

## CHAPTER III - LATTICE METHODS

JOSEPH G. CONLON

### 1. BINOMIAL TREES

Lattice methods such as binomial trees are in fact implementations of the explicit Euler method discussed in Chapter I, but with various kinds of grid which do not have to be uniform. The advantage of the method is its flexibility in grid design, which can be *adapted* to the problem at hand. In numerical analysis these numerical schemes are therefore often called *adaptive schemes*. A further advantage of the lattice method is that there are no *boundary conditions* to be determined. In the pricing of Asian options we saw that determining the appropriate boundary conditions was a non-trivial problem. There is however a disadvantage in dispensing with boundary conditions since it can have significant cost in terms of numerical efficiency. We shall see why this is the case later.

We consider first the simplest problem of numerically computing the BS price of a stock option. We first construct a discrete approximation for the evolution of the solution  $S(t)$  to the Black-Scholes SDE

$$(1.1) \quad dS(t) = S(t)[r dt + \sigma dB(t)], \quad t \geq 0, \quad S(0) = S_0.$$

We let  $\Delta t$  be the time discretization and assume that in the time step  $t \rightarrow t + \Delta t$  the stock price  $S$  at time  $t$  can move up to the value  $Su$  or down to the value  $Sd$ , where  $0 < d < 1 < u$ . Thus we are assuming that the random variable  $S(t + \Delta t)/S(t)$  is *Bernoulli*, in contrast to the situation in Chapter II where we saw that  $S(t + \Delta t)/S(t)$  is the exponential of a *Gaussian*. To define the Bernoulli variable completely we need to determine the probability  $p_u$  of going from  $S$  to  $Su$  and  $p_d$  of going from  $S$  to  $Sd$ . We do this by choosing  $u, d, p_u, p_d$  so that the first two moments of  $S(t + \Delta t)/S(t)$  in the Bernoulli approximation agree with corresponding continuous time quantities. We observed in Chapter II that the solution  $S(t)$  of (1.1) satisfies the identity

$$(1.2) \quad \frac{S(t + \Delta t)}{S(t)} = \exp \left[ (r - \sigma^2/2)T + \sigma\sqrt{T} \xi \right], \quad \text{where } \xi \text{ is standard normal.}$$

Now the moment generating function for the standard normal variable is given by (1.3)

$$E[e^{\theta\xi}] = e^{\theta^2/2}, \quad \text{so} \quad E\left[\frac{S(t + \Delta t)}{S(t)}\right] = e^{rT}, \quad E\left[\left\{\frac{S(t + \Delta t)}{S(t)}\right\}^2\right] = e^{(2r + \sigma^2)T}.$$

Evidently for the Bernoulli variable we have that

$$(1.4) \quad E\left[\frac{S(t + \Delta t)}{S(t)}\right] = up_u + dp_d, \quad E\left[\left\{\frac{S(t + \Delta t)}{S(t)}\right\}^2\right] = u^2p_u + d^2p_d.$$

Hence on equating the quantities in (1.3), (1.4) we obtain 3 equations for the 4 unknowns  $u, d, p_u, p_d$ ,

$$(1.5) \quad p_u + p_d = 1, \quad up_u + dp_d = e^{r\Delta t}, \quad u^2p_u + d^2p_d = e^{(2r+\sigma^2)\Delta t}.$$

Observe that we need to add one more equation to get unique values for  $u, d, p_u, p_d$  which satisfy (1.5).

Consider now how we can use this method to estimate the value of a call option with expiration date  $T$ . The value of the option is given by the formula

$$(1.6) \quad \text{value of call option} = e^{-rT} E[\max[S(T) - K, 0] \mid S(0) = S_0],$$

where  $S(t)$ ,  $t \geq 0$ , is the solution to (1.1). In the BS theory  $S(T)$  is the exponential of a Gaussian, but in this discrete approximation it is the exponential of a *binomial* variable. We choose  $\Delta t$  and integer  $M$  such that  $M\Delta t = T$ . At time  $t = 0$  we set  $S(0) = S_0$ , which is today's stock price. Then  $S(\Delta t)$  takes the 2 possible values  $S_0d$  and  $S_0u$ , with  $S(2\Delta t)$  taking the 3 possible values  $S_0d^2, S_0ud, S_0u^2$  etc. More generally we have that

$$(1.7) \quad S(m\Delta t) \text{ takes possible values } S_0d^m, S_0d^{m-1}u, S_0d^{m-2}u^2, \dots, S_0u^m.$$

