Math 623 (IOE 623), Winter 2007: Final exam

Name:

Student ID:

This is a closed book exam. You may bring up to ten one sided A4 pages of notes to the exam. You may also use a calculator but not its memory function.

Please write all your solutions in this exam booklet (front and back of the page if necessary). Keep your explanations concise (as time is limited) but clear. State explicitly any additional assumptions you make.

Time is counted in years, prices in USD, and all interest rates are continuously compounded unless otherwise stated.

You are obliged to comply with the Honor Code of the College of Engineering. After you have completed the examination, please sign the Honor Pledge below. A test where the signed honor pledge does not appear may not be graded.

I have neither given nor received aid, nor have I used unauthorized resources, on this examination.

Signed:

(1) Consider the PDE $u_t = (1+x)u_{xx} + 2xu_x$, $1 < x < 2,\ t > 0$, with the initial condition $u(x,0) = e^x,\ 1 < x < 2$, and the boundary conditions $u(1,t) = e,\ u(2,t) = e^2$. The initial value problem is to be solved numerically by a finite difference scheme, where the time step is denoted Δt and the space step Δx. The numerical scheme is to be second order accurate in Δx.

(a) Write down the forward Euler finite difference scheme for the initial value problem. How should Δt and Δx be related so that the numerical solution at time 1 is second order accurate in Δx ?

(b) Write down the Crank-Nicholson finite difference scheme for the initial value problem. How should Δt and Δx be related so that the numerical solution at time 1 is second order accurate in Δx ?

(c) If $\alpha = \Delta t/(\Delta x)^2$ in the forward Euler scheme, find the maximum value α can be and the scheme still remain stable. Explain your answer.

(2) We wish to find the value of an Asian option on a zero dividend stock which expires 8 months from today. The payoff on the option is the excess of the stock price at expiration over the continuous average of the stock price during the 8 month life span of the option. The current price of the stock is 45 and its volatility is 0.32 per annum. The risk free interest rate is 0.05 per annum.

(a) Write down the two variable partial differential equation for a function w of (t, ξ) on intervals $0 < t < 2/3$, $0 < \xi < \xi_{\max}$, together with boundary and terminal conditions, which one needs to solve to compute the value of the Asian option.

(b) Estimate an appropriate numerical value for $\xi_{\max}$. Justify your answer.

(c) Suppose the value of the option is 5. Calculate the corresponding value of the function $w(\xi, t)$, identifying the relevant values of ξ and t.

(3) We wish to estimate the values of put options on a zero dividend stock by using the Monte Carlo method. The current price of the stock is 32 and its volatility is 0.28 per annum. The risk free interest rate is 0.04 per annum. The expiration date of the option is 6 months from today. You are given that 3 values of independently drawn standard normal variables are: $.1453, -.3214, .4725$.

(a) Suppose the option is an at the money put option. Estimate the value of the option using the three sample values of the normal variable given above.

(b) Suppose the put option has strike price 25, so it is significantly out of the money. Use importance sampling and the three sample values of the normal variable given above to estimate the value of the option.

(4) In order to estimate the price of a European basket option on 2 stocks the risk neutral evolution of the 2 stocks is modelled as follows:

$$\begin{cases} \frac{dS_{1,t}}{S_{1,t}} = 0.04\ dt + 0.25\ dW_{1,t}, \\ \frac{dS_{1,t}}{S_{2,t}} = 0.04\ dt + 0.38\ dW_{2,t}, \end{cases}$$

where the $W_{1,t},\ W_{2,t},\ t > 0$, are correlated Brownian motions with $E[dW_{1,t}\ dW_{2,t}] = 0.45\ dt$.

(a) Obtain a representation for the evolution of the two stocks which has the form,

$$\begin{cases} \frac{dS_{1,t}}{S_{1,t}} = r_1\ dt + a\ dZ_{1,t}, \\ \frac{dS_{1,t}}{S_{2,t}} = r_2\ dt + b\ dZ_{1,t} + c\ dZ_{2,t}, \end{cases}$$

where the $Z_{1,t},\ Z_{2,t},\ t > 0$, are independent Brownian motions. Find the values of $r_1,\ r_2,\ a,\ b,\ c$.

(b) We wish to estimate the value of a call option on the two stocks. The current value of the stock S_1 is 35 and the current value of the stock S_2 is 48. The expiration date of the option is 6 months from today. The payoff on the option is the excess of the average of the two stocks at expiration over the strike value 40. You are given that two independent draws of a standard normal variable are: $.2574, -.3142$. Using the Monte Carlo method with antithetic variables, estimate the price of the option from the normal variable values given.

(5) The price of a stock follows a geometric Brownian motion with volatility .34. The risk free interest rate is .05 per annum. We wish to build a binomial tree to price a European call option which has expiration date 4 months from today. The strike price for the option is 25 and the value of the stock today is 22.

(a) The tree is defined by the four parameters, p_u, p_d, u, d. Assuming $p_u = p_d = 1/2, \Delta t = 1/81$, find the corresponding values of u, d.

(b) Write an algorithm for the binomial tree which will compute the value of the call option.

(6) We wish to construct a Black-Karazinski tree to value interest rate derivatives. In the tree we take $\Delta t = 1/4$ and the parameters in the stochastic equation for the short rate model are $\sigma = 0.21, \ \mathrm{a} = 0.18$.

(a) Suppose the interest rate corresponding to some node (m, j) $m \geq 0, |j| < m$, of the tree has value .04. Find the interest rate corresponding to the node $(m, j+1)$.

(b) A three month treasury bill has value .986 today and a six month treasury bill has value .976. Write down an equation whose solution gives us the interest rate corresponding to the node $(1, 0)$ of the tree.