Unconditional Variance, Mean Reversion and Short Rate Volatility in the Calibration of the Black-Derman and Toy Model and of Two-Dimensional Log-Normal Short Rate Models

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Abstract

Calibration of the BDT model to cap prices is notoriously simple, since an almost exact 'guess' of the correct time-dependent volatility can be obtained from the market implied volatility of caplets. This is a priori surprising, since the unconditional variance of the BDTshort rate process should be expected to depend on the (deterministic) history of the short rate volatility, and on the mean reversion of logarithm of the short rate. This apparent paradox is resolved in the first part of the paper, where general expressions (usable, for instance, to calibrate the Black-Karasinsky model) for the unconditional variance are obtained for a variety of one-factor models. The class of one-factor models for which the same type of relationship holds true is also introduced.

The results are then extended to more-than-one-factor models, with the introduction of a class of stable and arbitrage-free Generalized Brennan and Schwartz models. It is shown that, if no arbitrage is to be enforced, this very large class of log-normal-short rate models cannot be calibrated to cap prices using higher-dimensional extensions of the BDT procedure.

Introduction

Of the several one-factor models used for pricing interest rate options, the Black Derman and Toy (1990) (BDT in the following) is one of the best known, and of the most widely used. Amongst its most appealing features are the capability to price exactly an arbitrary set of received market discount bonds, the log-normal distribution of the short rate, and the ease of calibration to cap prices. The first feature (exact pricing of the yield curve) is shared by a variety of (non-equilibrium) models, such as the Ho and Lee (1986) or the Hull and White (1990). The second (log-normal distribution of rates) is also shared by the Black and Karasinski model. Only the BDT approach, however, allows log-normal rates *and* calibration to caplet prices (in absence of smile effects) that can be accomplished almost by inspection. This latter feature is, at the same time, the blessing and the bane of the BDT model, and directly stems from the inflexible specification of the reversion speed, which is completely determined by the future behaviour of the short rate volatility. This latter feature is well known, and has already been amply criticized on theoretical grounds in the literature. The decision as to whether, despite this rather artificial characteristic, the BDT model can be profitably used in practical applications for option pricing depends crucially on the type of option, and requires a considerable degree of experience and a rather subtle understanding of the implications of the model. A discussion of the related issues can be found, for instance, in Rebonato (1996).

In the light of the above, the present note

1. highlights the intimate connection, hitherto not fully appreciated, to the best knowledge of the authors, between the ease of calibration to cap prices and the particular link between the reversion speed and the logarithmic derivative of the short rate volatility;

2. extends the results to different classes of one-factor models;

3. shows to what extent these findings affect the calibration of an important class of no-arbitrage two-factor models.

Statement of the problem

For a generic Wiener process of the form

$$d \ln r(t) = \theta(t)dt + \sigma_{\text{inst}}(t)dz(t) \tag{1}$$

with $\theta(t)$ a *determinstic* drift, dz the increment of a Brownian process and $\sigma_{\text{inst}}(t)$ an instantaneous volatility (standard deviation per unit time) it is well known that the unconditional variance out to time T is given by

$$\text{Var}(\ln r(T)) = \int_0^T \sigma_{\text{inst}}(u)^2 du. \tag{2}$$

Furthermore, for a mean-reverting process of the form

$$d \ln r(t) = [\theta(t) + k(\psi(t) - \ln r(t))]dt + \sigma_{\text{inst}}(t)dz(t) \tag{3}$$

(with reversion speed k, reversion level $\psi(t)$, and $\theta(t)$ a *deterministic* drift component) the unconditional variance will, in general, depend on the reversion speed.

The continuous-time equivalent of the BDT model can be written as

$$d \ln r(t) = [\theta(t) - f'(t)(\psi(t) - \ln r(t))]dt + \sigma_{\text{inst}}(t)dz(t) \tag{4}$$

with

$$f'(t) = \partial \ln \sigma(t)/\partial t \tag{5}$$

and both $\theta(t)$ and $\sigma(t)$ deterministic functions of time. Equation (4) and its implications as to the model behaviour are well known in the literature (see, e.g. Rebonato (1996), where the link between the function $\theta(t)$ and the median of the short rate distribution is highlighted). For the present purposes it will suffice to say that it is only in the presence of a time decaying short rate volatility ($\partial \ln \sigma(t)/\partial t < 0$) that the resulting reversion speed ($-f'$) is positive and the model displays mean reversion.

