Chapter III - Lattice Methods

§ 2  Binomial Trees

Lattice methods are in fact implementations of the forward Euler method with various kinds of grids. The advantage over the fixed grids we discussed in Chapter I is obvious since we allow models more flexibility in grid design. In some sense we can "adapt" the grid to the problem we are studying, so these methods can come under the heading of "adaptive" methods. One big disadvantage is that these lattice methods do not have boundary conditions to be determined - which we have already seen can be problematic. The disadvantage is that differing with boundary conditions can cost a lot in terms of numerical efficiency. We shall see this later.

We consider first the basic method for computing the Black-Scholes price of an option. Thus, we assume stock price evolves as

\[(1.1)\quad S_{t+1} = S_t + \sigma S_t \Delta t + \xi \Delta W_t, \quad 0 \leq t \leq T.\]

We construct a discrete model for the evolution (1.1). Thus we take discrete time intervals 0, \(\Delta t, 2\Delta t, \ldots, T\). From an time step to the next, the stock can go to 2 possible values

\[S_t \rightarrow S_t \pm \sigma \Delta W_t.\]
We can draw several steps of the tree:

\[ S_0, u^3, \quad S_0, u^2, \quad S_0, u^1, \quad S_0, u, \quad S_0 \]

In this diagram, we have set \( u, d = 1 \) but this does not have to be the case. Generally, we have

\[ S_0 u^2 d \]

(1.2) Value of stock at time \( t = 0 \): \( S_0 \).

(1.3) Possible values of stock at time \( m T \) is \( S_0 u^m, S_0 u^{m-1}, S_0 u^{m-2}, \ldots, S_0 u, S_0 \), so at time \( m T \) there are \( m+1 \) possible values of stock price. If \( M N T = T \) then there are 2 possible paths for the stock price from its known value at \( t = 0 \) to its value at \( T \). We wish to assign Arithmatic probabilities \( p_u, p_d \) so that

\[ p_u + p_d = 1 \]

This discrete model is an approximation to the evolution of the continuous process (1.1). We have the 4 variables \( (u, d, p_u, p_d) \) and there is one linear constraint, namely

\[ p_u + p_d = 1 \]

We get 2 per the equation by requiring that the mean and variance of \( S (T+\Delta T) - S (T) \) be:

\[ \frac{\Delta S}{S} \]
matches for discrete and continuous models, which is
the same as assuming the mean and variance of
\( S(t+\Delta t) / S(t) \) match. Evidently we have

\[
E \left[ \frac{S(t+\Delta t)}{S(t)} \right] = \mu_u \Delta t + \sigma \Delta t d \quad \text{(discrete)}
\]

For the continuous case (1.1) recall that

\[
\frac{S(t+\Delta t)}{S(t)} = \exp \left[ (r - \frac{\sigma^2}{2}) \Delta t + \sigma \sqrt{\Delta t} \, d \right],
\]

with \( d \) standard normal. Here

\[
E \left[ \frac{S(t+\Delta t)}{S(t)} \right] = e^{\mu \Delta t}
\]

so we recall

equation for \( \mu, \sigma, d \) is

\[
\mu = \mu_u + \sigma d
\]

To ensure the variance note that

\[
E \left[ \left( \frac{S(t+\Delta t)}{S(t)} \right)^2 \right] = \mu_u \sigma^2 + \sigma^2 \Delta t d^2 \quad \text{(discrete)}
\]

\[
\text{also,} \quad E \left[ \left( \frac{S(t+\Delta t)}{S(t)} \right)^2 \right] = \exp \left[ (2r - \sigma^2) \Delta t \right]
\]

\[
E \left[ e^{2 \sigma \sqrt{\Delta t} \, d} \right] = \exp \left[ (2r - \sigma^2) \Delta t \right] \exp \left[ -\frac{1}{2} \Delta t \right]
\]

\[
= \exp \left[ (2r - \sigma^2) \Delta t \right].
\]

Thus we obtain a third equation,

\[
\mu = \mu_u + \sigma d
\]

We have now 3 equations for 4 unknowns so we are
(a) \( p_u = \frac{P}{A} = \frac{2}{3} \) where \( P = 2 \) and \( A = 3 \).

(1.12) \[ \frac{1}{2} (n + 1) = \frac{e^{\alpha t}}{\alpha t} \]

(1.13) \[ \frac{1}{2} (n^2 + 4) = e \left( \frac{\alpha^2 + 2\alpha}{2} \right) \alpha t \]

Now in equation (1.12) we obtain

(1.14) \[ \frac{1}{2} (n^2 + 4) + \frac{P}{A} d = 2 e \]

From (1.13) we conclude

(1.15) \[ nd = e^{\alpha t} \left[ 2 - \frac{\alpha^2}{2} \right] \]

Now if we multiply (1.12) by \( A \) and use (1.15) we have a quadratic eqn. for \( u \)

(1.16) \[ u^2 - 2e^{\alpha t} u + e^{2\alpha t} \left[ 2 - \frac{\alpha^2}{2} \right] = 0, \]

which has solutions

(1.17) \[ u = e^{\alpha t} \left( 2 \pm \sqrt{\frac{\alpha^2}{2} - 1} \right) \]

(1.18) \[ u_{n+1} = e^{\alpha t} \Rightarrow \quad p_u u_{n+1} = e^{\alpha t} \]

(1.19) \[ p_u u_{n+1} = p_u u_n + p_u \]

Here we have
\[ u_{n+1} = \left( e^{(2\sigma^2)\Delta t} \right)^{\frac{1}{2}} \left[ e^{\sigma \Delta W_n} \right] \mathbb{I} e^{\frac{\Delta t}{2} \left[ e^{\sigma \Delta W_{n+1}} - 1\right]} \Rightarrow \]
\[ e^{\frac{\Delta t}{2}} \left[ e^{\sigma \Delta W_{n+1}} - 1\right] = \left( e^{\sigma \Delta W_n} \right)^{\frac{1}{2}} \mathbb{I} e^{\frac{\Delta t}{2} \left[ e^{\sigma \Delta W_{n+1}} - 1\right]} \Rightarrow \]
\[ e^{\frac{\Delta t}{2}} \left[ e^{\sigma \Delta W_{n+1}} - 1\right] = \left( e^{\sigma \Delta W_n} \right)^{\frac{1}{2}} \mathbb{I} e^{\frac{\Delta t}{2} \left[ e^{\sigma \Delta W_{n+1}} - 1\right]} \Rightarrow \]
\[ e^{\frac{\Delta t}{2}} \left[ e^{\sigma \Delta W_{n+1}} - 1\right] = \left( e^{\sigma \Delta W_n} \right)^{\frac{1}{2}} \mathbb{I} e^{\frac{\Delta t}{2} \left[ e^{\sigma \Delta W_{n+1}} - 1\right]} \Rightarrow \]
\[ e^{\frac{\Delta t}{2}} \left[ e^{\sigma \Delta W_{n+1}} - 1\right] = \left( e^{\sigma \Delta W_n} \right)^{\frac{1}{2}} \mathbb{I} e^{\frac{\Delta t}{2} \left[ e^{\sigma \Delta W_{n+1}} - 1\right]} \Rightarrow \]
\[ e^{\frac{\Delta t}{2}} \left[ e^{\sigma \Delta W_{n+1}} - 1\right] = \left( e^{\sigma \Delta W_n} \right)^{\frac{1}{2}} \mathbb{I} e^{\frac{\Delta t}{2} \left[ e^{\sigma \Delta W_{n+1}} - 1\right]} \Rightarrow \]
\[ e^{\frac{\Delta t}{2}} \left[ e^{\sigma \Delta W_{n+1}} - 1\right] = \left( e^{\sigma \Delta W_n} \right)^{\frac{1}{2}} \mathbb{I} e^{\frac{\Delta t}{2} \left[ e^{\sigma \Delta W_{n+1}} - 1\right]} \Rightarrow \]

Thus \( A = \frac{1}{2} \left[ e^{-\sigma \Delta W_n} - e^{\sigma \Delta W_n} \right] \).

