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STABLE MANIFOLDS AND HOMOCLINIC POINTS NEAR RESONANCES IN THE RESTRICTED THREE-BODY PROBLEM

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Abstract. The restricted three-body problem describes the motion of a massless particle under the influence of two primaries of masses $1-\mu$ and μ that circle each other with period equal to 2π. For small μ, a resonant periodic motion of the massless particle in the rotating frame can be described by relatively prime integers p and q, if its period around the heavier primary is approximately $2\pi p/q$, and by its approximate eccentricity e. We give a method for the formal development of the stable and unstable manifolds associated with these resonant motions. We prove the validity of this formal development and the existence of homoclinic points in the resonant region. In the study of the Kirkwood gaps in the asteroid belt, the separatrices of the averaged equations of the restricted three-body problem are commonly used to derive analytical approximations to the boundaries of the resonances. We use the unaveraged equations to find values of asteroid eccentricity below which these approximations will not hold for the Kirkwood gaps with q/p equal to 2/1, 7/3, 5/2, 3/1, and 4/1. Another application is to the existence of asymmetric librations in the exterior resonances. We give values of asteroid eccentricity below which asymmetric librations will not exist for the 1/7, 1/6, 1/5, 1/4, 1/3, and 1/2 resonances for any μ however small. But if the eccentricity exceeds these thresholds, asymmetric librations will exist for μ small enough in the unaveraged restricted three-body problem.

Key words: asymmetric libration, homoclinic points, Kirkwood gaps, resonance, three-body problem

1. Introduction

The restricted three-body problem describes the motion of a massless particle under the influence of two primaries of masses $1-\mu$ and μ. The Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y - \frac{1-\mu}{(x^2+y^2)^{1/2}} - \mu\left(\frac{1}{((x-1)^2+y^2)^{1/2}} - x\right) \tag{1.1}$$

gives the equations of motion of the massless particle. In (1.1), it is assumed that the primary of mass $1-\mu$ is located at $(0,0)$, that the primary of mass μ is located at $(1,0)$, and that the frame of reference rotates with the second primary in the anticlockwise sense with period 2π.

In terms of the heliocentric Delaunay variables L, l, G, g, the Hamiltonian becomes

$$H=-\frac{(1-\mu)^2}{2L^2}-G-\mu\Omega'(L,G,l,g)=-\frac{1}{2L^2}-G-\mu\Omega(L,G,l,g)-\frac{\mu^2}{2L^2}, \tag{1.2}$$

where

$$\begin{aligned}\Omega' &= \frac{1}{(1+r^2-2r\cos\theta)^{1/2}}-r\cos\theta\\ \Omega &= \Omega'-1/L^2.\end{aligned} \tag{1.3}$$

The variables r,θ can be obtained in terms of L, l, G, g using the equations

$$\begin{aligned}e &= (1-G^2/L^2)^{1/2}\\ a &= L^2/(1-\mu)\\ l &= E-e\sin E\\ \cos\nu &= (\cos E-e)/(1-e\cos E)\\ \sin\nu &= (1-e^2)^{1/2}\sin E/(1-e\cos E)\\ \theta &= g+\nu\\ r &= a(1-e\cos E).\end{aligned} \tag{1.4}$$

The Cartesian coordinates x, y in the rotating frame used in (1.1) are given by $x=r\cos\theta$ and $y=r\sin\theta$. When $\mu=0$, the orbit of the massless particle in the inertial frame is an ellipse with eccentricity e and semimajor axis a; l, ν, and E are the mean, true, and eccentric anomalies; g is the argument of the perihelion in the rotating frame and G denotes angular momentum. The variables r and θ are the polar coordinates of the massless particle in the rotating frame. The choice of heliocentric variables in (1.2) is in accord with conventions used by astronomers.

We use L, l, G, g to investigate motion near resonances as the Hamilton's equations of (1.2) take an especially simple form for μ small. If $L=(p/q)^{1/3}$, $G=(p/q)^{1/3}(1-e^2)^{1/2}$, $0<e<1$, and $\mu=0$, the massless particle moves on an ellipse with period $2\pi p/q$ and eccentricity e in the inertial frame. It is assumed that p and q are relatively prime positive integers. The motion is periodic in the rotating frame as well with period equal to $2\pi p$. If l and g are multiples of π and the unperturbed orbit does not collide with the second primary, these periodic motions in the rotating frame perturb to periodic motions for $\mu>0$ and μ small (Barrar, 1965). Given p, q, e

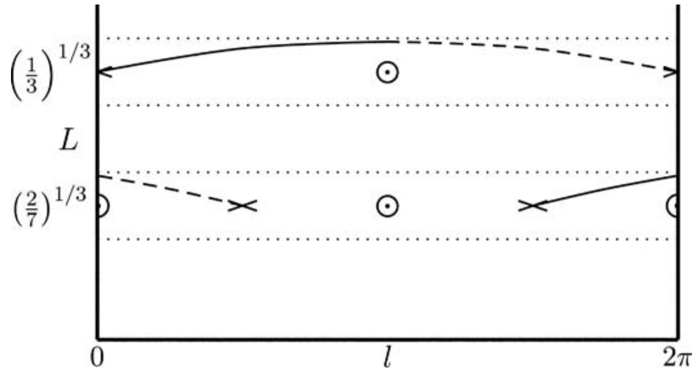

Figure 1. The l-L plane shown above is the Poincaré section $g=0$. Periodic points as well as portions of stable and unstable manifolds are shown schematically for $q/p=7/2$ and $q/p=3/1$.

it might appear that there are four possibilities as l and g can be either 0 or π, but in fact only two of these are distinct. These are the two q/p resonant periodic motions of the restricted three-body problem that correspond to eccentricity e.

The Hamiltonian H is conserved by the flow, and for a given H and μ small it is possible to solve (1.2) for G using the implicit function theorem. Thus the Poincaré section $g=0$ can be identified with the l-L plane. Each resonant periodic motion corresponding to p, q, e intersects this Poincaré section exactly p times. Typically, if one of the resonant periodic motions is of elliptic type the other is of hyperbolic type (Viswanath, 2005). In Figure 1, we have shown the periodic points on the L-l section for $q/p=3/1$ and $q/p=7/2$ with elliptic points marked as circles and hyperbolic points as crosses. The perturbing term Ω and the $O(\mu^2)$ term in (1.2) are unchanged by the transformation $L\leftarrow L$, $l\leftarrow -l$, $G\leftarrow G$, $g\leftarrow -g$, which is symplectic with multiplier -1. This discrete symmetry of the Hamiltonian (1.2) has the following implication for the return map to the l-L section: if (l_0, L_0) maps to (l_1, L_1) then the return map sends $(-l_1, L_1)$ to $(-l_0, L_0)$.

We obtain a scaled version of the return map near q/p resonance in Sections 2 and 3. The stable and unstable manifolds of the hyperbolic points of such a return map nearly coincide and the angle of transversality can be upper bounded by a quantity that is exponentially small in the small parameter μ. A discussion of exponential splitting of separatrices can be found in the work of Gelfreich and Lazutkin (2001), Fontich and Simó (1990), and Holmes et al. (1988). Instability is often associated with resonance, and it is therefore natural to look for transverse homoclinic points near resonances. However, it appears that there has been no construction of transverse homoclinic points near resonances in either the restricted three-body problem or in some other version of the planetary problem. The rescaling of the return map given in Section 3 brings this problem into sharper focus and it is possible to make an analogy to the discussion of the standard map given by Gelfreich and Lazutkin (2001).

In Section 4, we describe a procedure for the formal development of the stable manifold of a resonant periodic solution in powers of $\mu^{1/2}$. This procedure is specially adapted to the restricted three-body problem. In Section 5, we have included a proof of validity of this formal expansion. We also prove the existence of homoclinic points. A part of the verification essential for the proof is carried out in Section 6.