Whilst this is well known, it does seem to create a paradox, since it is 'empirically' known, and shown in the following, that, in discrete time, the unconditional variance of the short rate in the BDT model neither depends on the instantaneous volatility from time 0 to time $T - \Delta t$ (as one would have been led to expect from Equation (2)) nor does it depend on the reversion speed $-f'$ (as one might have surmised from Equation (4)). More precisely, one can easily show (see below) that

$$\text{Var}\ (\ln r(N\Delta t) = (N\Delta t)\sigma^2(N\Delta t), \tag{6}$$

where $\sigma^2(N\Delta t)$ is the (square of) the instantaneous short rate volatility at time $T = N\Delta t$. It is important to stress the crucial importance of Equation (6) for calibration purposes; it is only because the instantaneous volatility of the log-normal short rate is simply given by the expression above that calibration to caplet prices is so easy: market prices are in fact routinely quoted on the basis of the log-normal Black (1976) model, and the BDT forward induction construction implicitly carries out the Girsanov's drift transformation from the measure associated with the discount bond numeraire implied with the Black model to the equivalent measure asociated with the (discretely-compounded) money-market account implied by the BDT approach (see Rebonato (1996)). Since the market Black implied volatilities give direct information about the unconditional variance of the relevant forward rates (spot rates at expiry), from the quoted implied Black volatilities of caplets of different expiries the user can almost exactly[1] obtain their exact BDT pricing by assigning a time-dependent short rate volatility matching the implied Black volatilities: see Equation (6) above.

It is rather well known amongst practitioners that this is the case. What is not generally appreciated is how this can be, since Equations (2) and (4) would in general suggest that both the instantaneous short rate volatility from time 0 to time T and the reversion speed $-f'$ should affect the unconditional variance from time 0 to time T, that, in turn, determines, the Black price of the T-expiry caplet. The first part of this paper (Sections 3 and 4) will first show that the 'empirically known' result mentioned above regarding the unconditional variance is indeed correct, and then (Section 4), moving to the continuous limit, shed light on the origin and resolution of the resulting apparent paradox.

The unconditional variance of the short rate in BDT – the discrete case

A calibrated BDT lattice is fully described by a vector $\mathbf{r} = \{r_{i0}\}, (i = 0, k)$ whose elements are the lowest values of the short rate at time step i, and by a vector $\sigma = \{\sigma_i\}, (i = 0, k)$, whose elements are the volatilities of the short rate from timestep i to timestep $i + 1$. Every rate r_{ij}, in fact, can be obtained as $r_{ij} = r_{i0} \exp [2\sigma_i j\sqrt{\Delta t}]$. ($\Delta t$, as usual is the time step in years). Let us now define k random variables $y_1, y_2, \ldots, y_k$ by

1 Since the short rate enters the expression of the drift, the unconditional variance is not exactly equal to $\sigma^2_{Black}\,\tau$ (where τ is the time to expiry and σ_{Black} the market implied volatility). The approximation is however excellent (see Rebonato (1998)) for a discussion of this point.

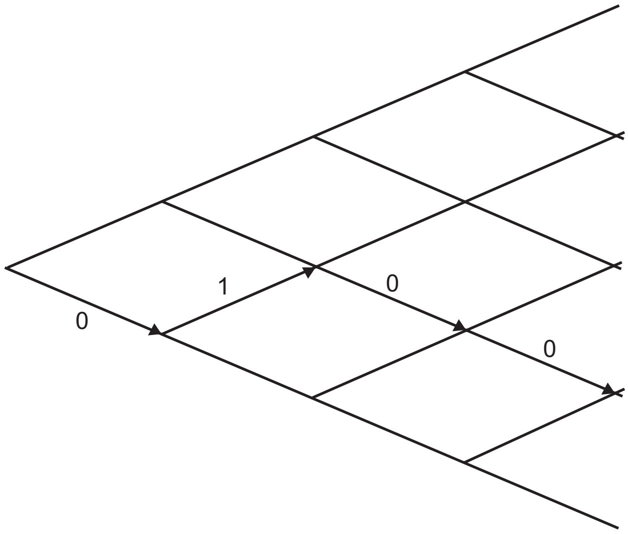

Figure 1 Values assumed by the random variables y_1, y_2, y_3 and y_4 for the down-up-down-down path highlighted

$$y_k = \begin{cases} 1 \text{ if an up move occurs at time } (k - 1)\, \Delta\, t \\ 0 \text{ if a down move occurs at time } (k - 1)\, \Delta\, t. \end{cases}$$