We can represent the situation by a lattice in 2 dimensions with integer coordinates  $(m, n)$  where  $0 \leq n \leq m$  and  $m = 0, 1, 2, \dots, M$ . To each lattice point we associate a time and stock price by:

$$(1.8) \quad \text{At } (m, n) \text{ the time } t = m\Delta t \text{ and the stock price } S(t) = u^n d^{m-n}.$$

We can use a recurrence equation to find the value of the option (1.6). Let  $V_n^m$  be the values of the option associated with lattice points  $(m, n)$ , so today's price for the option is  $V_0^0$ . Since we know the payoff on the option we have that

$$(1.9) \quad V_n^M = \max[S(T) - K, 0], \quad \text{where } S(T) = u^n d^{M-n}, \quad 0 \leq n \leq M.$$

The recurrence equation is then

$$(1.10) \quad V_n^m = e^{-r\Delta t} [p_u V_{n+1}^{m+1} + p_d V_n^{m+1}] \quad \text{for } 0 \leq n \leq m, \quad 0 \leq m < M.$$

We can solve (1.10) backwards in time starting at  $m = M$  and ending at  $m = 0$ , whence we can compute  $V_0^0$ .

We can derive (1.10) by using a no arbitrage argument. Suppose at time  $t$  the stock price is  $S$  and the value of the option is  $V$ . At time  $t + \Delta t$  the value of the stock and option are with probability  $p_u$  given by  $S^+, V^+$  where  $S^+ = Su$ , or with probability  $p_d$  given by  $S^-, V^-$  where  $S^- = Sd$ . We wish to construct a risk free portfolio consisting of one option minus  $\alpha$  of stock, whence we must have

$$(1.11) \quad V - \alpha S = e^{-r\Delta t} [V^+ - \alpha S^+] = e^{-r\Delta t} [V^- - \alpha S^-].$$

It follows that

$$(1.12) \quad \alpha = \frac{V^+ - V^-}{S^+ - S^-}, \quad V = e^{-r\Delta t} [V^+ + \alpha \{e^{r\Delta t} S - S^+\}].$$

Substituting  $S^+ = Su, S^- = Sd$  in (1.12) we conclude that

$$(1.13) \quad V = e^{-r\Delta t} \left[ \frac{e^{r\Delta t} - d}{u - d} V^+ + \frac{u - e^{r\Delta t}}{u - d} V^- \right].$$

Now the first two equations of (1.5) yield the formulas

$$(1.14) \quad p_u = \frac{e^{r\Delta t} - d}{u - d}, \quad p_d = \frac{u - e^{r\Delta t}}{u - d}.$$

Hence (1.13), (1.14) yield the recurrence formula

$$(1.15) \quad V = e^{-r\Delta t} [p_u V^+ + p_d V^-] ,$$

which implies (1.10). It is not difficult to see that (1.15) is a discretization of the Black Scholes PDE introduced in Chapter I.

We have already observed that in order to get unique values for  $u, d, p_u, p_d$  satisfying the three equations (1.5), we need to impose a fourth constraint equation. The set of four equations can be explicitly solved in the two cases where the fourth equation is either  $p_u = p_d = 1/2$  or  $ud = 1$ .

**(a)**  $p_u = p_d = 1/2$ : Observe that the final two equations of (1.5) become

$$(1.16) \quad u + d = 2e^{r\Delta t}, \quad u^2 + d^2 = 2e^{(2r+\sigma^2)\Delta t} .$$

If we square the first equation of (1.16) and subtract from it the second equation we obtain the formula

$$(1.17) \quad ud = e^{2r\Delta t} [2 - e^{\sigma^2 \Delta t}] .$$

Hence on multiplying the first equation of (1.16) by  $u$  and using (1.17) we obtain a quadratic equation for  $u$ ,

$$(1.18) \quad u^2 - 2e^{r\Delta t}u + e^{2r\Delta t}[2 - e^{\sigma^2 \Delta t}] = 0 .$$