Although investigations of the Kirkwood gaps in the asteroid belt have used physical models that include the secular variation of Jupiter's elements and the effect of Saturn, the averaged circular restricted three-body problem is still used to approximate the boundaries of the resonant regions. The more complicated models are essential to explain the dynamics within the resonant regions. An account of these models and their use can be found in the monograph by Morbidelli (2002). The use of the averaged equations of the circular restricted problem to sketch the boundaries of resonance can be found in Dermott and Murray (1983), Henrard and Lemaître (1983), and Lemaître (1984). It appears to be known that the boundaries obtained from the averaged circular restricted problem do not work well at low eccentricities (Yoshikawa, 1990, and 1991; Morbidelli, 2002). In fact for the $q/p = 2/1$ case, the left boundary cannot even be computed near $e = 0$ (Morbidelli, 2002). In Section 7, we interpret the calculation of the boundaries in terms of the unaveraged circular restricted problem. For the commonly studied Kirkwood gaps, we give values of the eccentricity below which the approximation of the boundary will not be valid.

Asymmetric librations near exterior resonances with $p > q$ in the averaged circular restricted problem have been investigated by Beaugé (1994) and more recently by Voyatis et al. (2005). In Section 8, we show the existence of these librations in the unaveraged circular restricted problem for q/p equal to 1/2, 1/3, 1/4, 1/5, 1/6, and 1/7. We also give minimum values of eccentricity for each of these exterior resonances which must be exceeded for the asymmetric librations to exist.

2. Perturbative Form of the Return Map

Let L range over the interval $[(p/q)^{1/3} - \delta_L, (p/q)^{1/3} + \delta_L]$ for some $\delta_L > 0$. We assume $p/q \neq 1/1$. Let e range over the interval $[e_{\min}, e_{\max}]$ such that

$L^2(1-e)>1+\delta$ if $p>q$ and such that $L^2(1+e)<1-\delta$ if $p<q$, for all allowed values of L and e and some $\delta>0$. In addition, assume $0<e_{\min}$ and $e_{\max}<1$. We take the range of the Hamiltonian H to be the set of values of $-(p/q)^{-2/3}/2-(p/q)^{1/3}(1-e^2)$ for $e_{\min}\leqslant e\leqslant e_{\max}$. Then for any allowed value of H and L, any real values of the angles l and g measured modulo 2π, and μ sufficiently small, the implicit function theorem enables us to solve (1.2) for G uniquely. In fact, G will be an analytic function of l, L, g, H, μ.

We use either $g=0$ or $g=\pi$ to define the Poincaré section. Since Hamilton's equations formed using the Hamiltonian (1.2) imply that $dg/dt=-1+O(\mu)$, the return map is well defined for L and H in the intervals specified by the previous paragraph, any value of l, and μ small. Since H is conserved by the flow, we may identify the Poincaré section for fixed H with the l-L plane as in Figure 1. The return map preserves the area element $dl\,dL$. The return map will be denoted by T_1 and T_1^p will be denoted by T_p. In this section, we will obtain the perturbative form of T_p.

The Hamilton's Equations of (1.2) imply that $\dot{l}=1/L^3-\mu\Omega_L+O(\mu^2)$, $\dot{g}=-1-\mu\Omega_G$, and $\dot{L}=\mu\Omega_l$. We seek a solution of these equations with the initial conditions $l(0)=l_0$, $L(0)=L_0$, $g(0)=g_0$, where g_0 is either 0 or π, and $G(0)=G_0$. It is understood that G must be obtained by solving (1.2). If the solution is represented as $l(t)=l_a(t)+l_b(t)\mu+O(\mu^2)$, $g(t)=g_a(t)+g_b(t)\mu+O(\mu^2)$, and $L(t)=L_a(t)+L_b(t)\mu+O(\mu^2)$, then $l_a(t)=l_0+t/L_0^3$, $g_a(t)=g_0-t$, and $L_a(t)=L_0$. In addition, l_b, g_b, and L_b must satisfy

$$\dot{l}_b=(-3/L_a^4)L_b-\Omega_L,\quad \dot{g}_b=-\Omega_G,\quad \dot{L}_b=\Omega_l,$$

where the partial derivatives of Ω must be evaluated at $l=l_0+t/L_0^3$, $g=g_0-t$, $L=L_0$, and $G=G_0$. By solving the equations above, we get

$$\begin{aligned} l(t)&=l_0+t/L_0^3+\mu\left(-\int_0^t\Omega_L\,dt-\frac{3}{L_0^4}\int_0^t\int_0^\tau\Omega_l\,dt\,d\tau\right)+O(\mu^2)\\ g(t)&=g_0-t-\mu\int_0^t\Omega_G\,dt+O(\mu^2)\\ L(t)&=L_0+\mu\int_0^t\Omega_l\,dt+O(\mu^2), \end{aligned}\tag{2.1}$$

where the partial derivatives of Ω must be evaluated at $l=l_0+t/L_0^3$, $g=g_0-t$, $L=L_0$, and $G=G_0$. The solution given by (2.1) is valid over any finite interval of time for initial conditions in the domain already indicated and for μ sufficiently small. In (2.1), as in (2.2) and (3.2) later, t is used as both the variable and the upper limit of integration.

To approximate T_p, it is necessary to find the time t_r at which $g(t)=-2\pi p+g_0$. Using the equation for $g(t)$ in (2.1), we get $t_r=2\pi p-\mu\int_0^{2\pi p}\Omega_G\,dt+O(\mu^2)$.

Using (2.1), we may deduce that T_p is given by

$$
\begin{aligned}
l_1 &= l_0 + 2\pi p/L_0^3 + \mu\Big(-\frac{1}{L_0^3}\int_0^{2\pi p}\Omega_G\,dt - \int_0^{2\pi p}\Omega_L\,dt \\
&\quad -\frac{3}{L_0^4}\int_0^{2\pi p}\int_0^{\tau}\Omega_l\,dt\,d\tau\Big) + O(\mu^2) \qquad (2.2)\\
L_1 &= L_0 + \mu\int_0^{2\pi p}\Omega_l\,dt + O(\mu^2),
\end{aligned}
$$

where the partial derivatives of Ω must be evaluated at $l = l_0 + t/L_0^3$, $g = g_0 - t$, $L = L_0$, and $G = G_0$. The expression for T_p given by (2.1) is valid for $L \in [(p/q)^{1/3} - \delta_L, (p/q)^{1/3} + \delta_L]$, for any real value of the angle l, for H within a range that ensures avoidance of collision with the second primary as specified earlier, and for μ sufficiently small. Since this domain of validity is compact, the $O(\mu^2)$ terms in (2.3) hold uniformly over the domain.

The lemmas below are related to the first return map T_1 and its pth iterate T_p.

LEMMA 2.1. *Assume that* (l_0, L_0) *maps to* (l_1, L_1) *under* T_1 *(or* T_p*). Then* $(-l_1, L_1)$ *maps to* $(-l_0, L_0)$ *under* T_1 *(or* T_p*).*

Proof. The Hamiltonian (1.2) is unchanged by the transformation $l \leftarrow -l$, $L \leftarrow L$, $g \leftarrow -g$, $G \leftarrow G$. Therefore, if $l(t)$, $g(t)$, $L(t)$, $G(t)$ is a solution of the Hamilton's equations of (1.2) for $0 \leqslant t \leqslant t^*$, then $-l(-t)$, $-g(-t)$, $L(-t)$, $G(-t)$, where $-t^* \leqslant t \leqslant 0$, is also a solution. The lemma follows if it is noted that the Poincaré section is defined using either $g = 0$ or $g = \pi$. □

The lemma below is useful for finding fixed points of T_p.

LEMMA 2.2. *Let*

$$\phi_p(l_0, L_0) = \int_0^{2\pi p}\Omega_l\,dt,$$

where the arguments of Ω_l *are evaluated with* $l = l_0 + t/L_0^3$, $g = g_0 - t$, $L = L_0$, *and* $G = G_0$. *Then* $\phi_p(l_0, (p/q)^{1/3}) = \phi_p(l_0 + 2\pi/p, (p/q)^{1/3})$ *and* $\phi_p(-l_0, (p/q)^{1/3}) = -\phi_p(l_0, (p/q)^{1/3})$.