From (1.18), we also have

\[ n = A + \sqrt{A^2 - 1}, \quad d = A - \sqrt{A^2 - 1}. \]

Thus

\[ A = \frac{1}{2} \left[ e^{\sigma \Delta W_n} - e^{-\sigma \Delta W_n} \right], \quad d = \frac{1}{2} \left[ e^{\sigma \Delta W_n} - e^{-\sigma \Delta W_n} \right]. \]

We can parameterize the binomial tree by

\[ (m, n), \quad m = 0, \ldots, M, \quad n = 0, \ldots, m. \]

Thus

\[ (m, n) \rightarrow \text{at time } n \Delta t \text{ stock price } = S_0 u^m d^n. \]

Suppose now we wish to value a European option with expiration time \( T \), \( M \Delta t = T \) where the value of the option at \( T \) and stock price \( S_0 u^m d^n \) is \( V^n \), \( n = 0, \ldots, m \). We have

\[ V^n = e^{\sigma \Delta W_n} \left( E^n + \Delta t \right). \]

\[ 0 \leq m < M, \quad 0 \leq n \leq m. \]

The equation

\[ V^n = e^{\sigma \Delta W_n} \left( E^n + \Delta t \right). \]

is a discretized version of the Black-Scholes equation. To derive it we construct a usual a
Thus for a rich free portfolio and no arbitrage we have

\[ V - d S = e^{-\sigma t} \left\{ V^+ - a S_f \right\} = e^{-\sigma t} \left\{ V - d S \right\}. \]

From the last 2 equalities we get

\[ a = \frac{\left\{ V^+ - V^- \right\}}{S \left( u - d \right)}, \quad \text{where} \]

\[ V = \frac{\left\{ e^{\sigma t} d \right\} + e^{-\sigma t} \left\{ V^+ - \frac{\left\{ V^+ - V^- \right\}}{u - d} \right\} u^2}{\left( u - d \right)^2} \]

\[ = e^{-\sigma t} \left\{ \frac{\left\{ e^{\sigma t} d \right\} V^+ + \left\{ u - e^{\sigma t} d \right\} V^-}{u - d} \right\} \]

\[ = e^{-\sigma t} \left\{ e^{\sigma t} d V^+ + (u - e^{\sigma t} d) V^- \right\}. \]

Clearly the recurrence (1.26) enables us to evaluate all the values \( V^n \) on the binomial tree.

Thus in fact the same as the present exact method we have used earlier, but not the advance of saving computational resources. Recall that the basic idea of the exact method was that these steps had to be small compared to space steps, i.e., \( \Delta t / \Delta x^2 = O(1) \), and we can see this holds here also.

Set \( u \) go to the logarithmic variables to

\[ \log S \to \log S + \log d \to \log S + \log u \]

where \( \Delta x = \log u \) is the space step. We note for definiteness the case (a) \( \rho u = \rho d = \frac{1}{2} \).
From (1.11) we have
\[ \log u = \frac{5}{2} \Delta t + \log \left[ 2 + 5 \sqrt{\Delta t} \right] + H.0. \]

Thus we have \( u(\Delta x) \approx 5 \sqrt{\Delta t} \) when \( \Delta x \approx 1/5^2 \). We compare this to the forward Euler method for the \( u \) equation with \( x = \log f \) as the space variable. Thus
\[ \frac{\partial^2 u}{\partial t^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial u}{\partial x} - \nu u = 0. \]

The forward Euler method for (1.32) gives
\[ \frac{V_{n+1} - V_n}{\Delta t} + \frac{1}{2} \sigma^2 \left[ \frac{V_{n+1} - 2V_n + V_{n-1}}{(\Delta x)^2} \right] 
+ (\nu - \frac{1}{2} \sigma^2) \left[ \frac{V_{n+1} - V_{n-1}}{2 \Delta x} \right] - \nu V_n = 0, \]
where
\[ V_n = V_{n+1} \left[ 1 - \frac{\sigma^2 \Delta t}{(\Delta x)^2} - \nu \Delta t \right] + \left( \frac{1}{\Delta t} \right) \text{ decay term.} \]

Evidently this is the condition for stability. A formal Euler is \( \Delta t \leq (\Delta x)^2 < \Delta t \sigma^2 \) which is what we obtained in the binomial tree.

We consider the absence of boundary conditions in the above algorithm. This is of course an advantage, especially in cases like linear options where it is not completely clear what the b.c. should be. There is naturally a disadvantage, which is that
\[ \text{the range of } x = \log f \text{ values in consider, is} \]
much large then we would naturally use the approx \( b.c. \)

So to see this consider the example (1.31), (1.32) we

just used. We have \( N \rightarrow D \rightarrow \epsilon \sqrt{D} \ldots \)

Suppose we have a lifetime \( 0 < \tau < T \) for the option. Thus

\( M = T / \Delta t \) and \( \tau \) ranges over \( |\tau| < M \Delta t \)

roughly which is \( |\tau| < (T / \Delta t) \epsilon \sqrt{\Delta t} = T \epsilon \sqrt{\Delta t} \)

so as \( \Delta t \to 0 \) the interval for \( \tau \) becomes extremely

large.

Now the binomial tree method gives the value

\( V_0 \) or explain as an expectation. In fact

from (1.26) we have

\[
V_0 = e^{-\frac{\epsilon T}{2}} \sum_{n=0}^{M} \binom{M}{n} \left( \frac{\epsilon}{T} \right)^{M-n} V_{n}
\]

\[
= e^{-\frac{\epsilon T}{2}} \mathbb{E} \left[ V(S(T)) \right]
\]

where expectation is with respect to the binomial

variable. We can rewrite this as

\[
V_0 = e^{-\frac{\epsilon T}{2}} \mathbb{E} \left[ V \left( S \exp \left( \sum_{m=1}^{M} X_m \right) \right) \right]
\]

where \( N \) the \( X_m, m=1, M \) are \( i.i.d \)

Bernoulli variables with law,

\[
X_m = \begin{cases} 0 & \text{prob } p_m, \\ 1 & \text{prob } 1-p_m \\ \end{cases}
\]

Since we have as well as (1.36),

\[
\log d = (\frac{-\epsilon}{2} \sigma^2 ) \Delta t - \epsilon \sqrt{\Delta t} + \epsilon \text{.} \quad 0.
\]

we have since we are assuming \( \sigma^2 = \epsilon^2 d = 1/2, \)

\[
\mathbb{E}[X_m] = (\epsilon \frac{-\epsilon}{2} \sigma^2 ) \Delta t + \epsilon \text{.} \quad 0.
\]

\[
\text{Var}[X_m] = \epsilon^2 a \Delta t + \epsilon \text{.} \quad 0.
\]
Now using the central limit theorem

$$\sum_{m=1}^{M} \frac{\epsilon_{m}}{\Delta T} \sum_{n=1}^{N} \left( \frac{\Delta W_{(m,n)}}{\sqrt{\Delta T}} \right) \sim \mathcal{N}(0, \sigma^{2})$$

we conclude on putting $M \Delta T = T$ and letting

$$\Delta T \rightarrow 0 \quad \text{with} \quad N \Delta T = T$$

$$\epsilon_{m} \rightarrow e^{-\gamma T} \mathbb{E} \left[ X^{T} \right]$$

which gives the continuous B-S price for the European option.

Next we wish to incorporate the payment of dividends into an binomial tree. The risk neutral stock evolution is now for a dividend payment rate $D$ given by replacing $\epsilon$ in (1.1) by $(\epsilon - D)$, where

$$dS_{t} = (\epsilon - D) dt + \sigma dW_{t}, \quad 0 \leq t \leq T$$

We proceed now as before replacing (1.8), (1.11) by

$$dX_{t} = (\epsilon - D) dT$$

$$p \nu \nu + \bar{b} d\nu = e^{-\gamma T} \mathbb{E} \left[ X^{T} \right]$$

The recurrence relation (1.26) remains as before, but now result a discrete dividend payment at time $\Delta T$ with the dividend is $X^{T} \mathcal{D}_{i} \nu$ for some $\nu$, $0 < \nu = 1$. To assemble this into account assume $m \Delta T < \mathcal{D}_{i} \nu < (m + 1) \Delta T$.