The two solutions of (1.18) are  $u, d$ , whence we have

$$(1.19) \quad u = e^{r\Delta t} \{1 + \sqrt{e^{\sigma^2 \Delta t} - 1}\}, \quad d = e^{r\Delta t} \{1 - \sqrt{e^{\sigma^2 \Delta t} - 1}\} .$$

**(b)**  $ud = 1$ : We multiply the second equation of (1.5) by  $u$ , whence we obtain

$$(1.20) \quad u^2 p_u + u d p_d = e^{r\Delta t} u, \quad \text{so } u^2 p_u + 1 - p_u = e^{r\Delta t} u, \quad \text{whence } (u^2 - 1)p_u = e^{r\Delta t} u - 1 .$$

Similarly we multiply the third equation of (1.5) by  $u^2$  to obtain

$$(1.21) \quad u^4 p_u + u^2 d^2 p_d = e^{(2r+\sigma^2)\Delta t} u^2, \quad \text{so } u^4 p_u + 1 - p_u = e^{(2r+\sigma^2)\Delta t} u^2, \\ \text{whence } (u^4 - 1)p_u = e^{(2r+\sigma^2)\Delta t} u^2 - 1 .$$

We conclude from (1.20), (1.21) that

$$(1.22) \quad u^2 + 1 = \frac{u^4 - 1}{u^2 - 1} = \frac{e^{(2r+\sigma^2)\Delta t} u^2 - 1}{e^{r\Delta t} u - 1} .$$

Evidently (1.22) gives us a cubic equation for  $u$ . Since it has constant term zero we can divide by  $u$  to obtain the quadratic equation

$$(1.23) \quad u^2 - [e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}]u + 1 = 0 .$$

Just as before the solutions to (1.23) are  $u, d$  and they are given by the formulas

$$(1.24) \quad u = A + \sqrt{A^2 - 1}, \quad d = A - \sqrt{A^2 - 1}, \quad \text{where } A = [e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}]/2 .$$

We can compare the method of computing the value of an option using (1.10) with the explicit Euler method. If we choose to use method **(b)** then setting  $x = \log S$  we see from (1.7) and the fact that  $ud = 1$  that the lattice points are equally spaced with  $\Delta x$  given by the formula

$$(1.25) \quad \Delta x = 2 \log u = 2 \log [A + \sqrt{A^2 - 1}] \quad \text{where } A = 1 + \sigma^2 \Delta t / 2 + O[(\Delta t)^2] .$$

We have from (1.25) that

$$(1.26) \quad \Delta x = 2 \log[1 + \sigma\sqrt{\Delta t} + O(\Delta t)] = 2\sigma\sqrt{\Delta t} + O(\Delta t) .$$

It follows from (1.26) that  $\Delta t/(\Delta x)^2 \simeq 1/4\sigma^2$  and hence satisfies the stability condition  $\Delta t/(\Delta x)^2 < 1/\sigma^2$  for the explicit Euler method which we established in Chapter I.

Next we consider the case of valuing options on stocks which pay dividends. If there is a continuous dividend payment rate of  $D$  then the value of the option is given again by (1.6), but now  $r$  is replaced by  $r - D$  in the stochastic evolution (1.1). Hence the equations (1.5) become

$$(1.27) \quad p_u + p_d = 1, \quad up_u + dp_d = e^{(r-D)\Delta t}, \quad u^2p_u + d^2p_d = e^{[2(r-D)+\sigma^2]\Delta t} .$$

We can also easily include discrete dividend payments in the model. Suppose for example there is a discrete dividend payment at time  $t_{\text{div}} < T$  with the dividend equal to  $\beta S(t_{\text{div}})$  for some  $\beta$  with  $0 < \beta < 1$ . Define an integer  $m_{\text{div}}$  by  $m_{\text{div}}\Delta t < t_{\text{div}} \leq (m_{\text{div}} + 1)\Delta t$ . To each lattice point  $(m, n)$  with  $m \leq m_{\text{div}}$  we assign a stock price  $S_n^m = u^n d^{m-n}$  as previously in (1.8). However for  $m > m_{\text{div}}$  we proceed differently. In particular we set

$$(1.28) \quad \begin{aligned} S_n^{m+1} &= S_n^m d(1 - \beta), & S_{n+1}^{m+1} &= S_n^m u(1 - \beta) \quad \text{if } m = m_{\text{div}} , \\ S_n^{m+1} &= S_n^m d, & S_{n+1}^{m+1} &= S_n^m u \quad \text{if } m > m_{\text{div}} . \end{aligned}$$