Proof. If we define $\phi_1(l_0, L_0) = \int_0^{2\pi}\Omega_l\,dt$, then (2.2) implies that $L_1 = L_0 + \phi(l_0, L_0)\mu + O(\mu^2)$ and that $l_1 = l_0 + 2\pi/L_0^3 + O(\mu)$ under the first return map T_1. If we apply T_1 to $(-l_1, L_1)$ and use Lemma 2.1 along with this expansion of T_1 in powers of μ, we get

$$\begin{aligned} L_0 &= L_1 + \phi_1(-l_1, L_1)\mu + O(\mu^2) \\ &= L_0 + \phi_1(l_0, L_0)\mu + \phi_1(-l_0 - 2\pi/L_0^3, L_0)\mu + O(\mu^2). \end{aligned}$$

Since (l_1, L_1) is obtained from (l_0, L_0) by a Hamiltonian flow that depends analytically upon the parameter μ, (l_1, L_1) must depend analytically on μ in a neighborhood of $\mu = 0$ for given (l_0, L_0). Since Ω is analytic in its arguments, $\phi(-l_1, L_1)$ must also depend analytically on μ for given (l_0, L_0). Thus the expression following the second equality sign in the display above stands for an expansion in powers of μ that converges for μ small. But the value of that sum must be L_0 for all μ sufficiently small. Therefore the coeffcients for μ^n for all $n \geqslant 1$ must all be zero. Thus we may consider the coefficient of μ and conclude that $\phi_1(l_0, L_0) + \phi_1(-l_0 - 2\pi/L_0^3, L_0) = 0$. If $L_0 = (p/q)^{1/3}$, we have

$$\phi_1(l_0, (p/q)^{1/3}) + \phi_1(-l_0 - 2\pi q/p, (p/q)^{1/3}) = 0. \tag{2.3}$$

From $T_p = T_1^p$, we get

$$\phi_p(l_0, L_0) = \sum_{j=0}^{p-1} \phi_1(l_0 + 2j\pi/L_0^3, L_0). \tag{2.4}$$

Now

$$\begin{aligned} \phi_p\big(l_0, (p/q)^{1/3}\big) &= \sum_{j=0}^{p-1} \phi_1\big(l_0 + 2\pi jq/p, (p/q)^{1/3}\big) \\ &= \sum_{j=0}^{p-1} \phi_1\big(l_0 + 2\pi j/p, (p/q)^{1/3}\big) \end{aligned}$$

The first equality above follows from (2.4). By elementary number theory, no two of the p numbers jq, with $0 \leqslant j \leqslant p-1$, are congruent to each other modulo p since p and q are prime to each other. Thus modulo p, these numbers must give the set of remainders $\{0, 1, \cdots, p-1\}$. Further $jq \equiv j' \mod p$ implies $\phi_1(l_0 + 2\pi jq/p, (p/q)^{1/3}) = \phi_1(l_0 + 2\pi j'/p, (p/q)^{1/3})$ since the angle l is measured modulo 2π. These observations imply the second equality above. Since

$$\sum_{j=0}^{p-1} \phi_1(l_0 + 2\pi j/p, (p/q)^{1/3}) = \sum_{j=0}^{p-1} \phi_1(l_0 + 2\pi(j+1)/p, (p/q)^{1/3}).$$

by similar reasoning, it follows that $\phi_p(l_0,(p/q)^{1/3})$ has period equal to $2\pi/p$ in l_0. That the quantity $\phi_p(l_0,(p/q)^{1/3})$ is an odd function of l_0 follows from (2.3). □

Lemma 2.2 can also be proved by another method. From its definition using (1.2), (1.4) and (1.5), Ω is periodic in l and g with period 2π. Further, $\Omega(L,G,-l,-g)=\Omega(L,G,l,g)$. Therefore we may Fourier expand Ω as $\sum c_{mn}(L,G)\cos(ml+ng)$, where the sum is over all nonnegative integers m and n. A detailed discussion of such expansions can be found in (Morbidelli, 2002). If the Fourier expansion of Ω is inserted into the definition of $\phi_p(l_0,L_0)$ in the statement of Lemma 2.2, we get

$$\phi_p(l_0,(p/q)^{1/3})=-2\pi p^2\sum_{k=1}^{\infty}k\,c_{kp,kq}\big((p/q)^{1/3},\\ (p/q)^{1/3}(1-e^2)^{1/2}\big)\sin(kpl_0+kqg_0).$$

The properties of $\phi_p(l_0,(p/q)^{1/3})$ asserted in Lemma 2.2 become obvious if it is noted that g_0 is either 0 or π by choice of the Poincaré section.

3. Scaling and Periodic Points of the Return Map

The variable λ defined by $L=(p/q)^{1/3}+\lambda\sqrt{\mu}$ can be used instead of L to blow up the region of the Poincaré section near p/q resonance. The image of the point (l_0,λ_0) in the l-λ plane under the pth return map T_p can be calculated using (2.2), and it is given by

$$\begin{aligned} l_1&=l_0-c_1\lambda_0\mu^{1/2}+(c_2\lambda_0^2+\chi(l_0))\mu+O(\mu^{3/2})\\ \lambda_1&=\lambda_0+\phi(l_0)\mu^{1/2}+\lambda_0\psi(l_0)\mu+O(\mu^{3/2}), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &c_1=6\pi q^{4/3}p^{-1/3}\quad\text{and}\quad c_2=12\pi q^{5/3}p^{-2/3}\\ &\phi(l_0)=\int_0^{2\pi p}\Omega_l\,dt\\ &\psi(l_0)=\int_0^{2\pi p}\Omega_{lL}\,dt+\frac{q}{p}\int_0^{2\pi p}\Omega_{lG}\,dt-\frac{3q^{4/3}}{p^{4/3}}\int_0^{2\pi p}t\Omega_{ll}\,dt\\ &\chi(l_0)=-\int_0^{2\pi p}\Omega_L\,dt-\frac{q}{p}\int_0^{2\pi p}\Omega_G\,dt-\frac{3q^{4/3}}{p^{4/3}}\int_0^{2\pi p}\int_0^{\tau}\Omega_l\,dt\,d\tau. \end{aligned} \tag{3.2}$$

In (3.2), the partial derivatives of Ω must be evaluated at $l=l_0+qt/p$, $g=g_0-t$ (where g_0 is either 0 or π depending upon the choice of the Poincaré section), $L = (p/q)^{1/3}$, and $G = (p/q)^{1/3}(1 - e^2)^{1/2}$. The expression for T_p given by (3.1) and (3.2) is valid for any real value of l_0 and $|\lambda_0| \leqslant \delta_L/\mu^{1/2}$. The domain of definition can therefore be taken as $|\lambda_0| \leqslant C_\lambda$ with any positive constant C_λ for sufficiently small μ. To derive the expression for $\psi(l_0)$ given in (2.3), we must use the second line of (2.2) and notice that the equation $L_0 = (p/q)^{1/3} + \lambda_0\mu^{1/2}$ and (1.2) imply $G_0 = (p/q)^{1/3}(1 - e^2)^{1/2} + (q/p)\lambda_0\mu^{1/2} + O(\mu)$. A term equal to $2\pi q$ has been dropped from the first line of (3.2) as l_0 and l_1 are angles measured modulo 2π.

The function $\phi(l_0)$ equals $\phi_p(l_0, L_0)$ defined by Lemma 2.1 when $L_0 = (p/q)^{1/3}$. By Lemma 2.1, $\phi(l_0)$ is an odd function with period equal to $2\pi/p$. Therefore, $\phi(0) = 0$ and $\phi(\pi/p) = 0$. We make the following assumption about $\phi(l_0)$:

Assumption A: For $l_0 \in [0, 2\pi/p)$, the only points where $\phi(l_0) = 0$ are $l_0 = 0$ and $l_0 = \pi/p$. At these points, the derivative $\phi'(l_0)$ is nonzero.

We turn to the verification of this assumption in Section 6. An example of a function which is odd with period $2\pi/p$ and which satisfies the assumption above is $\sin(pl_0)$. In fact, it will be shown later that $\phi(l_0)$ is proportional to $\sin(pl_0)e^{|p-q|}$ for small e.