We construct the binomial tree as before up to $m \Delta T$ last. Then make a change from $m = m \Delta T$ to $m = m \Delta T + \Delta T$, as follows:
\[(1.44) \quad \sum \frac{m_i \Delta x}{n} = \sum \frac{m_i \Delta x}{n} d_i (1 - f_i) \]

\[\sum m_i \Delta x = \sum m_i u_i (1 - f_i) \]

Where \( m_i \) are the counts in each bin for \( m > n \), \( m_i \Delta x \) is the width of the bin. This is sufficient to take into account of the distribution. Thus, we can use the recurrence \((1.26)\) with \( u_i, \beta_i \) defined as \( m_i \Delta x \) and \( u_i, \beta_i \), \( f_i, \beta_i \) satisfying the expectation and variance conditions \((1.2), (1.1)\). The reason for this is that \((1.27)\) can be written, taking account \( m \) the definition as

\[(1.45) \quad X - 2 \bar{S} = e^{-\alpha} \left[ \frac{N}{N - 2} \left( \frac{1}{\beta} + \frac{1}{\beta_3} \right) \right] \]

where \( \alpha, \beta \) are the possible values. The path moves \( \Delta t \) to in time \( \Delta t \). From \((1.44)\) \( \bar{S} = \frac{(1 - \beta)}{\beta} \) where \( \bar{S} = \frac{u_i (1 - \beta)}{\beta} \)

\[(1.29) \quad \text{as previously.} \]

The binomial AAE can be generalized. As binomial AAE for example, where \( d < e < n \).

If we want AAE to remain, then we need \( \mu d = x^2 \). Now we have 3 probabilities \( P_1, P_2, P_3 \) and 3 equations as before

\[(1.46) \quad P_1 + P_2 + P_3 = 2 \]

\[P_1 u + P_2 x^2 + P_3 d^2 = e \]

\[P_1 u^2 + P_2 x^2 + P_3 d^2 = e(\sigma^2 + \sigma^4) \Delta t \]
If the variance is not 1, we may change units so that the variance is 1. Let $Y = \log S$ for the other stock.

From (1.1) we see that $Y_t = \log S_t$ satisfies

$$\frac{dY_t}{\sigma} = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$ 

As before we have parameters $\mu, \sigma, \Phi, \phi$ and

$$\frac{dY_t}{\sigma^2} = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$ 

Let $U, V$ be normally distributed with mean 0, variance 1, and $C = \sqrt{\frac{\sigma^2}{2}}$. Then

$$Y + \log u + \log v \sim \mathcal{N}((u, v), C^2).$$

Suppose now we have N correlated stocks $S_1, \ldots, S_N$ satisfying the SDE

$$\frac{dS_i}{S_i} = \mu dt + \sigma dW_{i,t},$$

where $W_{i,t}, \ldots, W_{N,t}$ are Brownian motions with correlation.

Then

$$\mathbb{E}[dW_i, dW_j] = \delta_{ij} dt,$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
As previously in $R = [A_i, a_j]$, we do the Cholesky factorization, $R = AA^T$ when $A = [a_{ij}]$. Thus,

$$d x_i, t = \rho x_i - \sum_{j=1}^{d} \sigma a_i a_j \, d Z_{j, t}$$

where $\text{d} Z_{j, t}, j = 1, \ldots, d$ are independent $G.M.$

Next we transform the log variables

$$d \log L_i, t = \left[ 1 - \frac{1}{2} \left( \bar{\sigma}^2 \bar{x}_i + \bar{\sigma}^2 \bar{x}_j \right) \right] dt$$

$$+ \sum_{j=1}^{d} \bar{\sigma} a_i a_j \, d Z_{j, t}$$

where the matrix $\bar{\sigma}$ is given by

$$\bar{\sigma} = \left[ \bar{\sigma}_{ij} \right], \quad \bar{\sigma}_{ij} = \bar{\sigma}_{i j} = \sigma_{i j}, \quad \text{thus was constant.}$$

The stage we reached to apply the MC method.

To construct a tree for this system, we need to go on stage further. We make a transformation

$$\left[ \log L_i, t, \ldots, \log L_d, t \right] \rightarrow \left[ \bar{L}_i, t, \ldots, \bar{L}_d, t \right]$$

so that the variables $\bar{L}_i, t, \ldots, \bar{L}_d, t$ are independent. To do this, we solve

$$\left( \bar{L}_i, t \right)^T = \bar{\sigma} \bar{x}_i \left[ \log L_i, t, \ldots, \log L_d, t \right]^T$$

and hence if we apply $\bar{\sigma}^{-1}$ to (2.8) we get equations

$$d \bar{L}_i, t = \rho \bar{L}_i, t + d \bar{Z}_i, t$$

$$d \bar{L}_d, t = \mu \bar{L}_d, t + d \bar{Z}_d, t$$

where the $\mu$s are given by
(2.11) \[
\left[ p_0, \ldots, p_n \right]^T = \tilde{\xi}^{-1} \left[ c - \frac{1}{2} \left( \tilde{\xi}_1 + \cdots + \tilde{\xi}_n \right), \ldots, c - \frac{1}{2} \left( \tilde{\xi}_1 + \cdots + \tilde{\xi}_n \right) \right]^T.
\]

Now we construct a set for \( Y_1, \ldots, Y_n \), which can be parameterized by \((m, n_1, \ldots, n_n), n_i = 0, \ldots, m\). Thus if \( Y_{1,0}, \ldots, Y_{1,m} \) is the initial point on the tree we have then

(2.12) \[
Y_{i, m, n_i} = Y_{i, 0} + n_i \log n + (m - n_i) \log 1.
\]

**Example:**
Carnivore 2 which evolves as

(2.13) \[
\frac{ds_{1, t}}{s_{1, t}} = -\lambda t + \delta_1 dW_t, \quad \frac{ds_{2, t}}{s_{2, t}} = -\lambda t + \delta_2 \left[ \rho dW_t + \sqrt{1 - \rho^2} dZ_t \right],
\]

where \( W_t, Z_t \) are independent BMs. In this case

(2.14) \[
R = \begin{bmatrix} A & \ast \\ \ast & B \end{bmatrix}, \quad A = \begin{bmatrix} \Delta & 0 \\ \rho & \sqrt{1 - \rho^2} \end{bmatrix}, \quad B = \begin{bmatrix} \Delta & 0 \\ \rho & \sqrt{1 - \rho^2} \end{bmatrix}.
\]

(2.15) \[
AA^T = \begin{bmatrix} A & \ast \\ \ast & B \end{bmatrix} = R.
\]

So (2.13) is already in the form (2.8)

(2.16) \[
\begin{align*}
\Delta \log s_{1, t} &= \left( \delta - \frac{1}{2} \sigma_1^2 \right) dt + \sigma_1 dW_t, \\
\Delta \log s_{2, t} &= \left( \delta - \frac{1}{2} \sigma_2^2 \right) dt + \sigma_2 \left[ \rho dW_t + \sqrt{1 - \rho^2} dZ_t \right].
\end{align*}
\]
Our matrix $\tilde{\mathbf{t}}$ is given by

$$
(2.17) \quad \tilde{\mathbf{t}} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \sqrt{1-\eta^2} & 0 \\ 0 & \sqrt{1-\eta^2} \end{bmatrix}
$$

Once $\mathbf{A}$ is now regular, $\mathbf{M}^{\dagger} \mathbf{A} = \text{product of diagonal elements} = \frac{1}{\sqrt{1-\eta^2}}$ where

$$
(2.18) \quad \mathbf{M}^{\dagger} \tilde{\mathbf{t}} = \mathbf{M} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \quad \mathbf{M} \mathbf{A} = \sigma_1 \sigma_2 \sqrt{1-\eta^2}.
$$