For instance, for the path highlighted in Figure 1, $y_1 = 0$, $y_2 = 1$, $y_3 = 0$, $y_4 = 0$. It will further be assumed i) that the variables y_j are independent, and ii) that the probability $P[y_k = 1] = P[y_k = 0] = \frac{1}{2}$. The variable $X_k = \sum_{j=1,k} y_j$ therefore gives the 'level' of the short rate at time $k\Delta t$, and the value of the short rate at time $k\Delta t$ in the state labelled by X_k is given by

$$r_{k,X(k)} = r_{k0} \exp\, [2\sigma_i X_k \sqrt{\Delta t}]. \tag{7}$$

Our task is now to evaluate the expectation and variance of the logarithm of this quantity, denoted by $E[\ln\, r_{k,X(k)}]$ and $\mathrm{Var}[\ln\, r_{k,X(k)}]$, respectively. To this effect one must first find the distribution of X_k. By evaluating its characteristic function one can easily show that the probability of X_k assuming value j is given by

$$P[X_k = j] = C_k^j / 2^k, \tag{8}$$

with

$$C_k^j = k!/((k-j)!\,j\,!). \tag{9}$$

Therefore

$$P[r_{k,X(k)} = r_{k0} \exp\, [2\sigma_k j \sqrt{\Delta t}]] = P[X_k = j] = C_k^j / 2^k. \tag{10}$$

We are now in a position to evaluate $E[\ln\, r_{k,X(k)}]$ ($r_{k,X(k)}$ will be abbreviated as r_k in the following to lighten notation):

$$E[\ln\, r_k] = \sum_{j=0,k} \frac{1}{2}^k C_k^j (\ln\, r_k + 2\sigma_k \sqrt{\Delta t} j) = \ln\, r_k \frac{1}{2}^k 2^k + \frac{1}{2}^k 2\sigma_k \sqrt{\Delta t} \sum_{j=1,k} j C_k^j. \tag{11}$$

Given, however, the definition of C_k^j,

$$j C_k^j = k C_{k-1}^{j-1},$$

and, after substituting in (11), one obtains

$$E[\ln r_k] = \ln r_k + k\sigma_k\sqrt{\Delta t}. \tag{12}$$

Similarly for the variance

$$\begin{aligned} E[(\ln r_k)^2] &= \sum_{j=0,k} \frac{1}{2}^k C_k^j (\ln r_k + 2\sigma_k\sqrt{\Delta t} j)^2 \\ &= (\ln r_k)^2 + 2k\sigma_k(\ln r_k)\sqrt{\Delta t} + 4\sigma_k^2\Delta t \sum_{j=1,k} j^2 C_k^j. \end{aligned} \tag{13}$$

But the last term is simply equal to

$$j^2 C_k^j = k(k-1)C_{k-2}^{j-2} + kC_{k-1}^{j-1}, \tag{14}$$

and therefore the last summation adds up to

$$\sum_{j=0,k} j^2 C_k^j = k(k-1)2^{k-2} + k2^{k-1}. \tag{15}$$

It follows that the unconditional variance is given by

$$\begin{aligned} \mathrm{Var}[\ln r_k] &= E[(\ln r_k)^2] - (E[\ln r_k])^2 \\ &= (\ln r_k)^2 + 2k\sigma_k(\ln r_k)\sqrt{\Delta t} + \sigma_k^2\Delta t k(k+1) - (\ln r_k + k\sigma_k\sqrt{\Delta t})^2 \\ &= \sigma_k^2 k\Delta t. \end{aligned} \tag{16}$$

The expression above therefore shows that the unconditional variance of the logarithm of the short rate in the BDT model only depends on the final instantaneous volatility of the short rate, despite the continuous-time limit of the model displaying both mean-

Table 1 Caplet prices per unit notionals and ATM strikes for the GBP sterling curve of expiries reported on the left-hand column, as evaluated using the Black model (column Black), and the BDT model calibrated as described in the text (column BDT)

	Sterling Curve Nov 1995	
Expiry	Black	BDT
01-Nov-95		
31-Jan-96	0.000443	0.000431
01-May-96	0.000773	0.000757
31-Jul-96	0.001148	0.001133
31-Oct-96	0.001559	0.001548
30-Jan-97	0.002002	0.001994
01-May-97	0.002422	0.002416
01-Aug-97	0.002746	0.002742
31-Oct-97	0.003024	0.003020
30-Jan-98	0.003265	0.003263
02-May-98	0.003471	0.003471
01-Aug-98	0.003449	0.003452
31-Oct-98	0.003406	0.003411

reversion and a non-constant short rate volatility, and therefore validates the 'empirical' procedure, well-known amongst practitioners, to calibrate to caplet market prices. The table above shows the results of calibrating the BDT tree using the Black implied volatilities.