The assumption about $\phi(l_0)$ can be put to use to find fixed points of T_p in the l-λ plane. We can use (3.1) and write

$$\begin{aligned}(l_1 - l_0)/\mu^{1/2} &= -c_1\lambda_0 + (c_2\lambda_0^2 + \chi(l_0))\mu^{1/2} + O(\mu)\\ (\lambda_1 - \lambda_0)/\mu^{1/2} &= \phi(l_0) + \lambda_0\psi(l_0)\mu^{1/2} + O(\mu).\end{aligned} \tag{3.3}$$

When $\mu = 0$, the right hand sides of the two equations in (3.3) are both zero if $\lambda_0 = 0$ and if l_0 is an integral multiple of π/p. The implicit function theorem, along with the assumption about $\phi(l_0)$ stated above, allows us to infer that the right hand sides in (3.3) are 0 for μ sufficiently small,

$$l_0 = j\pi/p + O(\mu), \quad \text{and} \quad \lambda_0 = \big(\chi(j\pi/p)/c_1\big)\mu^{1/2} + O(\mu), \tag{3.4}$$

where j is an integer. Thus the points given by (3.4) are fixed points of T_p for μ sufficiently small. Since each application of the first return map T_1 increments l_0 by $2\pi q/p + O(\mu)$, where q and p are relatively prime, we may group the fixed points given by (3.4) into two sets, the first with $j = 0, 2, \ldots, 2(p-1)$ and the second with $j = 1, 3, \ldots, 2(p-1)+1$. Then any fixed point in the first set moves to all other points in that set upon successive applications of T_1 and returns to itself after the pth application; and likewise with the second set. The assumption about $\phi(l_0)$ implies that if one set of periodic points is elliptic then the other set is hyperbolic as will become clear shortly. A discussion of the existence of these periodic points is given by Meyer and Hall (1992).

If Assumption A about $\phi(l)$ fails to hold, points l_0 with $\phi(l_0)=0$ and $\phi'(l_0)\neq 0$ will still correspond to periodic points of (3.1) for μ sufficiently small. If $l_0\neq 0$ and $l_0\neq\pi$, the corresponding periodic points can be the centers of asymmetric librations as described in the last section.

4. Formal Expansion of the Stable Manifold

The expression for the map T_p given by (3.1) and (3.2) can be rewritten by shifting the center of the l-λ plane to (l^*,λ^*), where (l^*,λ^*) is the fixed point of T_p given by (3.4), with j being some integer. The map T_p applied to the l-λ plane centered at such a fixed point takes the form

$$\begin{aligned} l_1 &= l_0 - c_1\lambda_0\mu^{1/2} + (c_2\lambda_0^2 + \chi(l_0))\mu + r(l_0,\lambda_0,\sqrt{\mu})\mu^{3/2} \\ \lambda_1 &= \lambda_0 + \Phi(l_0)\mu^{1/2} + \lambda_0\Psi(l_0)\mu + s(l_0,\lambda_0,\sqrt{\mu})\mu^{3/2}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \chi(l_0) &= \chi(l_0+j\pi/p) - \chi(j\pi/p), \qquad \Phi(l_0)=\phi(l_0+j\pi/p), \\ \Psi(l_0) &= \psi(l_0+j\pi/p). \end{aligned} \tag{4.2}$$

The derivations of (2.2) and (3.1) imply that the remainder terms r and s in (4.1) are analytic in l_0, λ_0, and $\sqrt{\mu}$ for any real l_0, λ_0 with $|\lambda_0|\leqslant C_\lambda$, and $\sqrt{\mu}$ sufficiently small in magnitude. Besides, $r(0,0,\sqrt{\mu})=s(0,0,\sqrt{\mu})=0$ because $(0,0)$ is a fixed point of (4.1).

The Jacobian dT_p of (4.1) at the origin is given by

$$dT_p = \begin{pmatrix} 1 & -c_1\mu^{1/2} \\ \Phi'(0)\mu^{1/2} & 1 \end{pmatrix} + \begin{pmatrix} \chi'(0) & \\ & \Psi(0) \end{pmatrix}\mu + O(\mu^{3/2}). \tag{4.3}$$

For $\mu>0$ and μ small, the fixed point is hyperbolic if $\Phi'(0)<0$ and elliptic if $\Phi'(0)>0$. From (4.2), it follows that $\Phi'(0)=\phi'(j\pi/p)$. The assumption of Section 3 implies that $\phi'(0)$ and $\phi'(\pi/p)$ are of opposite signs. Thus if the set of fixed points given by (3.4) is of hyperbolic or elliptic type for even j, the set of fixed points given by odd j must be of the opposite type. We shall assume that the fixed point used to shift the coordinate system and obtain (4.1) to be of hyperbolic type, which means $\Phi'(0)<0$. Let $\alpha=\sqrt[4]{-\Phi'(0)/c_1}$. Then a calculation using (4.3) shows that the eigenvalues $1-\alpha^2c_1\mu^{1/2}+\big((\chi'(0)+\Psi(0))/2\big)\mu+O(\mu^{3/2})$ and $1+\alpha^2c_1\mu^{1/2}+\big((\chi'(0)+\Psi(0))/2\big)\mu+O(\mu^{3/2})$ of dT_p correspond to eigenvectors of slopes

$$\alpha^2 + \frac{\chi'(0) - \Psi(0)}{2c_1}\mu^{1/2} + O(\mu) \quad \text{and} \quad -\alpha^2 + \frac{\chi'(0) - \Psi(0)}{2c_1}\mu^{1/2} + O(\mu), \tag{4.4}$$

respectively. The slope of the stable manifold of the fixed point of the map (4.2) at the origin must be given by the first of the two expressions in (4.4). When we derive an approximation to that stable manifold, (4.4) will serve to check the correctness of that approximation.

To find the stable manifold of this fixed point, we rewrite (4.2) in the following form:

$$\begin{aligned}
\lambda_0\mu^{1/2} &= -\frac{l_1 - l_0}{c_1} + \frac{c_2\lambda_0^2 + \chi(l_0)}{c_1}\mu + \cdots \\
\lambda_1^2 &= \lambda_0^2 + 2\lambda_0\Phi(l_0)\mu^{1/2} + (\Phi(l_0)^2 + 2\lambda_0^2\Psi(l_0))\mu + \cdots
\end{aligned} \tag{4.5}$$

Let (l_0, λ_0) be a point on the stable manifold. Its iterates (l_1, λ_1), (l_2, λ_2), and so on are also on the stable manifold. In addition, $l_n \to 0$ and $\lambda_n \to 0$ as $n \to \infty$. Using (4.5), we may write

$$\begin{aligned}
\lambda_n^2 &= \lambda_0^2 + 2\big(\lambda_0\Phi(l_0) + \lambda_1\Phi(l_1) + \cdots + \lambda_{n-1}\Phi(l_{n-1})\big)\mu^{1/2} + O(\mu) \\
&= \lambda_0^2 - \frac{2}{c_1}\big((l_1 - l_0)\Phi(l_0) + (l_2 - l_1)\Phi(l_1) + \cdots + (l_n - l_{n-1})\Phi(l_{n-1})\big) + O(\mu).
\end{aligned}$$

Note that

$$\begin{aligned}
(l_{j+1} - l_j)\Phi(l_j) &= \int_{l_j}^{l_{j+1}} \Phi(l)\,dl - \frac{(l_{j+1} - l_j)^2}{2}\Phi'(l_j) + \cdots \\
&= \int_{l_j}^{l_{j+1}} \Phi(l)\,dl + O(\mu).
\end{aligned} \tag{4.6}$$

Using (4.6) and noting that $l_{j+1} - l_j$ is $O(\mu^{1/2})$, we have

$$\lambda_n^2 = \lambda_0^2 - \frac{2}{c_1}\int_{l_0}^{l_n} \Phi(l)dl + O(\mu^{1/2}).$$

Taking the limit $n \to \infty$, we find that formally the stable manifold is given by $\lambda = u(l) + O(\mu^{1/2})$, where

$$u(l)^2 = -\frac{2}{c_1}\int_0^l \Phi(l)\,dl. \tag{4.7}$$

The positive root must be used if $l > 0$ and the negative root if $l < 0$. It can be verified that this expression for $u(l)$ agrees with (4.4) for the slope at the origin. Figure 2 plots $u(l)$ and $U(l) = u(l)^2$.