We use the same recurrence formula (1.10) as before to value the option. Note that today's value of the option  $V_0^0$  depends on the dividend since  $V_n^M$ ,  $0 \leq n \leq M$ , is given by  $V_n^M = \max[S_n^M - K, 0]$ ,  $0 \leq n \leq M$ , and from (1.28) we see that  $S_n^M$ ,  $0 \leq n \leq M$ , depends on  $\beta$ .

In order to see that the recurrence (1.10) with the modified stock prices (1.28) yields the value of the option we need to show that (1.15) is valid across the dividend time. Thus we take  $t = m_{\text{div}}\Delta t$  with associated stock price  $S$  and option price  $V$ .  $S^-, V^-$  where  $S^- = Sd$ . Now to construct a risk free portfolio consisting of one option minus  $\alpha$  of stock we modify (1.11) as follows:

$$(1.29) \quad V - \alpha S = e^{-r\Delta t}[V^+ - \alpha\{S^+ + \beta S\}] = e^{-r\Delta t}[V^- - \alpha\{S^- + \beta S\}] ,$$

where  $S^+, S^-$  are the possible values the stock moves to after time  $\Delta t$ . If we choose now  $S^+ = Su(1 - \beta)$  as in (1.28) then we have that  $S^+ + \beta S = S[u(1 - \beta) + \beta] = Su$  to a high degree of accuracy since  $(u - 1)\beta \simeq 0$ . Similarly we may assume that  $S^- + \beta S = Sd$ , and so (1.29) yields again the recurrence equation (1.15). We can compare this method of including dividends with the formulas (1.27) for the continuous dividend payment. In the continuous case we have that if  $(u_D, d_D)$  are the values of  $(u, d)$  for dividend rate  $D$  then  $e^{rD\Delta t}u_D = u_0$ . Thus we have that  $S^+ = Su_D = Se^{-rD\Delta t}u_0 = Su_0(1 - \beta)$  with  $\beta = rD\Delta t$ .

Finally we note that the central limit theorem implies that the price of the discrete time option converges as  $\Delta t \rightarrow 0$  to the BS price. To see this we define a Bernoulli variable  $X$  by

$$(1.30) \quad X = \log u \quad \text{with probability } p_u, \quad X = \log d \quad \text{with probability } p_d .$$

Using method (a) we see from (1.19) that

$$(1.31) \quad \log u = r\Delta t + \log[1 + \sigma\sqrt{\Delta t}\{1 + O(\Delta t)\}], \quad \log d = r\Delta t + \log[1 - \sigma\sqrt{\Delta t}\{1 + O(\Delta t)\}] .$$

It follows from (1.31) and the equality of probabilities  $p_u = p_d = 1/2$  that

$$(1.32) \quad E[X] = (r - \sigma^2/2)\Delta t + O[(\Delta t)^{3/2}], \quad \text{var}[X] = \sigma^2\Delta t + O[(\Delta t)^2] .$$

Let  $X_1, X_2, \dots$ , be i.i.d. variables with distribution given by (1.30). Then if  $T = M\Delta t$  we have that

$$(1.33) \quad \log S(T) = \log S_0 + \sum_{j=1}^M X_j .$$

From the CLT we see that

$$(1.34) \quad \lim_{M \rightarrow \infty} \frac{\sum_{j=1}^M X_j - E[X_j]}{\sigma\sqrt{M\Delta t}} = \xi = \text{standard normal variable} .$$

We conclude then from (1.33), (1.34) that

$$(1.35) \quad \log S(T) \simeq \log S_0 + (r - \sigma^2/2)T + \sigma\sqrt{T}\xi \quad \text{as } \Delta t \rightarrow 0 .$$

Hence the price of the discrete option (1.6) converges to the BS price of the option as  $\Delta t \rightarrow 0$ .

UNIVERSITY OF MICHIGAN, DEPARTMENT OF MATHEMATICS, ANN ARBOR, MI 48109-1109  
*E-mail address:* conlon@umich.edu