>
<th>$(T_{i+2}, T_{i+3})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1(t)$</td>
<td>$\eta_1$</td>
<td>Dead</td>
<td>Dead</td>
<td>Dead</td>
</tr>
<tr>
<td>$F_2(t)$</td>
<td>$\eta_2$</td>
<td>$\eta_2$</td>
<td>Dead</td>
<td>Dead</td>
</tr>
<tr>
<td>$F_3(t)$</td>
<td>$\eta_3$</td>
<td>$\eta_2$</td>
<td>$\eta_2$</td>
<td>Dead</td>
</tr>
<tr>
<td>$F_4(t)$</td>
<td>$\eta_4$</td>
<td>$\eta_2$</td>
<td>$\eta_2$</td>
<td>$\eta_2$</td>
</tr>
</tbody>
</table>

Making this assumption we see from (4.59) that

\[
\int_0^{T_{i+1}} \sigma_i(t)^2 dt = \eta_i^2 T_{i+1}.
\]

\[
\int_0^{T_{i+2}} \sigma_i(t)^2 dt = \eta_i^2 (T_{i+2} - T_{i+1}) + \eta_i^2 T_{i+1}
\]

\[
\int_0^{T_{i+3}} \sigma_i(t)^2 dt = \eta_i^2 (T_{i+3} - T_{i+2}) + \eta_i^2 (T_{i+2} - T_{i+1}) + \eta_i^2 T_{i+2}.
\]
Here we assume the $S_i$ of equation (4.5) are constant $\beta_i$ in $(T_{i-1}, T_i)$ call it, equation (4.5.7) uniquely determines $\beta_i$.

We consider now how to use the CFM to value a swap. Suppose the swap is over the time period $(T_1, T_f)$ with payments at times $T_i$, $i = 1, 2, \ldots, f$. Then if $K$ is the strike price in the swap, the payoff at time $T_2$ is

$$[S_{T_2}, \beta(T_2) - K] \sum_{i=1}^{f} (T_i - T_{i-1}) P(T_i, T_i)$$

where $S_{T_2}$ is the swap rate at time $T_2$ for the swap over the time period $(T_1, T_f)$. The value of the swap at time $T_0$ is then

$$P(0, T_f) \{ \left[ S_{T_2}, \beta(T_2) - K \right] \sum_{i=1}^{f} (T_i - T_{i-1}) P(T_i, T_i) \}$$

where we are taking expectations with respect to the risk neutral measure $E_T$. Since we model give the dynamics of forward rates, we need to write all the quantities in the expectation in terms of forward rates $F_i(t)$. Using the identity

$$P(T_{x}, T_{y}) = \frac{\beta_x + (T_x - T_{x-1}) F_x(T_x)}{P(T_y, T_{x-1})}$$

we see that

$$P(T_2, T_4) = \prod_{i=x}^{y} \left[ 2 + (T_i - T_{i-1}) F_i(T_i) \right]$$

We can similarly compute the swap rate $S_{T_2}$ by

On first using
$$\begin{align*}
(1) \beta^T (A)^T \alpha + \gamma^T \frac{\beta}{\gamma} (A)^T \gamma - \\
\left[ (A)^T \alpha \right] \left[ (A)^T \gamma \right] = [A]_{\beta \alpha} \cdot [A]_{\gamma \gamma} = 0
\end{align*}$$
Once we have done this we proceed $F_{t+1}(t), \delta \leq t \leq T_2$, by using the same MC simulation of $F_{t+1}(t)$. Doing this for $F_{t}(t), \delta \leq t \leq T_2$, and computing the payoff (4.59) gives us one MC simulation for the value of the option. Note that since all the $F_{t}(t)$ are generated by the same Common random number they are independent variables. The degree of correlation of the $F_{t}(t)$ affect the value of expected log return which is interest rate cap as a linear sum of variables which depend only on $F_{t}(t)$. It is clear to compute the coefficients of correlation between the $F_{t}(t)$. We put

\[ (4.61) \quad \rho_{i,j} = \frac{dF_{t}(t) dF_{t+j}(t)}{\text{std}(dF_{t}(t)) \text{std}(dF_{t+j}(t))} \]

where "std" stands for standard deviation of the change variable as information up to time $t$. Explicitly from our assumption (4.65) and (4.66) we have that $\rho_{i,j} = 1$ for all $i, j$. Thus all forward rates are perfectly correlated, i.e. our assumption that the yield curve is driven by a single random factor, is