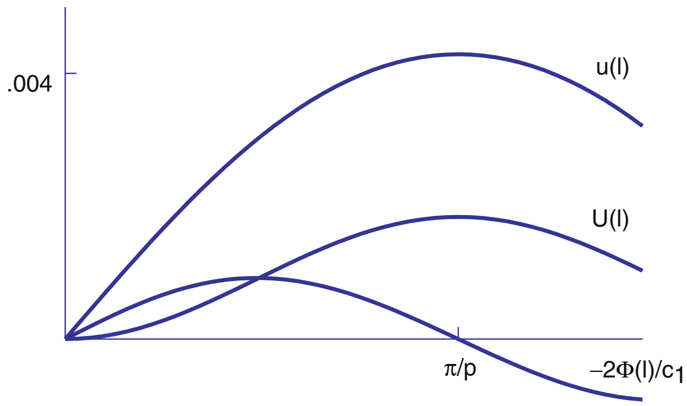

Figure 2. In the figure above, $u(l)$ has been scaled down by 0.1 to make it fit. The plots correspond to the case $q/p=3/1$, $e=0.1$.

To find the next term in the expansion of the stable manifold, we use (4.5) to get

$$\lambda_n^2=\lambda_0^2+2\mu^{1/2}\sum_{j=0}^{n-1}\lambda_j\Phi(l_j)+\mu\sum_{j=0}^{n-1}\Phi(l_j)^2+2\lambda_j^2\Psi(l_j)+\cdots \tag{4.8}$$

and use (4.5) and (4.6) to get

$$\begin{aligned}\lambda_j\Phi(l_j)\mu^{1/2}&=-\frac{1}{c_1}(l_{j+1}-l_j)\Phi(l_j)+\frac{c_2\lambda_j^2+\chi(l_j)}{c_1}\Phi(l_j)\mu+\cdots\\&=-\frac{1}{c_1}\int_{l_j}^{l_{j+1}}\Phi(l)\,dl+\frac{1}{c_1}\big(c_1^2\lambda_j^2\Phi'(l_j)/2+(c_2\lambda_j^2\\&\quad+\chi(l_j))\Phi(l_j)\big)\mu+\cdots\end{aligned} \tag{4.9}$$

Using (4.8) and (4.9) and by turning a sum into an integral as before, we get the expansion $\lambda=u(l)+v(l)\mu^{1/2}+O(\mu)$ for the stable manifold, where

$$v(l)=-\frac{1}{c_1u(l)}\int_0^l\frac{c_1u(l)\Phi'(l)}{2}+\frac{c_2u(l)^2+\chi(l)}{u(l)}\Phi(l)+\frac{\Phi(l)^2}{2u(l)}+u(l)\Psi(l)\,dl. \tag{4.10}$$

This formula for $v(l)$ agrees with (4.4) with regard to the slope at the origin. This procedure can be repeated to calculate more terms in the expansion of the stable manifold.

5. Homoclinic Points Near Resonances

Assume that the stable manifold of the origin under the map (4.1) is the graph of the function $\lambda = M_0(l)$. Then $M_0(l)$ must satisfy a functional equation of the form

$$\lambda_0 + \Phi(l_0)\mu^{1/2} + \cdots = M_0(l_0 - c_1\lambda_0\mu^{1/2} + \cdots).$$

Assume that $M_0(l) = u(l) + v(l)\mu^{1/2} + w(l)\mu$, where $u(l)$ and $v(l)$ are given by (4.7) and (4.10), respectively. Then $w(l)$ must satisfy the functional equation

$$(1 + c_1 u'(l_0)\mu^{1/2})w(l_0) = w(l_1) + s_2(l_0, \sqrt{\mu}w(l_0), \sqrt{\mu})\mu^{1/2}, \tag{5.1}$$

where s_2 is analytic in its arguments for $0 \leqslant l_0 \leqslant 3\pi/2p$, $\sqrt{\mu}w$ real and bounded by a large constant, and $\left|\sqrt{\mu}\right| \leqslant \sqrt{\mu_0}$ for some $\mu_0 > 0$. In addition, $s_2(0, 0, \sqrt{\mu}) = 0$. In Lemma 5.1 below, we prove that (5.1) has a unique C^1 solution $w(l)$, with $0 \leqslant l \leqslant 3\pi/2p$, for $0 < \mu \leqslant \mu_0$ and some $\mu_0 > 0$. We also prove that $|w(l)|$ and $\left|w'(l)\right|$ are bounded by constants which are independent of μ but which may depend upon μ_0. Therefore, the stable manifold of (4.1) is the graph of $\lambda = u(l) + v(l)\mu^{1/2} + w(l)\mu$ for $0 \leqslant l \leqslant 3\pi/2p$ and $0 < \mu \leqslant \mu_0$, where $u(l)$ and $v(l)$ are defined by (4.7) and (4.10).

As discussed by Zehnder (1973), the functional equation (5.1) for w is obviously a contraction for $0 \leqslant l \leqslant \pi/p - \epsilon$, $\epsilon > 0$. But the construction of homoclinic points requires the existence of w to be proved over a larger interval as in the lemma below. The proof of the lemma uses a technique found in Zehnder (1973). Another approach can be due to (Gelfreich and Lazutkin, 2001).

LEMMA 5.1. *For some* $\mu_0 > 0$ *and any* $\mu \in (0, \mu_0]$, *there exists a unique* C^1 *function* $w(l)$ *such that* (5.1) *is satisfied for* $0 \leqslant l_0 \leqslant 3\pi/2p$, *with* $|w(l)|$ *and* $\left|w'(l)\right|$ *bounded by constants for* $0 \leqslant l \leqslant 3\pi/2p$. *The constants are independent of* μ *but may depend upon* μ_0.

Proof. We will look for a continuous solution of (5.1) that satisfies $w(0) = 0$, $|\exp(-Kl)w(l)| \leqslant C^*$, and $\mathrm{Lip}(\exp(-Kl)w(l)) \leqslant L^*$, where $\mathrm{Lip}(\cdot)$ is the Lipschitz constant. The choice of the positive constants K, C^*, and L^* will be made later in the proof.

The function $r(l, \lambda, \sqrt{\mu})$ from the first line of (4.1) and the function $s_2(l, \sqrt{\mu}\,w, \sqrt{\mu})$ from (5.1) determine the initial choice of μ_0. Let $w^* = \mu\, w$ and consider $r^*(l, w^*, \sqrt{\mu}) = r(l, u(l) + \sqrt{\mu}v(l) + w^*, \sqrt{\mu})$. The constant $\mu_0 > 0$ is chosen so that $r^*(l, w^*, \sqrt{\mu})$ is analytic in its arguments over the compact domain D where $0 \leqslant l \leqslant 3\pi/2p$, w^* is real and

$|w^*| \leqslant C$, and $|\sqrt{\mu}| \leqslant |\sqrt{\mu_0}|$. By taking μ_0 small enough, we can assume C to be as large as we please. It is enough to assume C to be twice the height of $u(l)$ depicted in Figure 2, for example. Now let $w^* = \sqrt{\mu}\, w$ and assume the choice of μ_0 to be such that $s_2(l, w^*, \sqrt{\mu})$ is also analytic in its arguments in the compact domain D. We note $r^*(0, 0, \sqrt{\mu}) = s_2(0, 0, \sqrt{\mu}) = 0$.

The proof, which is organized into a number of steps, introduces many constants. The constants that depend on the domain D will be denoted by subscripting D. The constants that do not depend upon the domain D will be denoted by subscripting C. The constants that depend upon D are typically upper bounds for the magnitudes of derivatives of r^* and s_2 over the domain D. The constant μ_0 may be made smaller by some of the steps in the proof. But the bounds obtained using the domain D as specified above will of course apply even if μ_0 is made smaller. All constants introduced in the proof are strictly positive.