Thus

$$
(2.19) \quad \tilde{\mathbf{t}}^{-1} = \frac{1}{\sigma_1 \sigma_2 \sqrt{1-\eta^2}} \begin{bmatrix} \sqrt{1-\eta^2} & 0 \\ 0 & \frac{1}{\sigma_1 \sqrt{1-\eta^2}} \end{bmatrix}, \quad \text{where}
$$

$$
(2.20) \quad \begin{bmatrix} \tilde{m}_1 \\ \tilde{m}_2 \end{bmatrix} = \tilde{\mathbf{t}}^{-1} \begin{bmatrix} \frac{\sigma_i}{\sigma_1} \\ \sigma_i \end{bmatrix}
$$

$$
(2.21) \quad \begin{array}{c}
\tilde{m}_1 = \frac{1}{\sigma_1} \left( \sigma - \frac{\sigma_i}{2} \right), \\
\tilde{m}_2 = \frac{\eta}{\sigma_1 \sqrt{1-\eta^2}} \left( \sigma - \frac{\sigma_i}{2} \right) + \frac{1}{\sigma_2 \sqrt{1-\eta^2}} \left( \sigma - \frac{\sigma_i}{2} \sigma_i \right),
\end{array}
$$

We now we have

$$
(2.22) \quad d \gamma_1 = \tilde{m}_1 \, dt + d \gamma_2, \quad d \gamma_2 = \tilde{m}_2 \, dt + d \gamma_2.
$$

To construct the line with nodes $(m_1, n_1, n_2)$ we proceed similarly to $(2.2) \Rightarrow (2.4)$. Thus we choose $d_x, d_y, d_{\mathbf{u}}, d_{\mathbf{v}}, d_{\mathbf{p}}$ to satisfy

$$
(2.23) \quad p_u + p_d = x, \quad (2.24) \quad p_u (\log u) + p_d (\log d) = m_1, \quad (2.25) \quad p_u (\log u)^2 + p_d (\log d)^2 = m_2 (\Delta t)^2 + \Delta t.
$$
Let \( V_{n_1, n_2} \) be the value of the option corresponding to the node \((m, n_1, n_2)\). Then, we get a recurrence as previously

\[
V_{n_1, n_2} = e^{-rt} \left[ \phi_{2, n_1, n_2} V_{n_1, n_2+1}^{(m+1)} + \phi_{1, n_1, n_2} V_{n_1+1, n_2}^{(m+1)} + \phi_{3, n_1, n_2} V_{n_1, n_2+1}^{(m+1)} \right],
\]

This is the analogue of (2.26). The parameters \( \phi_{1, n_1, n_2}, \phi_{2, n_1, n_2}, \phi_{3, n_1, n_2} \) are determined by (2.27) - (2.25) for \( Y_1, Y_2 \). To get the terminal data, we have say for an arithmetic mean call option with payoff

\[
(2.27) \quad \text{payoff} = \left[ \frac{s_{1, 1} + s_{2, 1}}{2} - K \right]^+. \]

Terminal condition on \( V_{n_1, n_2} \) given by

\[
V_{n_1, n_2}^{(m)} = \left[ \frac{1}{2} \exp \left\{ \tilde{\sigma}_{1, 1} Y_{1, m, n_1, n_2} \right\} + \frac{1}{2} \exp \left\{ \tilde{\sigma}_{2, 1} Y_{2, m, n_1, n_2} \right\} \right]^+ \]

\[
= \frac{1}{2} \exp \left\{ \sigma_{2} Y_{2, m, n_1, n_2} \right\} + \frac{1}{2} \exp \left\{ \sigma_{2} \sqrt{1-\rho^2} Y_{2, m, n_1, n_2} \right\} - K \right]^+. \]

In terms of a lattice, the picture is:
Note there are several possibilities including staying at the same rate.

3.1. 
\[ p(t, T) = \text{value at time } t \text{ of a new bond issued with a face value of } 1 \text{ which matures at time } T. \]

\[ \] We can use these prices to compute the forward rate of interest \( F(t, T_1, T_2) \) with \( t < T_1 < T_2 \), i.e.,

\[ F(t, T_1, T_2) = \text{(simple interest)} \times \text{rate of interest on a } 0 \text{ bond at time } T_1 \text{ with repayment at } T_2. \]

\[ \] Clearly, the no-arbitrage value of \( F \) is

\[ F(t, T_1, T_2) = \frac{p(t, T_1) - p(t, T_2)}{(T_2 - T_1) p(t, T_2)} \]

In a more complicated case, suppose at time \( T_0 \) we wish to find the swap rate i.e., fixed versus floating of money borrowed at time \( T_0 \) to \( T_2 \), repaid at time \( T_n \) with interest payments at times \( T_0, \ldots, T_n \).
with \( T_0 < T_1 < T_2 < \cdots < T_n \).

Just as before, we have that the fixed rate \( r \) is given by

\[
(3.4) \quad r = \frac{p(0,T_0) - p(0,T_n)}{\sum_{i=1}^{n} (T_i - T_{i-1}) p(0,T_i)}\]

Next, suppose we wish to borrow as in the previous example and repay at the floating rate with interest payments at \( T_1, T_2, \ldots, T_n \) but with a cap \( R \) say on the interest payments. Consider how to value the cost of this cap. Thus at time \( T_i \) the option pays

\[
(3.5) \quad (T_i - T_{i-1}) [L(T_{i-1}, T_i) - R^+] \]

where \( L(t,t) \) is the spot rate at time \( t \) for borrowing with repayment at \( t \).

The simplest way to price this is to use Black's model. Thus suppose the interest payments are for a period \( \delta \). Then on putting \( L(t) = L(t, t+\delta) \), Black proposes that \( L(t) \) satisfies

\[
(3.6) \quad L(t) = F(0, t, t+\delta) S(t), \quad \text{where}
\]

\[
(3.7) \quad \frac{dS(t)}{S(t)} = \sigma dW(t), \quad \text{and} \quad S(0) = 1,
\]

\( W(t) \) is as usual Brownian motion. Here the
value of the call at (3.5) is given by

\[ (3.8) \quad \text{Value} = (\text{Black-Scholes price of a call option on a stock with strike price } K \text{ and volatility } \sigma \text{ with strike price at time } t \text{ given by} \quad F(t, T_i, T_0), \]

\[ \text{expiration date } T_{i-1}, \quad (T_i - T_{i-1}) P(t, T_i) \]

We can estimate \( \sigma \) in (3.7) from historical data or by using Black’s formula to compute implied volatility from market data on interest rate caps or other interest rate derivatives.

### 4 Hull-White Model

In Black’s model we use the bond prices \( P(0, T), T > 0 \), to calibrate the model. There are of course precisely known. To them we add all the volatility \( \sigma \) which we can estimate in various ways. In the Hull-White model there are 2 parameters \( \alpha, \beta \) which we need to estimate. In this model the short term rate – adjusted by market price of risk – evolves as

\[ (4.1) \quad d\Theta(t) = (\alpha \Theta(t) - \beta \Theta(t)) dt + \sigma \Theta(t) dW(t), \]

with \( W(t) \) and \( \sigma \) as before. Now \( \Theta(t) \) is determined from zero coupon bond prices by

\[ (4.2) \quad P(0, T) = E \left[ \exp \left( -\int_0^T \Theta(t) dt \right) \right], \quad T > 0. \]

With \( \alpha, \sigma \) given this determines \( \Theta(t) \).
precisely. The value of the constant (2.5) is then given by

$$E \left\{ \exp \left[ \frac{T_i}{T_n-T_{n-1}} \right] \left( \frac{\exp \left[ \int_{T_{n-1}}^{T_n} f(t)dt \right] - 1}{T_n-T_{n-1}} \right)^{-2} \right\}$$