§ 5 Importing correlations into the CPM model

Let us consider what we have been assuming so far about the yield curve. First we have assumed that the market of fixed rates $F_{t}(t)$ are deterministic and secondly about the instantaneous correlation matrix $\rho_{i,j}$ for forward rates has rank 1. A typical graph of Black coefficient implicit volatility,
Thus we expect volatility to peak around 2 years and then drop steadily, eventually fading off. If we make the time correlation matrix assumption, then we get $C_\ell(t)$ for $t > 0$ as simply

$$N(t, t)$$ along the time axes.

We can also use the multivariate correlation matrix $\rho$ given by (2.57). We have assumed $\rho$ is rank 1 so $p_\rho = \tilde{s}$ for all $i, j$. This seems like a reasonable assumption if $|i-j|$ is small but perhaps not so good an assumption if $|i-j|$ is large. In other words, we might reasonably expect period rates $F_i(t)$, $F_j(t)$ to be well correlated if the time points $(T_{i-1}, T_i)$ and $(T_{j-1}, T_j)$ at which they correspond are close, but not otherwise. The basic idea here is to introduce parameters $\rho$ and match these parameters so that the model gives the market values of swaplets. Suppose the swaplet is at the time interval $(T_\ell, T_\beta)$ as before. The present value involved in valuing the swaplet is $F_i(t)$, $i = \ell, \ldots, \beta$. Thus $M = \beta - \ell$ present values.

The corresponding correlation matrix $\rho$ is then $M \times M$ positive definite. The simplest way to go beyond...
A way of doing this is as follows: we write \( f = A A^T \) by its Cholesky decomposition. Then if the rank of \( A \) does not exceed \( 2 \), it follows that the rank of \( f \) also does not exceed \( 2 \). We take \( A \) to be of the form

\[
A = \begin{bmatrix}
\cos \theta_1 & \sin \theta_1 & 0 \\
\cos \theta_2 & \sin \theta_2 & 0 \\
\cos \theta_3 & \sin \theta_3 & 0 \\
\end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix}
\cos \theta_1 & \sin \theta_1 & 0 \\
\cos \theta_2 & \sin \theta_2 & 0 \\
\cos \theta_3 & \sin \theta_3 & 0 \\
\end{bmatrix}
\]

It is easy to see that \( f = [S_{ij}] \) when \( S_{ij} = \cos (\theta_i - \theta_j) \). Thus \( f \) has at most rank \( 2 \) and if all \( \theta_i \) are the same \( f \) has rank \( 1 \). The next step is to calibrate \( f_{ij} \), \( E \) to the market price of expectations. This can be problematic because in matching market prices the matrix \( f \) may end up with some undesirable properties. For example it is reasonable to expect that all forward rates are positively correlated if \( S_{ij} \) \( \geq 0 \) for all \( i,j \). In matching market prices we can easily end up with \( S_{ij} < 0 \) for some \( i,j \).