1. For $0 \leqslant l_0 \leqslant 3\pi/2p$, by (4.1) l_1 as a function of l_0 is given by

$$l_1 = l_0 - c_1\lambda_0\mu^{1/2} + (c_2\lambda_0^2 + \chi(l_0))\mu + r(l_0, \lambda_0, \sqrt{\mu})\mu^{3/2}, \tag{5.2}$$

   where $\lambda_0 = u(l_0) + v(l_0)\mu^{1/2} + w(l_0)\mu$. It is possible to think of (5.2) as defining l_1 in terms of l_0 and w. By the assumption about $\phi(l)$ in Section 3 and (4.7), it follows that $u'(0) > 0$ and that $u(l) > 0$ for $0 < l \leqslant 3\pi/2p$. As $u(0) = 0$, there must be a constant C_1 such that $c_1 u(l) \geqslant C_1 l$ for $0 \leqslant l \leqslant 3\pi/2p$. Both the μ and $\mu^{3/2}$ terms in (5.2) vanish when $l_0 = 0$ and $w = 0$. Further, $|w(l_0)| \leqslant \operatorname{Lip}(w) l_0$. Thus the magnitudes of the two terms can be upper bounded by $\big(C_2 + C_3 \operatorname{Lip}(w)\mu\big) l_0 \mu$ and $\big(D_1 + D_2 \operatorname{Lip}(w)\mu) l_0 \mu^{3/2}$, respectively. Therefore, we may assert $l_1 \leqslant \big(1 - (C_1/2)\mu^{1/2}\big) l_0$ for $0 \leqslant l \leqslant 3\pi/2p$ and $0 \leqslant \mu \leqslant \mu_0$, with μ_0 made smaller if necessary.
2. Let l_1^* be obtained using (5.2) with l_0 replaced by l_0^* but with the same w. Assume $0 \leqslant l_0, l_0^* \leqslant 3\pi/2p$ and let $C_4 = \operatorname{Lip}(c_1 u(l))$. Then, as in the previous step, it follows that $|l_1 - l_1^*| \leqslant \big(1 + 2C_4\mu^{1/2}\big) |l_0 - l_0^*|$ for $0 \leqslant \mu \leqslant \mu_0$ and μ_0 sufficiently small.
   An additional fact about $|l_1 - l_1^*|$ will be needed. Assume $0 \leqslant l_0, l_0^* \leqslant \pi/2p$. By the mean value theorem, $c_1(u(l_0) - u(l_0^*)) = c_1 u'(\bar{l})(l_0 - l_0^*)$, and by the assumption about $\phi(l)$ in Section 3 and (4.6), $u'(l) \geqslant C_5$ for $0 \leqslant l \leqslant \pi/2p$. Therefore, $|l_1 - l_1^*| \leqslant \big(1 - (C_5/2)\mu^{1/2}\big) |l_0 - l_0^*|$ or simply $|l_1 - l_1^*| \leqslant |l_0 - l_0^*|$ in this situation, for $0 \leqslant \mu \leqslant \mu_0$ and μ_0 sufficiently small.
3. From (5.1), we may obtain the following iteration:

$$\exp(-Kl_0)w_{n+1}(l_0) = F_{w_n}(l_0)\exp(-Kl_1)w_n(l_1) + \frac{\exp(-Kl_0)s_2(l_0, \sqrt{\mu}\, w_n(l_0), \sqrt{\mu})}{1+c_1u'(l_0)\mu^{1/2}}\mu^{1/2}, \tag{5.3}$$

where l_1 is obtained from l_0 using (5.2) but with w replaced by w_n, where w_n belongs to the class of functions for w specified at the beginning of this proof, and where the contraction factor F_{w_n} is given by

$$F_{w_n}(l_0) = \frac{\exp(K(l_1 - l_0))}{1+c_1u'(l_0)\mu^{1/2}}.$$

The constant K will be chosen so as to make F_{w_n} a sufficiently strong contraction.

First consider $0 \leqslant l_0 \leqslant \pi/2p$. Let the minimum value of $c_1u'(l_0)$ for l_0 in this range be C_6. Since $l_1 - l_0 \leqslant -C_1 l_0 \mu^{1/2}/2$ by the first step, it follows that $F_{w_n}(l_0) \leqslant \big(1-(C_6/2)\mu^{1/2}\big)$ for $0 \leqslant l_0 \leqslant \pi/2p$, $0 \leqslant \mu \leqslant \mu_0$, and μ_0 sufficiently small.

Next consider $\pi/2p \leqslant l_0 \leqslant 3\pi/2p$. Let the minimum value of $c_1u'(l)$ for l_0 in this range be $-C_7$. Then

$$F_{w_n}(l_0) \leqslant \frac{\exp(-KC_1\pi\mu^{1/2}/4p)}{1-C_7\mu^{1/2}}.$$

Choose K so that $KC_1\pi/4p \geqslant 5C_4 + 2C_7$ and conclude that $F_{w_n}(l_0) \leqslant (1 - 4C_4\mu^{1/2})$ for $\pi/2p \leqslant l_0 \leqslant 3\pi/2p$, $0 \leqslant \mu \leqslant \mu_0$, and μ_0 sufficiently small.

4. By assumption, $|\exp(-Kl)w_n(l)| \leqslant C^*$ for $0 \leqslant l \leqslant 3\pi/2p$. Using (5.3), we may upper bound $|\exp(-Kl)w_{n+1}(l)|$ by $(1-C_8\mu^{1/2})C^* + D_3\mu^{1/2}$, where $C_8 = \min(C_6/2, 4C_4)$ and D_3 is an upper bound of the coefficient of $\mu^{1/2}$ in the last term of (5.3). By choosing $C^* \geqslant D_3/C_8$, we assert that $|\exp(-Kl)w_{n+1}(l)|$ is also upper bounded by C^*.
5. By assumption, $\mathrm{Lip}\big(\exp(-Kl)w_n(l)\big) \leqslant L^*$. Let

   $$Q = \exp(-Kl_0)w_{n+1}(l_0) - \exp(-Kl_0^*)w_{n+1}(l_0^*),$$

   where $0 \leqslant l_0^* \leqslant l_0 \leqslant 3\pi/2p$. We will upper bound $|Q|$.

   Using (5.3), both the terms of Q can be replaced by expressions in terms of w_n. The resulting expression for Q equals $A_1B_1 - A_2B_2$ — where $A_1 = F_{w_n}(l_0)$, $B_1 = \exp(-kl_1)w_n(l_1)$, $A_2 = F_{w_n}(l_0^*)$, and $B_2 = \exp(-kl_1^*)w_n(l_1^*)$ — plus another term which equals the difference of two quantities times $\mu^{1/2}$. This other term will be denoted by $Q_r\mu^{1/2}$.

   To bound $|Q|$, first consider the case $l_0 \geqslant \pi/2p$. We write $|A_1B_1 - A_2B_2| \leqslant |A_1|\,|B_1 - B_2| + |B_2|\,|A_1 - A_2|$. By the third step and the assumption about l_0, $|A_1| \leqslant \big(1-4C_4\mu^{1/2}\big)$, and $|B_1 - B_2| \leqslant L^*\big|l_1 - l_1^*\big| \leqslant L^*\big(1+2C_4\mu^{1/2}\big)\big|l_0 - l_0^*\big|$, where the last inequality follows from the second step above. Therefore, $|A_1(B_1-B_2)| \leqslant \big(1-C_4\mu^{1/2}\big)L^* \left|l_0-l_0^*\right|$. A simple estimate shows that $\mathrm{Lip}(F_{w_n}(l)) \leqslant C_9\mu^{1/2}$ for sufficiently small μ_0. Therefore, $|B_2(A_1-A_2)| \leqslant C^*C_9\mu^{1/2}\left|l_0-l_0^*\right|$. To upper bound Q_r, note that the coefficient of $\mu^{1/2}$ in (5.3) has a Lipschitz constant with respect to l_0 that can be bounded as $D_4+D_5\,\mathrm{Lip}(w_n)\mu^{1/2}$ or $D_4+D_5L^*\mu^{1/2}$. Therefore

$$|Q| \leqslant \Big(\big(1-C_4\mu^{1/2}\big)L^* + C^*C_9\mu^{1/2} + D_4\mu^{1/2} + D_5L^*\mu\Big)\left|l_0-l_0^*\right|.$$

If $L^* \geqslant 2(C^*C_9+D_4)/C_4$, then $|Q| \leqslant L^*\left|l_0-l_0^*\right|$ for $0 \leqslant \mu \leqslant \mu_0$ and μ_0 sufficiently small.

Consider the case $0 \leqslant l_0^* \leqslant l_0 \leqslant \pi/2p$. In this case, the argument is identical to that given in the previous paragraph, except that the bound on $|A_1|$ must be replaced by $\big(1-(C_6/2)\mu^{1/2}\big)$ form the third step, and $|B_1-B_2| \leqslant L^*\left|l_1-l_1^*\right| \leqslant L^*\left|l_0-l_0^*\right|$ from the additional fact in the second step. In this case, if $L^* \geqslant 4(C^*C_9+D_4)/C_6$, then $|Q| \leqslant L^*\left|l_0-l_0^*\right|$ for $0 \leqslant \mu \leqslant \mu_0$ and μ_0 sufficiently small. The choice $L^* = 2(C^*C_9+D_4)\max(1/C_4, 2/C_6)$ implies $\mathrm{Lip}\big(\exp(-Kl)w_{n+1}(l)\big) \leqslant L^*$.