If the interval $T_{n-1} \rightarrow T_n$ is small then this is approximately,

$$E \left\{ \left( \int_{T_{n-1}}^{T_n} f(t)dt - R \right)^{+2} \right\}$$

We shall show now how to make a lattice model for (4.1), and calibrate it according to (4.2). First we construct a lattice for the equations related to (4.1) given by

$$d \xi(t) = -\alpha \xi(t)dt + \sigma dW(t)$$

Note that (4.5) is mean reverting to 0. The nodes are of the form

$$\left( m, j \right), \quad 0 \leq m \leq M, \quad j \leq \min \left\{ m, J \right\}$$

Thus the lattice looks like:

```
\[x(3 \Delta t, 3 \Delta t)\]
```

Thus we begin from $t = 0$ and the mesh we allow

$$5 \leq \Delta t \leq 5 \Delta \xi$$

clearly it should be chosen large enough depending on $\alpha, \sigma$. This
obviously depends on the width $A$ the moment measure for (4.5) satisfies the equation,

\[(4.7) \quad \frac{1}{2} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{\sqrt{\pi}} (r \phi r) = 0, \quad \text{which has solution}
\]

\[(4.8) \quad \phi r r = k \exp \left[ -\frac{ar^2}{\sqrt{2}} \right], \quad \text{where the constant } k \text{ is chosen so that}
\]

\[(4.9) \quad \int_{0}^{\infty} \phi r r dz = 1.
\]

Here the moment measure is concentrated in the region

\[(4.10) \quad |y| < 3, \quad \text{where} \quad \frac{y^2}{2} = \frac{ar^2}{\sqrt{2}} \quad \text{i.e. within 3 standard deviations} \Rightarrow
\]

\[(4.11) \quad 15 < 3 \sqrt{2}a \Rightarrow 15 \Delta \Rightarrow 3 \sqrt{2}a.
\]

Next we define transition probabilities on the lattice consistent with (4.5). There are 3 possibilities - remain in, up or down, i.e.

- Up probability = $\phi u \Rightarrow (m, j) \Rightarrow (m+1, j+1)$
- Down probability = $\phi d \Rightarrow (m, j) \Rightarrow (m+1, j-1)$
- Remain same probability = $\phi r \Rightarrow (m, j) \Rightarrow (m+1, j)$.

Evidently, we have

\[(4.12) \quad \phi u + \phi d + \phi r = 1.
\]

The rule since we cannot go up, we do this as follows:
\[(m, J) \to (m+1, J) \text{ with prob. } p_u\]
\[(m, J) \to (m+1, J-1) \text{ with prob. } p_s\]
\[(m, J) \to (m+1, J-2) \text{ with prob. } p_d\]

Similarly, at \(J = -J\) we have,
\[(m, -J) \to (m+1, -J) \text{ with prob. } p_u\]
\[(m, -J) \to (m+1, -J+1) \text{ with prob. } p_s\]
\[(m, -J) \to (m+1, -J+2) \text{ with prob. } p_d\]

We need now 2 more constraints to determine \(p_u, p_s, p_d\). As previously in determine these by equating
1st and 2nd moments for the SDE (4.5) to the corresponding values for the discrete model. Note that
(4.5) is the SDE for the Ornstein-Uhlenbeck process and hence has Gaussian solutions. To see
this we simply integrate (4.5), where

\[
(4.13) \quad d \left[ e^{at} \right] = e^{at} \ dW(t)
\]
\[
(4.15) \quad e^{a(t+\Delta t)} - e^{at} = \int_{t}^{t+\Delta t} a e^{at} \ dW(t)
\]

\[
(4.16) \quad \int_{t}^{t+\Delta t} e^{at} \ dW(t) = k \ i \ \text{ for some } k \ \text{ where } i \in \mathbb{N}(0,1).
\]

\[
(4.17) \quad e^{a(t+\Delta t)} - e^{at} = \left[ e^{-a \Delta t} - 1 \right] e^{at} + \sqrt{\Delta t} \ \text{ to leading order as } (4.15) \text{ gives}
\]

\[
(4.18) \quad E \left[ \left( e^{a(t+\Delta t)} - e^{at} \right) \right] = -a \Delta t \ e^{at},
\]
\[
E \left[ t^2 (t + \Delta t)^2 - t^2 (t)^2 \right] = \sigma^2 \Delta t + a^2 \Delta t^2 (\Delta t)^2,
\]

for the discrete model if we are at time \( t \).

\[
E \left[ s^2 (t + \Delta t)^2 - s^2 (t)^2 \right] = \rho u \Delta t + \phi (0) + h (\Delta t - \Delta t),
\]

where \( s^2 (t) \sim \mathcal{N} \), hence \( s^2 (t) \sim \mathcal{N} \Delta t \).

\[
(\rho u - \phi (0)) \Delta t = -a \Delta t \frac{\Delta t}{\Delta t}
\]

is the second moment we have similarly.

\[
(\rho u + \phi (0)) \Delta t = \sigma^2 \Delta t + a^2 \Delta t^2 (\Delta t)^2,
\]

Observing now from (4.22) that since \( \rho u + \phi (0) \geq 1 \)

we must have

\[
\sigma^2 \Delta t / (\Delta t)^2 < 1.
\]

We shall actually choose

\[
\sigma^2 \Delta t / (\Delta t)^2 = 1/3
\]

because this will make the 3rd moment for the continuous and discrete systems agree as well as the 1st and 2nd.

In fact from (4.10), we have

\[
E \left[ \left( \sum_{i=1}^{n} \epsilon_i (t + \Delta t) - \epsilon_i (t) \right)^2 \right] =
-3 \sigma^2 a s \epsilon_i (t) (\Delta t)^2 - a^2 (\epsilon_i (t))^3 (\Delta t)^3,
\]

so equating 3rd moments we obtain

\[
(\rho u - \phi (0)) \Delta t = \sigma^2 \Delta t + a^2 \Delta t^2 (\Delta t)^2,
\]

\[
(\rho u - \phi (0)) \Delta t = \sigma^2 \Delta t + a^2 \Delta t^2 (\Delta t)^2.
\]
If we divide now (4.21) into (4.26) we obtain

\[(\Delta r)^2 = 3 \sigma^2 \Delta t\]

As leading order which is the same as (4.27). Now using (4.21), (4.22), (4.24) we have for \(P_n, \bar{r}_S, \bar{r}_D\) formulas,

\[
P_n = \frac{1}{6} + \frac{1}{2} \left[ a^2 \bar{r}^2 (\Delta t)^2 - a \bar{r} \Delta t \right],
\]

\[
P_D = \frac{1}{6} + \frac{1}{2} \left[ a^2 \bar{r}^2 (\Delta t)^2 + a \bar{r} \Delta t \right],
\]

\[
P_S = \frac{2}{3} - a^2 \bar{r}^2 (\Delta t)^2.
\]

For \(J = 1\) we will get different values since now the discrete mean and variance we give by

\[
E \left[ s(t+\Delta t) - s_t(t) \right] = P_n(0) + P_S(-\Delta t) + P_D(-2\Delta t)
\]

where we obtain

\[
(P_S + 2P_D) \Delta t = a \Delta t J \Delta t.
\]

\[
E \left[ (s(t+\Delta t) - s_t(t))^2 \right] = P_S \Delta t^2 + P_D \Delta t^2
\]

where we obtain

\[
(P_S + 4P_D) \Delta t^2 = 6 \Delta t^2 + a^2 J^2 (\Delta t^2)^2.
\]

Thus, given \(a\) gives

\[
P_D = \frac{1}{6} + \frac{1}{2} \left[ a^2 J^2 (\Delta t)^2 - a J \Delta t \right],
\]

\[
P_S = -\frac{1}{3} - a^2 J^2 (\Delta t)^2 + 2a J (\Delta t),
\]

\[
P_n = \frac{7}{6} + \frac{1}{2} \left[ a^2 J^2 (\Delta t)^2 - 3a J \Delta t \right].
\]

For \(J = -\frac{1}{2}\) we have the same formula as in (4.33) with \(P_D\) and \(P_n\) interchanged. Note that since we need \(P_n, P_S, P_D\), the formula (4.33) puts a limit on the
\[ (4.34) \quad \beta s = \frac{2}{\lambda} - [a J A T - 1]^2, \quad \text{where we put} \]

\[ (4.35) \quad a J_A T < \lambda + \sqrt{\frac{2}{\lambda}}, \]

\[ \text{Note that the column } (4.11) \text{ plus } (4.28) \]
gives

\[ (4.26) \quad J_T (a J_A T)^{1/2} x \approx (212)^{1/2}. \]

Evidently (4.36) implies (4.35) for small \( \Delta T \).