The unconditional variance of the short rate in BDT – the continuous-time equivalent

The above derivation has shown that, in discrete time, the unconditional variance of the short rate is indeed given by expression (6). What is not apparent, however, is *why* the reversion speed and/or the instantaneous short rate volatility from time 0 to time $T - \Delta t$ does not appear in the equation. To see why this is the case it is more profitable to work in the continuous-time equivalent of the BDT model, Equation (4). This can be re-written as a diffusion of the general form:

$$d\ \ln r(t) = [a(t)(b(t) - \ln\ r(t))]dt + \sigma(t)dz(t), \tag{17}$$

where $a(t)$, $b(t)$ and $\sigma(t)$ are *deterministic* functions of time. The SDE (*) can easily be solved (see Appendix I) giving

$$\mathrm{Var}[\ln\ r(T)] = \exp[-2\int_0^t a(s)ds]\int_0^t \sigma(t)^2 \exp[2\int_0^t a(s)ds]dt. \tag{18}$$

As it can be appreciated from this result, the unconditional variance of the logarithm of the short rate out to time T does in general indeed depend on the reversion speed, and on the values of the instantaneous volatility $\sigma(t)$ from time 0 to time T. This result is completely general, but it is instructive to specialize it to the case of the BDT model. In this case $a(t) = -f'$, and therefore, recalling that $f(t) = \ln\ \sigma(t)$, the unconditional variance of the short rate out to time T becomes

$$\begin{aligned}\mathrm{Var}[\ln\ r(T)] &= \exp[2f(T) - f(0)]\int_0^t \sigma(t)^2\ \exp[-2(f(t) - f(0))]dt \\ &= \exp[2f(T)]\int_0^T \sigma(t)^2 \exp[-2f(t)]dt. \end{aligned} \tag{19}$$

Finally, recalling that $f(t) = \ln\ \sigma(t)$, one can immediately verify that, in the BDT case, the unconditional variance is indeed simply given by

$$\mathrm{Var}(\ln\ \sigma(T)) = \sigma(T)^2 \int_0^T du = \sigma(T)^2 T, \tag{20}$$

i.e. *for any mean-reverting process for which the reversion speed is exactly equal to the negative of the logarithmic derivative of the instantaneous volatility with respect to time (i.e., $a(t) = -\partial \ln\ \sigma(t)/\partial t$) neither the reversion speed nor the past instantaneous volatilty enter the expression for the unconditional variance, which only depends on the instantaneous short rate volatility at the final time*. This result fully resolves the 'BDT paradox', and indicates the necessary and sufficient conditions under which *any* locally log-normal model can be simply calibrated to cap prices by using its terminal instantaneous volatility.

Extensions to two-factor approaches

The importance of using more-than-one- factor models is widely recognized in the financial community, especially for pricing options that depend in an important way on the imperfect correlation amongst rates. From the discussion above, on the other hand, it is easy to see that retaining a log-normal distribution of rates is very important if one is to achieve easy calibration of any model to market cap prices across several strikes. The impact of non-log-normal distributions on model cap prices has been clearly shown, for instance, for the normal-short-rate Hull and White Generalized Vasicek model (see Rebonato (1996)).

In order to retain the log-normality of the short rate, and to extend the analysis to more than one factor, one is naturally led to consider the general framework introduced by Brennan and Schwartz (1982, 1983), who showed that, if one of the two state variables is chosen to be the consol price or yield, the accompanying market price of risk can be made to disappear from the resulting parabolic partial differential equation which describes the 'real-world' evolution of the price of a generic security. It would be very useful for practical applications if one could specify a two-factor log-normal short rate model along the same conceptual lines, so that the model calibration could be accomplished as readily as in the BDT case. In the following it is shown that this is not possible, if arbitrage is to be prevented, but approximate expressions for the unconditional variance of the short rate for this class of models are currently under study.