Now let us assume that we have found the instantaneous correlation matrix \( f \). How do we interpret this with the LFM? Our basic assumption is that the evolution of \( F(t) \) is driven by a \( EM \) \( dW_k(t) \), and that the \( EMs \) are correlated according to \( f \), where

\[
(4.68) \quad S_{ij} \, dt = E \left[ dW_i(t) \, dW_j(t) \right].
\]
In the enrichment process $Q(t)$, we have as in (4.46):

$$
\frac{d F_{t+1}(t)}{F_{t}(t)} = Q(t) M(t). \tag{4.69}
$$

If we go through our previous calculations for $F_{t+1}(t)$ we see that

$$
\frac{d F_{t+1}(t)}{F_{t+1}(t)} = \frac{(T_{t+1}T_{t+2}) F_{t+1}(t)}{1 + (T_{t+1}T_{t+2}) F_{t+1}(t)} \\(4.70\)
+ Q_{t+1}(t) dW_{t+1}(t).
$$

To simplify the evaluation of $F_{t+2}(t)$ in the measure $Q(t)$, we shall need to compute

$$
E \left[ \frac{F_{t+2}(t + \Delta t)}{F_{t+2}(t)} \right] = 1 + O(\Delta t). \tag{4.71}
$$

We proceed as before with our second goal of establishing the $Q$ on the RHS of (4.71), and so in (4.46) the RHS of (4.71) is given by

$$
E \left[ \exp \left[ - \int_{t}^{t+\Delta t} c(t) dt \right] \frac{F(t+\Delta t, T_{t+2}) - F(t, T_{t+2})}{F(t, T_{t+2})} \right] \tag{4.72}
$$

Now we have

$$
\frac{P(t, T_{t+2})}{P(t, T_{t})} = \frac{P(t+\Delta t, T_{t})}{P(t+\Delta t, T_{t+2})}. \tag{4.73}
$$
\[
\begin{align*}
\{ a + (T_{t+1} - T_t) F_{t+1}(t) \}^2 & \{ a + (T_{t+1} - T_t) F_{t+1}(t + dt) \}^2 \\
\{ a + (T_{t+2} - T_{t+1}) F_{t+2}(t) \}^2 & \{ a + (T_{t+2} - T_{t+1}) F_{t+2}(t + dt) \}^2 \\

= a + (T_{t+1} - T_t) \left[ \frac{F_{t+1}(t + dt) - F_{t+1}(t)}{1 + (T_{t+1} - T_t) F_{t+1}(t)} \right] + \\
(T_{t+2} - T_{t+1}) \left[ \frac{F_{t+2}(t + dt) - F_{t+2}(t)}{1 + (T_{t+2} - T_{t+1}) F_{t+2}(t)} \right] + \text{product of last two terms}.
\end{align*}
\]

Next we use the analogue of (4.51). Thus

\[
(4.74) \quad \{ p(t + dt, T_{t+1}) - p(t + dt, T_{t+2}) \}^2 \\
\{ p(t, T_{t+1}) - p(t, T_{t+2}) \}^2 \quad \frac{p(t, T_{t+1}) p(t + dt, T_{t+2})}{p(t, T_{t+2}) F_{t+2}(t)}
\]

Thus we conclude that

\[
(4.75) \quad E \left[ \left[ \frac{F_{t+2}(t + dt)}{F_{t+2}(t)} \right] \right] =
\]

\[
\frac{1}{p(t, T_{t+2})} E \left[ \exp \left[ - \int_0^{t+dt} s(t') dt' \right] \frac{p(t, T_{t+1}) p(t + dt, T_{t+2})}{p(t, T_{t+2}) F_{t+2}(t)} \right]
\]

\[
\{ a + \frac{F_{t+2}(t + dt) - F_{t+2}(t)}{F_{t+2}(t)} \}^2 \quad \frac{p(t, T_{t+1}) p(t + dt, T_{t+2})}{p(t, T_{t+2}) F_{t+2}(t)}
\]

\[
\frac{1}{p(t, T_{t+2})} E \left[ \exp \left[ - \int_0^{t+dt} s(t') dt' \right] \frac{p(t, T_{t+1}) p(t + dt, T_{t+2})}{p(t, T_{t+2}) F_{t+2}(t)} \right]
\]