Next we wish to show how to calibrate to satisfy (4.2). In the process the SDE (4.5)
shall be transformed to (4.1) with the function \( \theta \) determined by the zero curve. To see why
this is so, put \( \Theta (t) = \Theta (0) - \sigma (t) \), with \( \sigma (t) \) satisfying (4.1) and \( \Theta (t) \) satisfying (4.5)
with the same \( \beta M = \frac{1}{2} \). Then we have

\[ (4.37) \quad d \Theta (t) = [\Theta (t) - a \Theta (t)] dT, \quad \text{where} \]

\( d \Theta (t) \) is deterministic. Thus we shall change our initial \( \Theta \) to an \( \Theta (0) \) as follows: On \( A \) put \( m \Delta T \) if \( \Theta (t) \) and a value is \( m \Delta T \) with

\[ l(t) \leq \min \{ m, J_T \} \].

Also on \( A \) put the value of \( \Theta \) at \( a m \Delta T \). The \( \Theta \)-like \( \Theta \) is

\[ \Theta = \Theta (t) + d^m = \Delta A T + d^m. \]

The values \( d^m \), \( m = 0, 1, 2, \ldots \), are determined according to (4.2) by the price of a
common bond.
We show how to find the values of \( m \). First note that we approximate the integral of \( r(t) \) as

\[
(4.39) \quad \int_0^T r(t) \, dt \approx r(0) \Delta t + r(\Delta t) \Delta t + r(2 \Delta t) \Delta t,
\]

where \( M \Delta t = T \). Then from (4.2) we have

\[
\Delta t \approx \frac{1}{2} \log \frac{3}{2}, \quad \text{yielding}
\]

\[
(4.40) \quad d^0 = -\frac{\Delta t}{\Delta t} \Delta t \log \frac{3}{2}.
\]

We are taking here \( r(0) \approx d^0 \). Now there are two possible values for \( r(\Delta t) \sim d^2 + s^2(\Delta t) \) since

\[
(4.41) \quad \int_0^{\Delta T} \Delta t = d^2 + s^2(\Delta t) \Delta t
\]

and using the fact that \( s^2(\Delta t) = \Delta t \)

with probability \( p_1 \), \( s^2(\Delta t) = 0 \) with prob. \( p_0 \) and

\[
(4.42) \quad p^2 = \exp \left[ -\left( d^0 + 2^0 + d^2 + s^2(\Delta t) \right) \Delta t \right] p_0
\]

\[
+ \exp \left[ -\left( 2^0 + 2^2 + s^2(\Delta t) \right) \Delta t \right] p_0 + \exp \left[ -\left( d^0 + 2^0 + d^2 \right) \Delta t \right] p_1
\]

Here we use the notation \( p^m \) in the place of a zero coupon bond maturing at time \( m \Delta t \). Similarly

\[
(4.42) \quad \text{determines the value of } d^4. \text{ Proceeding in a similar way we can determine all } d^m, \quad m = 0, 1, \ldots
\]

To produce an algorithm to determine the
\[ Q_{\Delta} = E \left[ \exp \left( -\int_0^{\Delta t} c(t) dt \right) \right] \]

Thus we have

\[ Q_1 = e^{-\Delta \omega} \quad Q_0 = e^{-\Delta \omega} \]

\[ Q_{\Delta} = e^{-\Delta \omega} \quad Q_{\Delta+1} = e^{-\Delta \omega} \]

Evidently, we have that

\[ Q_{m+1} = \sum_{\Delta = -\min(m, \Delta)}^{\min(m, \Delta)} Q_{\Delta} \exp \left[ -l \Delta(\Delta + 1) \Delta t \right] \]

Now if we can compute the \( Q_{\Delta} \) in norm \( A \) of the \( A^0, \ldots, A^{m-1} \) from (4.45),

...
\[(4.48) \quad Q^m_j = \exp \left[ -\left( a^m + \Delta_j \Delta_i \right) \Delta t \right] P_i \quad Q^m_i + \exp \left[ -\left( a^m + (J-1) \Delta_j \right) \Delta t \right] P_j \quad Q^m_j + \exp \left[ -\left( a^m + (J-2) \Delta_j \right) \Delta t \right] P_i \quad Q^m_i \]

Evidently if we combine (4.45) and (4.47) we shall get an algorithm which computes all the \( Q^m_i \) and all the \( A-B \) securities. Note that the \( Q^m_i \) are artificial prepaid intervals. Roughly speaking we have

\[(4.48) \quad Q^m_j = \text{price of a security which at time } m \Delta t \text{ pays } P_j \text{ at note } (m,j) \text{ and zero at notes } (m,i), \quad i \neq j.\]

Note that (4.48) needs to be modified if

\[\Delta = J \text{ or } -J, \quad \text{also } J-1, J-2.\]

Suppose for example \( m > J \). Then we have:

\[\text{Thus } \quad Q^m_{m+1} = \sum d_j \quad \text{where } \quad Q^m_j, \quad j = J-3, J-2, J-1, J.\]

Let us finally compare the Hull-White model with the Black model. In both cases, we input the entire zero curve but in the Black model interest rate fluctuations are with Amie proportioned to \( St \) whereas in the H-W model it remains bounded since the O-N process is mean reverting.
One problem about equation (4.1) which is the
equation for the evolution of the short rate
adjusted by the market price of risk is that
\( r(t) \) can become negative. One way to avoid
this is to assume that \( r(t) = \exp\left[ x(t) \right] \),
where \( x(t) \) evolves like (4.1), so

\[
(4.49) \quad d x(t) = \left[ \Theta(t) - a x(t) \right] dt + \sigma d W(t).
\]

This is called the Black-Karasinski model. To
make a discrete model calibrated by the zero
yield curve we proceed just as in A-W. Thus we
first construct a tree for the process \( x(t) \)
satisfying

\[
(4.50) \quad d x(t) = - a x(t) dt + \sigma d W(t).
\]

Then we put \( x(t) = x(0) + \Delta(t) \). Then writing
\( d^n = d(n \Delta t) \) we compute \( x^n \) by
recurrence. Note now that (4.45) becomes

\[
(4.51) \quad p^{m+1} = \sum_{j=\min(m,n),j} Q^{m}_{j} \exp \left[ - f(d^{m+n} x) \Delta t \right]
\]

where \( f(x) = e^x \). The \( Q^{m}_{j} \) we know
we cannot explicitly solve (4.51) for \( a_m \]
but we can solve numerically. Similarly,
the recurrence relations for the A-D securities are
obtained as in (4.47). Thus when in the
recurrence,

\[
(4.52) \quad Q^{m+1}_{j} = \exp \left[ - f(d^{m+n} x) \Delta t \right] Q^{m}_{j}
\]
\[ + \exp \left[ - f\left( x^n + (\delta + 1) x^2 \right) \cdot \Delta t \right] \cdot \delta \cdot Q_{\frac{n}{2} + 1}^m \]

\[ + \exp \left[ - f\left( x^n + (\delta - 1) x^2 \right) \cdot \Delta t \right] \cdot \delta \cdot Q_{\frac{n}{2} - 1}^m \]
§ 5 Black-Derman-Toy's Model

The BDT seeks to input more market data into the interest rate model and also to avoid the possibility of interest rates becoming negative which is an artifact of the HW model. We shall first construct the model on a lattice and then will discuss the continuum limit as $\Delta t \to 0$. Our lattice is not thus at each time step in corresponding to $m\Delta t$ there is a minimum interest rate $r_0$, but time in there on $m+1$ possible interest rates $r_i$.