6. If $w_n(0)=0$, then $w_{n+1}(0)=0$ since $s_2(0,0,\sqrt{\mu})=0$. This observation together with the choice of C^* and L^* in the fourth and fifth steps implies that w_{n+1} belongs to the same class of functions as w_n. The third step with some other estimates given above implies that the map $w_n \to w_{n+1}$ given by (5.3) is a contraction for $0<\mu \leqslant \mu_0$. We conclude that there is a unique continuous solution w of (5.1) such that $|\exp(-Kl)w(l)| \leqslant C^*$ and $\mathrm{Lip}\big(\exp(-Kl)w(l)\big) \leqslant L^*$ for $0 \leqslant l \leqslant 3\pi/2p$.
7. If $w(l)$ is continuously differentiable its derivative can be easily bounded in terms of K, C^*, and L^*. To complete the proof, it suffices to show that $w(l)$ is continuously differentiable. The standard stable manifold theorem states that $w(l)$ will be analytic in l in a neighborhood of $l=0$. The stable manifold over the interval $0 \leqslant l \leqslant 3\pi/2p$ can be obtained by repeated applying T_p^{-1} to a local segment. Therefore $w(l)$ must be continuously differentiable. □

If $\phi(l_0)$ defined by (3.2) satisfies the assumption in Section 3, the fixed points of the map T_p given by (3.4) are hyperbolic for j even or for j odd. One of these hyperbolic points was shifted to the origin in (4.1), and we proved that the stable manifold of the origin is given by the graph of $\lambda = u(l)+v(l)\mu^{1/2}+O(\mu)$ for $0 \leqslant l \leqslant 3\pi/2p$. In the L-l plane, the stable manifold is the graph of

$$L = \left(\frac{p}{q}\right)^{1/3} + \frac{\chi(j\pi/p)}{c_1}\mu + u(l-j\pi/p)\mu^{1/2} + v(l-j\pi/p)\mu + O(\mu^{3/2}) \tag{5.4}$$

for $0 \leqslant l - j\pi/p + O(\mu) \leqslant 3\pi/2p$, where χ is given by (3.1), u is given by (4.7), and v is given by (4.10).

The choice of the Poincaré section as either $g = 0$ or $g = \pi$ is yet to be made. To facilitate the construction of homoclinic points, it is also useful to pick j in (3.4) carefully. There are four cases.

- If p is odd, then $g = 0$ is chosen as the Poincaré section. If $\phi'(0) > 0$, then $j = -1$.
- If p is odd and $\phi'(0) < 0$, then $j = (p - 1)$.
- If p is even, first try $g = 0$ as the Poincaré section. If $\phi'(0) > 0$, then $j = -1$.
- If p is even and $\phi'(0) < 0$ with $g = 0$ as the Poincaré section, choose the Poincaré section $g = \pi$ and $j = -1$.

THEOREM 5.2. *Let p and q be relatively prime positive integers and let $p/q \neq 1/1$. Assume that e lies in the interval* $[e_{\min}, e_{\max}]$ *defined at the beginning of Section* 2. *Let $\phi(l_0)$ be defined by* (3.2) *with $L = (p/q)^{1/3}$ and $G = (p/q)^{1/3}(1 - e^2)$. Assume that $\phi(l_0)$ satisfies Assumption A of Section* 3, *namely, for $l_0 \in [0, 2\pi/p)$, $\phi(l_0) = 0$ only if $l_0 = 0$ or $l_0 = \pi/p$ and $\phi'(l_0) \neq 0$ at those two points. Identify the Poincaré section for the flow of the Hamiltonian* (1.2) *of the restricted three-body problem with a region of the L-l plane by using $H = -(p/q)^{-2/3}/2 - (p/q)^{1/3}(1 - e^2)$ and by choosing $g = 0$ or $g = \pi$ as indicated above. Then the pth return map T_p given by* (2.2) *has a homoclinic point on this Poincaré section at (l_h, L_h), where $L_h = (p/q)^{1/3} + u(\pi/p)\mu^{1/2} + O(\mu)$ and $l_h = 0$ or $l_h = \pi$, for $0 < \mu \leqslant \mu_0$ and μ_0 sufficiently small.*

Proof. We give a proof for the second case listed above. The other cases are treated similarly. In this case, p is odd, the Poincaré section is $g = 0$, and $j = (p - 1)$. By Lemma 5.1 the representation of the stable manifold given by (5.4) is valid for $0 \leqslant l - (p - 1)\pi/p + O(\mu) \leqslant 3\pi/2p$, and therefore the stable manifold crosses the line $l = \pi$. By Lemma 2.1, this stable manifold can be reflected about the line $l = \pi$ to obtain an unstable manifold. Thus we find a homoclinic point with $l = \pi$. □

The homoclinic point constructed in Theorem 5.2 can be mapped using the first return map T_1 to obtain a ring of p homoclinic points with $L_h > (p/q)^{1/3}$. In Section 4, we constructed the stable manifold of the fixed point of (4.2) at the origin over $0 \leqslant l_0 \leqslant 3\pi/2p$. A similar construction applies over the interval $-3\pi/2p \leqslant l_0 \leqslant 0$. That construction can be used to find a ring of p homoclinic points with $L_h < (p/q)^{1/3}$.

6. Verification of the Condition on $\phi(l_0)$

Equations (1.3) and (1.4) define Ω as a function of L, l, G, g. If the angles l and g are replaced by $-l$ and $-g$, Ω is unchanged. Therefore, Ω can be Fourier expanded as

$$\Omega = \sum_{m,n} c_{mn} \cos(ml + ng), \tag{6.1}$$

where m can be any non-negative integer and n can be any integer. The coefficients c_{mn} are functions of L and G. We use $L = (p/q)^{1/3}$ and $G = (p/q)^{1/3}(1-e^2)^{1/2}$. The Fourier expansion is valid if $1 + r^2 - 2r\cos(\theta) > 0$ and is therefore valid if e is sufficiently small for $p/q \neq 1/1$. The coefficients c_{mn} can be expanded as power series in e, and it is possible to determine the precise radius of convergence of these series. For our purposes, it suffices to note that all these series converge in some neighborhood of $e = 0$. If Ω is differentiated with respect to L or G the condition $e > 0$ has to be imposed. However, the partial derivatives of Ω with respect to l and g and Ω itself are analytic in a neighborhood of $e = 0$.

LEMMA 6.1. *If the Fourier coefficients c_{mn} of* (6.1) *are expanded in powers of e, the lowest power of e with a possibly nonzero coefficient is $e^{|m-n|}$.*

Proof. The quantity Ω can be expanded as $\sum_{m=0}^{\infty} c_m(r)\cos(m\theta)$. First consider $e = 0$. Then $r = (p/q)^{1/3}$ and $\theta = \nu + g = l + g$. Therefore the only nonzero terms in the expansion (6.1) occur when $m = n$. If $e \neq 0$, then r depends upon l and ν is no longer equal to l. This lemma follows when the dependence of r and ν on l is taken into account. The way to do this can be found on pages 44 and 170 of (Plummer, 1918) or on page 35 of (Morbidelli, 2002). □

By (3.2), $\phi(l_0) = \int_0^{2\pi p} \Omega_l \, dt$, where Ω_l must be evaluated at $l = l_0 + qt/p$, $g = g_0 - t$, $L = (p/q)^{1/3}$, and $G = (p/q)^{1/3}(1-e^2)^{1/2}$; $g_0 = 0$ or $g_0 = \pi$ depending upon the choice of the Poincaré section. Using (6.1), we get

$$\phi(l_0) = -2\pi p^2 \sum_{m=1}^{\infty} m \, c_{mp,mq} \sin(kpl_0 + kqg_0).$$