The specific model proposed by Brennan and Schwartz has been shown (Hogan(1993)) to suffer from instability of the long yield. This feature, however, stems from the arbitrary 'real-world' specification of the dynamics of the state variables chosen by Brennan and Schwartz. Their central insight regarding the market price of long yield risk remains valid, and is made use of in the following in the context of the risk-neutral (as opposed to 'real-world') measure. More precisely, the derivations presented in the following apply to the measure Q (often referred to as 'risk-neutral') under which asset prices divided by the rolled-up money-market account are martingales. It will be recalled at this point that, in any n-dimensional tree-based n-nomial methodology where, with obvious extension of the BDT algorithm, pay-offs are first averaged and then discounted to the 'originating' node by the short rate corresponding to that node, one is effectively discounting final pay-offs by the (discretely) rolled-up money market account (see Rebonato (1996)). Therefore the risk-neutral measure defined above is indeed the appropriate measure to consider for lattice-based methodologies.

If one wants to retain, at the same time, the Brennan and Schwartz general approach and local log-normality for the short rate, one is naturally led to choose (see Rebonato (1997)) as state variables the consol yield, L, and the ratio, K, of the short rate, r, to the consol yield:

$$r = KL \tag{21}$$

If, in addition both K and L are assumed to be log-normally distributed, not only would positivity of the short rate be automatically ensured, but its distribution would also turn out to be log-normal. Therefore

$$dK/K = \mu_K(t, K, L)dt + \sigma_K dz_K \qquad (22')$$

$$dL/L = \mu_L(t, K, L)dt + \sigma_L dz_L \qquad (22'')$$

$$E[dz_K, dz_L] = \rho \qquad (22''')$$

The expressions above contain the unknown drifts for the consol yield and for the relative spread K. Following the spirit of the Brennan and Shwartz approach, however, the no-arbitrage risk-neutral dynamics for the consol bond, $C = 1/L$, can then be obtained by imposing that, since the consol bond is an asset, it must grow in Q at $(r - L)$, i.e. at the short rate minus its dividend yield, if discounting is effected using the money-market account. Applying Ito's lemma one therefore obtains:

$$dL/L = [L(1 - K) + \sigma_L^2]dt + \sigma_L dz_L \qquad (23)$$

or, equivalently,

$$dL = [L(L - R) + \sigma_L^2 L]dt + L\sigma_L dz_L. \qquad (24)$$

Notice that, from this no-arbitrage condition, **the long (consol) yield flees the short rate with fleeing speed L**; this would seem to imply an intrinsically unstable behaviour for the joint dynamics of the state variables. This is, however, not necessarily the case, as can be see in the following. Ito's lemma, in conjunction with the above equation and the definitions i) and ii), in fact gives for the SDE for R

$$dr/r = [a'(t) + (L - r) + \mu_K]dt + \sigma_r dz_r, \qquad (25)$$

with $\sigma_r^2 = \sigma_K^2 + \sigma_L^2; +2\sigma_{K\sigma L}\rho$, i.e. the short rate is log-normally distributed and it reverts to the long yield with reversion speed 1. It is important to notice that this condition directly follows from nothing else but the distributional assumptions and the no-arbitrage requirement. Notice also that Equation (25) implies a reversion of the short rate to the long rate with reversion speed equal to 1. This is particularly significant, since Hogan (1993) shows that the reversion speed above must be ≥ 1 for the coupled system of equations describing the evolution of r and L to be stable.

The equations obtained up to this point have been fully determined by the no-arbitrage conditions and the distributional assumptions. Nothing, however, has been said about the drift of K. At this point one could want to impose that the drift of K should be equal to an arbitrary function f time only and a linear function of K and L only:

$$\mu_K(t, K, L) = b_0(t) + b_1 K + b_2 L, \qquad (26)$$

with $b_1 < 0$ so as to ensure the reversion of K to a constant level. Whatever the merits of this choice, Equation (26) can always be seen as obtained by retaining the first-order term of the expansion of the (unknown) true drift. Different choices of functional dependence of the drift of K on the state variables give rise to different models, which can therefore aptly described as belonging to the Generalized Brennan and Schwartz family.