$\frac{r_m^{j+1}}{r_m^j} < 1 \leq m$. There is just 2 parameters $r_m$ which determine the $r_m^{j+1} = \prod_{i=0}^{j} r_i$, since $r_0$ is given since we assume the lattice is logarithmically uniformly spaced. We have then

\begin{equation}
(5.1) \quad r_m^j = r_0 \exp \left[ 2 \delta \sum_{i=0}^{j} \beta^m \delta t \right], \quad 0 \leq j \leq m.
\end{equation}

The transition probabilities we just choose

$$
\frac{1}{2} \quad \text{so on \ N-W we have:}
$$

\begin{align*}
&\frac{1}{m-1} \quad \text{we compute now} \\
&\frac{1}{m+1} \\
&\frac{1}{m} \quad \text{since}
\end{align*}

$$
\log r_m^j (m\Delta t) - \log r_m^{j-1} (m-1\Delta t) = \beta \Delta t + \frac{\beta^m}{2} \Delta t
$$

each with prob $\frac{1}{2}$ that
Thus \( \Delta T \to \beta^m \) should correspond to the volatility of \( \log r(t) \) at time \( m \Delta T \). Since \( \beta^m \) again depend on \( m \) we can altering this volatility to depend on \( m \) unlike in Black’s model or HW. Thus the formulas in the model are the \( \gamma^m \) and the \( \beta^m \). We have how to calibrate these to the observed market data.

We define the Arrow-Debreu securities as

\[
\text{in HW, thus}
\]

\[
Q^m_j = \mathbb{E} \left[ \exp \left\{ - \int_0^{m \Delta T} r(t) \, dt \right\} e^r(m \Delta T) \right]
\]

where the price \( P^{m+1} \) of the zero coupon bond maturing at time \((m+1) \Delta T\) is given by

\[
P^{m+1} = \sum_{j=0}^m Q^m_j \exp \left[ - \gamma^m_j \Delta T \right]
\]

(5.4)

As before the \( Q^m_j \) are obtained by recursion. Thus:

\[
Q^m_j = \frac{1}{2} \exp \left[ - \gamma^m_j \Delta T \right] Q^m_{j+1}
\]

\[
+ \frac{1}{2} \exp \left[ - \gamma^m_{j-1} \Delta T \right] Q^m_{j-1}
\]

Note again that the relations (5.5) need to be modified for \( j = 0 \), \( j = m+1 \) when \( m \) is

\[
Q^{m+1} = \frac{1}{2} \exp \left[ - \gamma^m_m \Delta T \right] Q^m_m
\]
\( Q_{m+1}^m = \frac{1}{2} \exp \left[ - \frac{1}{2} \frac{\sigma_m^2}{\theta^2} \right] Q_m \)

This system is not sufficient to determine the yield curve, since we have 2 parameters \( r_m^0 \), \( f_m^m \) for each \( m \). To calibrate fully we need to estimate from market data the volatility \( \sigma_m \). The yield to the

\( r_m^m \) is the value at node \( n \) of zero coupon bond maturing at time \( n \Delta t \).

We can of course calculate (5.8) once we have determined the lattice. Now let \( y_t(t) \) be the yield on a bond determined by the equation

\( y_t(t) = \exp \left[ - \int_t^T y_t(u) (u-t) du \right] \).

Initially \( y_t(0) \) is known and for \( 0 < t < T \), \( y_t(t) \) is stochastic once we make assumptions on the bond prices at time \( t \) depend on the (stochastic) short term rate \( t \). We define the volatility \( \sigma_t \) of the yield as

\( \frac{1}{\theta} \log \frac{y_T(t)}{y_T(0)} = \mu dt + \sigma_t dW(t) \),

where \( dW(t) \) is BM. We can estimate \( \sigma_t \) from the \( P_m^m \) in (5.8). Thus if we write

\( P_m^m = \exp \left[ - \frac{1}{2} \frac{\sigma^2}{\theta^2} (m-1)^2 N(T) \right] \).
\[ \sigma_{T}^{2} \Delta t \sim \frac{1}{4} \left[ \log \frac{y_{1}^{m}}{y_{0}^{m}} \right]^{2}, \quad m \Delta t = T \]

where we are assuming \( \log \frac{y_{1}}{y_{0}} \) takes the values \( \pm \log c_{1} y_{0}^{m} \) with equal probability \( \frac{1}{2} \) \( \). Putting \( \sigma_{m} = \sigma_{1} \) we have \( \sigma_{m} \)

\[ (5.13) \quad \sigma_{m}^{2} = \frac{1}{2 \Delta t} \log \frac{y_{1}^{m}}{y_{0}^{m}} \]

Similarly as \( (5.4) \) we can define A-D recurrences \( \widehat{Q}_{\frac{\Delta t}{2}}^{m} \), \( i = 0, 1 \) so that

\[ (5.14) \quad \widehat{Q}_{\frac{\Delta t}{2}}^{m+1} = \sum_{\overline{m}}^{m} \widehat{Q}_{\frac{\Delta t}{2}}^{n} \exp \left[ -i \frac{\pi}{\Delta t} \right] \]

The \( \widehat{Q}_{\frac{\Delta t}{2}}^{m} \) satisfy the same recurrence relations as the \( \widehat{Q}_{\frac{\Delta t}{2}}^{m} \), i.e. \( (5.5), \ (5.6), \ (5.7) \), but the values at \( m = 0, \ n = 1 \) differ. Thus the \( \widehat{Q}_{\frac{\Delta t}{2}}^{m} \) are determined by setting

\[ (5.16) \quad \widehat{Q}_{0}^{m} = \pi \quad \text{and using } (5.5) - (5.7) \text{ to determine the } \widehat{Q}_{\frac{\Delta t}{2}}^{m}. \]

\[ (5.17) \quad \widehat{Q}_{0}^{m} = \frac{1}{2} \exp \left[ -i \frac{\pi}{\Delta t} \right] = Q_{2}^{m} \]

On the other hand, \( Q_{\frac{\Delta t}{2}}^{m} \) is determined for \( m > 2 \) by the values at \( m = 1 \) and the

\[ (5.18) \quad Q_{\frac{\Delta t}{2}}^{0} = 1 \quad Q_{\frac{\Delta t}{2}}^{1} = 0 \quad Q_{\frac{\Delta t}{2}}^{2} = 0 \]

Observe now that the \( Q_{\frac{\Delta t}{2}}^{m} \) are determined once we know the \( \widehat{Q}_{\frac{\Delta t}{2}}^{m} \), \( \beta_{m} \), \( \alpha_{m} \).
We denote now $\gamma^m$, $\beta^m$ from the equations (5.12), (5.14) using our denotations (5.13). In particular, we solve for $\gamma^m$, $\beta^m$ the system,

\begin{equation}
\gamma^{m+1} = \sum_{\Delta=0}^{m} Q_{\Delta} \exp \left[ -\frac{\gamma^m}{\Delta} \Delta T \right],
\end{equation}

\begin{equation}
\beta^{m+1} = \frac{1}{2\sqrt{\Delta T}} \log \left( \frac{\log \gamma^{m+1}}{\log \gamma^{m+1}} \right).
\end{equation}

Thus we will solve the entire system. Let us consider the continuous limit $\Delta T \to 0$ of R.D.T. to the right. We need to use the chain rule involving as

\begin{equation}
\alpha_{\Delta} \to \frac{\gamma^{m+1}}{\gamma^m}, \quad \text{where...}
\end{equation}

\begin{equation}
\gamma^m = \gamma_0 \exp \left[ 2 J \beta^m \sqrt{\Delta T} \right], \quad \text{where...}
\end{equation}

\begin{equation}
\log \frac{\gamma^{m+1}}{\gamma^m} = \log \frac{\gamma_0^{m+1}}{\gamma_0^m} + 2 J (\beta^m - \beta^{m+1}) \sqrt{\Delta T},
\end{equation}

\begin{equation}
\log \frac{\gamma^{m+1}}{\gamma^m} = \log \frac{\gamma_0^{m+1}}{\gamma_0^m} + 2 J (\beta^m - \beta^{m+1}) \sqrt{\Delta T}.
\end{equation}