Let $c_{p,q} = c^*(p,q)e^{|p-q|} + O(e^{|p-q|+1})$. Then by Lemma 6.1, $\phi(l_0) = \pm 2\pi p^2 c^*(p,q)\sin(pl_0)e^{|p-q|} + \cdots$. Thus the assumption about $\phi(l_0)$ in Section 3 and in Theorem 5.2 will be verified for $e > 0$ and e small if we can show that $c^*(p,q) \neq 0$. An expression for $c^*(p,q)$ can be obtained from the discussion of the quantity

$$C(e, p, q) = -\frac{6\pi q^{4/3}}{p^{1/3}} \int_0^{2\pi p} \Omega_{ll}\, dt$$

given in (Viswanath, 2005). If $p < q$, for example,

$$c^*(p, q) = -\frac{(-1)^{q-p} q^{2/3}}{6.2^{q-p} \pi p^{8/3}} \left(\sum_{k=0}^{q-p} \binom{D+q}{k} \frac{p^{q-p-k}}{(q-p-k)!} \right) (\alpha b_q(\alpha))$$

evaluated at $\alpha = (p/q)^{1/3}$; above D stands for the differential operator $\alpha \frac{d}{d\alpha}$ and $b_n(\alpha)$ are defined by the expansion

$$(1 + \alpha^2 - 2\alpha \cos\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} b_n(\alpha) \exp(in\theta).$$

The $b_n(\alpha)$ are hypergeometric functions whose series converge for $|\alpha| < 1$. The value of $c^*(p, q)$ can be obtained using the expression given above or by other means.

In Figure 3, we have plotted $\phi(l)$ with $q/p = 3/1$. From that figure, it is clear that the assumption about $\phi(l_0)$ is valid for even large values of e.

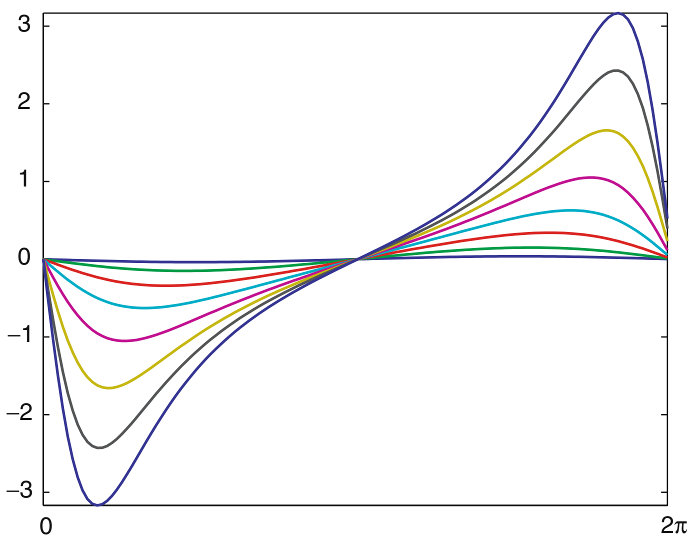

Figure 3. Plots of $\phi(l)$ against l with $q/p = 3/1$ and with e ranging from 0.1 to 0.8.

7. Resonance Boundaries

Discussion of the averaged circular restricted problem and its use in sketching the boundaries of q/p Kirkwood gaps in the a-e plane can be found in (Dermott and Murray, 1983; Henrard and Lemaître, 1983; Lemaître, 1984; Yoshikawa, 1990 and 1991; Morbidelli, 2002). The basic procedure is to average the Hamiltonian (1.2) by retaining only the terms in the Fourier expansion (6.1) of Ω with $ml+ng=k(pl-qg)$ for some integer k. These are the resonant terms. In some instances such as in (Yoshikawa, 1990 and 1991), certain additional terms are added to model the effect of the eccentricity of Jupiter and the secular variation of its elements.

This averaged Hamiltonian has 1 degree of freedom. Its fixed points and separatrices are used to approximate the boundaries of resonance. For small values of the asteroid eccentricity e, some of the nonresonant terms dropped during averaging have larger coefficients than any of the terms retained during averaging. To some extent the influence of the nonresonant terms is captured by the formal change of variables used to average the Hamiltonian, but this change of variables is often not taken into account. Even if it is, the averaging will not be valid at small values of e.

The return map (3.1) corresponds to the unaveraged circular restricted problem. Only the resonant terms of Ω contribute to $\phi(l)$ defined by (3.2), but the nonresonant terms contribute to both $\psi(l)$ and $\chi(l)$. As e approaches 0, the magnitude of $\phi(l)$ becomes much smaller than that of the other two functions in (3.2). Thus for fixed μ and small e the periodic points (3.4) will not exist.

These periodic points, when they exist, correspond to the fixed points of the averaged Hamiltonian. The separatrices of the averaged Hamiltonian correspond to the stable manifolds discussed in Sections 4 and 5. If for certain values of μ, e, p, q the unaveraged circular restricted problem does not have the periodic points given by (3.4), the fixed points and separatrices of the averaged Hamiltonian must be treated as artifacts of the averaging procedure.

In Table I, we have given the minimum values of e required for the periodic points (3.4) to exist for some of the commonly studied resonances

TABLE I

This table gives the values of e for certain q/p below which the periodic points of Section 3 fail to exist for $\mu=10^{-3}$. Boundaries of resonance obtained using the averaged circular restricted problem will not be valid below these values.

q/p	3/2	2/1	7/3	5/2	3/1	4/1
e	0.10	0.13	0.08	0.08	0.07	0.09

in the asteroid belt. The value of μ used is close to that of Jupiter. The boundaries of resonance obtained by averaging can be valid only above these values of the asteroid eccentricity.

8. Asymmetric Librations

If the assumption about $\phi(l)$ in Section 3 holds, the circular restricted problem has two resonant periodic solutions for μ small. One of these is of elliptic type and therefore there will be solutions that librate around the periodic points that correspond to it in the Poincaré section given by the l-L plane. These are symmetric librations.

As shown in Figure 4, the assumption about $\phi(l)$ in Section 3 can fail for some exterior resonances as e increases. In both the plots shown in that figure, the periodic point with $l \approx \pi/p$ is initially of elliptic type as the slope of $\phi(l)$ is positive, and when e increases, it undergoes a pitch-fork bifurcation and becomes a point of hyperbolic type. Elliptic points appear on the Poincaré section at values of l that are not $O(\mu)$ close to any integral multiple of π/p. The librations around these points are termed asymmetric.

A study of asymmetric librations in the exterior resonances using averaged equations can be found in (Beaugé, 1994). In Table II, we have listed values of e above which asymmetric librations occur in the unaveraged equations for some exterior resonances. If the averaged equations imply the existence of asymmetric librations below these values of e, those must be

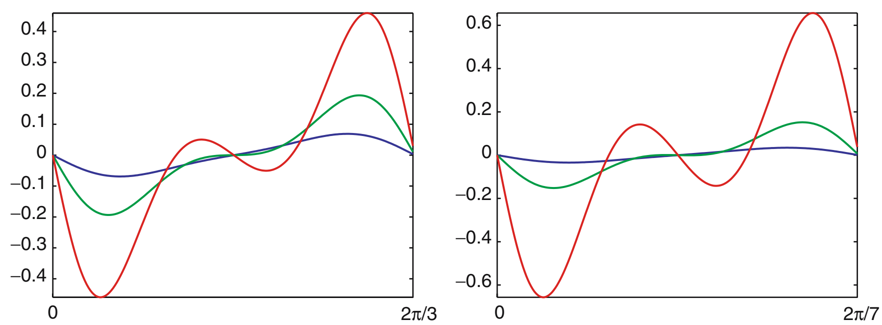

Figure 4. The two figures show plots of $\phi(l)$ vs. l for $l \in [0, 2\pi/p)$ for $q/p = 1/3$ and $q/p = 1/7$, respectively. The values of e are 0.08, 0.12 and 0.16 in the left plot, and 0.30, 0.36 and 0.40 in the right plot.

TABLE II

This table gives the values of e above which asymmetric librations exist in the unaveraged circular restricted problem for certain exterior resonances.

q/p	1/7	1/6	1/5	1/4	1/3	1/2
e	0.365900	0.320133	0.265532	0.199749	0.121094	0.036083

considered artifacts of averaging. However, if e exceeds these values, asymmetric librations will be found for μ small enough.

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