\[
\{ \frac{F_{t+2}(t + dt) - F_{t+2}(t)}{F_{t+2}(t)} \}^2 \quad \frac{p(t, T_{t+1}) p(t + dt, T_{t+2})}{p(t, T_{t+2}) F_{t+2}(t)}
\]

\[
\frac{1}{p(t, T_{t+2})} E \left[ \exp \left[ - \int_0^{t+dt} s(t') dt' \right] \frac{p(t, T_{t+1}) p(t + dt, T_{t+2})}{p(t, T_{t+2}) F_{t+2}(t)} \right]
\]
Now just as in (4.52) we make the approximation

\[ (4.76) \quad \mathbb{E}(t, T_n) / \mathbb{E}(t, T_{n+1}) \approx \mathbb{E}(t, T_{n+1}) / \mathbb{E}(t, T_{n+2}). \]

Then (4.75) yields the equation

\[ (4.77) \quad \mathbb{E} \left[ Q_t \left( \frac{F_{n+2}(t+\Delta t)}{F_{n+1}(t)} \right) \right] = 2 + \]

\[ \mathbb{E} \left[ Q_{n+2} \left( \frac{F_{n+2}(t+\Delta t) - F_{n+2}(t)}{F_{n+2}(t)} \right) \right] \left[ \frac{\mathbb{P}(t, T_{n+1}) \mathbb{P}(t+\Delta T, T_{n+2})}{\mathbb{P}(t, T_{n+2}) \mathbb{P}(t+\Delta T, T_{n+1})} - 1 \right]. \]

Now we use (4.72) to conclude that

\[ (4.78) \quad \mathbb{E} \left[ Q_t \left( \frac{F_{n+2}(t+\Delta t)}{F_{n+1}(t)} \right) \right] = 2 + \]

\[ \frac{\sigma_{n+2}(t)^2 (T_{n+2} - T_{n+1}) F_{n+2}(t)}{2 + (T_{n+2} - T_{n+1}) F_{n+2}(t) - (T_{n+1} - T_{n}) F_{n+1}(t) \Delta t} \]

+ \mathcal{O}(\Delta t) \] where \( \mathcal{O}(\Delta t) \) takes account of the product \( \mathbb{P}(t, T_n) \) in (4.72). Hence the dynamics of \( F_{n+2}(t) \) are given by

\[ (4.79) \quad \frac{dF_{n+2}(t)}{F_{n+2}(t)} = \left( \frac{(T_{n+2} - T_{n+1}) F_{n+2}(t)}{2 + (T_{n+2} - T_{n+1}) F_{n+2}(t)} \right) \frac{\sigma_{n+2}(t)^2 \Delta t}{2 + (T_{n+1} - T_n) F_{n+1}(t)} \]

\[ + \frac{(T_{n+1} - T_n) F_{n+1}(t)}{1 + (T_{n+1} - T_n) F_{n+1}(t)} \frac{\sigma_{n+1}(t) \sigma_{n+2}(t) \int_{T_{n+1}}^{T_{n+2}} dW(t)}{\mathcal{O}(\Delta t)} + \frac{\sigma_{n+1}(t) \sigma_{n+2}(t) \int_{T_{n+1}}^{T_{n+2}} dW(t)}{\mathcal{O}(\Delta t)}. \]
We can generalize the formula (4.79) to any $k > i$. Thus

$$\frac{d F_k(t)}{F_k(t)} = \gamma_k(t) \sum_{j=n+1}^{k} \frac{(T_d - T_{d-1}) F_j(t)}{1 + (T_d - T_{d-1}) F_j(t)} \, dt$$

$$+ \gamma_k(t) \, dW_k(t).$$

Once we know the correlation matrix $\mathcal{C}$ and the volatilities $\gamma_k(t)$, then we may value the European option just as before. Next we need to generate i.i.d. correlated B.M.'s for the swap rate $\mathcal{X}$ over the interval $(T_d, T_f)$. This can be done using Cholesky decomposition, as before.