We now write

\begin{equation}
\log \gamma^{m+1} - \log \gamma^m + 2 J (\beta^m - \beta^{m+1}) \sqrt{\Delta T}
= \left( \frac{\beta^{m+1}}{\beta^m} \right) \log \gamma^m + \log \gamma_0^{m+1} - \frac{\beta^{m+1}}{\beta^m} \log \gamma_0^m,
\end{equation}

as putting $m \Delta T = t$, $\gamma_t = \gamma^m$,

\begin{equation}
\log \gamma_t - \log \gamma_t = \frac{\sigma_t (t + \Delta t) - \sigma_t (t)}{\sigma_t (t)} \log \gamma_t,
\end{equation}

\begin{equation}
\sigma_t (t + \Delta t) - \sigma_t (t) = \sigma_t (t + \Delta t) - \sigma_t (t).
\end{equation}

\begin{equation}
\sigma (t) + \text{function of } t.
\end{equation}
where \( \sigma(t) = \beta^m \) is the volatility. Hence the continuous drift is given by the equation,

\[
(5.23) \quad d \left[ \log \gamma(t) \right] = \left[ \theta(t) + \frac{\gamma(t)}{\sigma(t)} \log \gamma(t) \right] dt + \sigma(t) dW(t),
\]

### 6. Estimating Volatility from Market Data

The various models we have discussed require input on the volatility of the yield curve. How do we obtain reliable estimates of these? One approach is to do a Principal Components Analysis (PCA) of the yield curve. Thus suppose we have the following graph for the 0 yield curve:

![Graph showing yield fluctuation and percentage change over maturity]

In this graph short-term rates are quite low and long-term maturities are somewhat large. Consider what is likely to happen if there is an increase in the short-term yield. One would expect an increase in the yield of all maturities i.e. a parallel movement in the curve. Thus we would regard the principal component in the fluctuation of the yield curve to be a parallel movement roughly. There will be of course some subtle effects where there will be oscillation i.e.
Some activities will move in one direction while others will move in a different direction. The basic idea of CCA is that most of the volatility is in the slowly varying components. Thus by an analogy of a few components we can get a good estimate on the overall volatility.

Consider, for example how we could proceed to input the volatilities for CDT. Thus we need to input the volatility of \( \Phi(t, T) \), where

\[
(6.1) \quad \Phi(t, T) = \exp \left[ - \frac{(T-t)^2}{2} \right].
\]

If we assume \( \Phi(t, T) \) is a stationary r.v., i.e., depending only on \( T-t \), we can estimate \( \Phi(t, T) \) from historical data. Thus if we have time \( t_i \), \( i = 1, 2, \ldots, n \) and

\[
(6.2) \quad \text{percentage change in } \Phi(t_i, T) \text{ between time } \tau_{i-1} \text{ and } \tau_i \text{ is }
\]

\[
\Delta \Phi_i = \frac{\Phi(t_i, T - t_{i-1}) - \Phi(t_{i-1}, T - t_{i-1})}{\Phi(t_{i-1}, T - t_{i-1})}
\]

Then if we have

\[
(6.3) \quad \exp \left[ \log \frac{\Phi(t_i + \Delta t, T)}{\Phi(t_i, T)} - \log \frac{\Phi(T, T)}{\Phi(T, T)} \right] = e^{\Delta \Phi_i}
\]

where we estimate \( \Delta \Phi \) as

\[
(6.4) \quad \Delta \Phi(T, T) = \frac{1}{N-1} \sum_{i=1}^{N} \Delta \Phi_i, \text{ i.e., unbiased estimate}
\]

Note it might be better to replace (6.4) by a weighted average - weighting more recent events higher.
We can do the estimates \((6.4)\) for various values of \(T-t\), say \(K\) values, \(k=1, \ldots, K\), let

\(\Lambda_k\) denote the values \((6.2)\) corresponding to the \(k\)th value of \(T-t\). Then we can compute the covariance matrix \(\Sigma\), which is a \(K \times K\) matrix,

\[
\Sigma = \sum_{k=1}^{K} \frac{1}{N-1} \sum_{i=1}^{N} (u_i - \mu_k) (u_i - \mu_k)^T
\]

Now \(\Sigma\) is a symmetric positive definite so we can compute its eigenvectors \(\Phi_1, \ldots, \Phi_K\), where the \(\Phi_k\), \(1 \leq k \leq K\), are orthonormal,

\[
\Phi_k^T \Phi_k = \delta_{k,k'} = 1, \quad k = k', \quad 0, \quad \text{otherwise}
\]

The eigenvector \(\Phi_k\) has eigenvalue \(\lambda_k\), \(1 \leq k \leq K\),

\[
\Sigma \Phi_k = \lambda_k \Phi_k, \quad 1 \leq k \leq K
\]

If we form the diagonal matrix \(\Lambda\) and a diagonal matrix \(\Phi\) given by

\[
\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_K \end{bmatrix} \quad \Phi = \begin{bmatrix} \Phi_1 \end{bmatrix}^T
\]

Then we have

\[
\Sigma = \Phi \Lambda \Phi^T
\]

We define our new variables \(x_1, \ldots, x_K\) by

\[
x_k = u_k^T \Phi_k, \quad \text{where} \quad u_k = [u_{1k}, u_{2k}, \ldots, u_{Nk}]^T
\]

where each \(u_{ik}\) is given by the \(i\)th member of the \(k\)th value of \(u_i\),

\[
u_k = \begin{bmatrix} \gamma_i(T-t_i) \end{bmatrix} - \begin{bmatrix} \eta_i(T) \end{bmatrix}
\]

for the \(k\)th value of \(T-t\).
We have
\[ E \left[ \sum_{k} x_k x_k' \right] = \mathbb{E} \left( (u^T k) (u^T k') \right) = \mathbb{E} \left( u^T u' k k' \right) = \mathbb{E} \left( \delta_k' \right) \]
\[ = \mathbb{E} \left( \sigma_k' \right) \quad \text{since} \quad \mathbb{E} \left( \delta_k \right) = 0 \]


Thus, the variables \( x_k \) are independent, only guaranteed if they are jointly Gaussian, with \( \mathbb{E} \left( \delta_k k' \right) = \mathbb{E} \left( \delta_k' \right) \).

We arrange the random variables \( \delta_k \) in this form:

\[ \delta_1, \delta_2, \ldots, \delta_K \]

Then \( \delta_k \) is the first principal component etc. One expects that all the \( \delta_k \) have the same sign, representing a parallel shift in the yield curve with the heights of the more oscillatory. The corresponding \( \delta_k \) decrease sharply so that generally a small number of components are responsible for the volatility of the yield curve.

If we use the representation from (6.71) that

\[ u = \sum_{k=1}^{K} \left( u^T k \right) \delta_k \]

from (6.71), we conclude that

\[ u = \sum_{k=1}^{K} \chi_k \delta_k \]

Now recall that for EMT, we need an input \( s^2 \) for all values of \( T-T' \). To implement this, from (6.15) we can cast \( u \). The sum in (6.15) at a few components is

\[ \chi_k = 1, 2, 3, 4 \]

Now using interpolation we can obtain an expression

\[ u(x) = \sum_{k=1}^{K} \chi_k \delta_k(x), \quad 0 < t < T-T' \]

and hence obtain the corresponding \( s^2(x) \) since

\[ E \left[ x_k^2 \right] = \chi_k \quad \text{by} \quad \mathbb{E} \left[ \chi_k \delta_k^2 \right] = \mathbb{E} \left[ \delta_k^2 \right] \]