Whatever the choice for μ_K might be, Equation (25) is crucial to the following argument: the system of SDEs (22)–(25) has been shown to stem from the very conditions of no-arbitrage, and, therefore, for the chosen numeraire and distributional assumptions, these results are inescapable for any viable model. This fact, however, poses a grave problem insofar as ease of calibration is concerned. On the one hand, in fact, it has been observed above that necessary condition for the variance of a mean-reverting process to depend only on the (terminal value of) the instantaneous volatility and not on the reversion speed is that the latter should be identical to the negative of the logarithmic time derivative of the instantaneous volatility (See Eq. (20)). We then argued that it was because of this very feature that calibration to cap prices in the BDT model is so straightforward. On the other hand, even if the drift of K were assumed to be purely a function of time, there is no way to retain stability of the dynamics of r and L and avoiding arbitrage without the term $r(L - r)$, which implies a reversion speed for the short rate (rather than its logarithm) that is *not* related to the logarithmic derivative of the instantaneous volatility. Therefore the unconditional variance of the short rate out to time T cannot be simply related to the instantaneous volatility of the short rate at time T, as it is the case for the BDT model. In other words, *it is not possible to extend the BDT model to more than one state variable as proposed above in such a way that one of its most important features (the ease of calibration) is preserved.*

Conclusions

The inflexible nature of the reversion speed in the BDT model is, at the same time, its blessing and its bane. The positive features, connected with the ease of calibration, are too well known to be dwelt upon, and are, to a large extent, responsible for the wide acceptance of the model amongst practitioners. The negative aspects, however, should not be underestimated. These are more pernicious than the usual limitations of one-factor or low-dimensionality models (see, e.g. Rebonato and Cooper (1996), for a discussion of the latter): one of the distinctive features of the BDT model is in fact the inextricable link it implies between its reversion speed and the logarithmic derivative of the short rate volatility. The time- decaying volatility needed in the BDT model in order to ‘contain’ an excessive dispersion of rates can, in fact, well succeed in obtaining an *unconditional* distribution of rates consistent with the one implied by the cap market; but, since an explicit deterministic mean reversion is absent from the model for any non-decaying behaviour of the short rate volatility, this is obtained at the expenses of a lower and lower forward-forward volatility. This undesirable feature can have a limited impact for relatively short maturity options, but must always be born in mind by users who extend their analyses well beyond the common ‘volatility hump’ observed in most cap markets. Unfortunately, this paper has shown that it is impossible to remove this undesirable feature and to retain at the same time the original ease of calibration.

In an attempt to obviate those shortcomings of the BDT model shared by all one-factor models, an obvious extension along the lines of the Brennan-and- Schwartz approach was introduced in the second part of the paper, and a class of arbitrage-free log-normal short rate models which do not display the Hogan instability was obtained in the second part of the paper. It was shown, however, that, despite the log-normality of the short rate, if arbitrage is to be avoided these models cannot have the same type of unconditional variance displayed by the BDT model. On the one hand this allows for ‘true’ mean reversion to occur in the SDE for the short rate even in the presence of constant volatility; on the other hand, however, the resulting calibration to cap prices is prima facie considerably more arduous. Research in the validity of approximate expressions that could make the cap calibration almost as easy as for the BDT approach is currently under way.

Appendix: Evaluation of the variance of the logarithm of the instantaneous short rate

From (*) one can write

$$d\ \ln r(t) + a(t)\ \ln\ r(t)dt = a(t)b(t)dt + \sigma(t)dz(t).$$

This implies that

$$\exp[-\int_0^t a(s)ds]d[\ \ln\ r(t)\ \exp[\int_0^t a(s)ds]] = a(t)b(t)dt + \sigma(t)dz(t).$$

But the quantity $\ln\ r(T)\ \exp[\int_0^t a(s)ds]$ can be written as

$$\ln\ r(T)\ \exp[\int_0^T a(s)\,ds] = \ln\ r(0) + \int_0^T a(t)b(t)\ \exp[\int_0^t a(s)\,ds]dt$$
$$+ \int_0^T \sigma(t)\ \exp[\int_0^t a(s)\,ds]dz(t),$$

and, therefore

$$\ln\ r(T) = \exp[-\int_0^T a(s)\,ds]\ \ln\ r(0) + \exp[-\int_0^T a(s)\,ds] \int_0^T a(t)b(t)\ \exp[\int_0^t a(s)\,ds]dt$$
$$+ \exp[-\int_0^T a(s)\,ds] \int_0^T \sigma(t)\ \exp[\int_0^t a(s)\,ds]dz(t).$$

Remembering that, for any deterministic function $f(t)$, $\mathrm{Var}[\int_0^t f(u)\,dz(u)] = \int_0^t f(s)^2\,dt$, it then follows that

$$\mathrm{Var}[\ln\ r(T)] = E[(\ln\ r(T)^2] - (E[\ln\ r(T)])^2$$
$$= \exp[-2\int_0^T a(s)\,ds] \int_0^T \sigma(t)^2\ \exp[2\int_0^t a(s)\,ds]dt.$$

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