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Delay embedding of periodic orbits using a fixed observation function

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HIGHLIGHTS

- Beginning steps in embedology with a fixed observation function.
- Proof of genericity of embedding of periodic solutions in $\mathbb{R}^3$.
- Neuroscience citations and the Lorenz example are used to show how periodic solutions arise in applications.

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ABSTRACT

Delay coordinates are a widely used technique to pass from observations of a dynamical system to a representation of the dynamical system as an embedding in Euclidean space. Current proofs show that delay coordinates of a given dynamical system result in embeddings generically with respect to the observation function (Sauer et al., 1991). Motivated by applications of the embedding theory, we consider flow along a single periodic orbit where the observation function is fixed but the dynamics is perturbed. For an observation function that is fixed (as a nonzero linear combination of coordinates) and for the special case of periodic solutions, we prove that delay coordinates result in an embedding generically over the space of vector fields in the C^{r-1} topology with $r \geq 2$.

1. Introduction

Suppose a physical system is described by the differential equation $\frac{dx}{dt} = f(x)$, where $f : \mathbb{R}^d \to \mathbb{R}^d$. Often the state vector x is unobservable in its entirety, and that is especially true if d is large. Thus, reconstructing the flow from observations is not straightforward. The technique of delay coordinates makes it possible to look at a single scalar observation and reconstruct the dynamics. We denote the scalar that is observed by πx. The observation function π could be a projection to a single coordinate, for example, when the velocity of a fluid flow is recorded at a single point and in a single direction. It could be some other linear function of x. More generally, the observation function πx could be nonlinear.

If $\phi_t(x)$ is the time-t flow map, the idea behind delay coordinates [1–3] is to use the delay vector

$$\xi(x; \tau, n) = \left(\pi x, \pi\phi_{-\tau}(x), \ldots, \pi\phi_{-(n-1)\tau}(x)\right),$$

which is observable, as a surrogate for the point x in phase space. For a suitable choice of delay τ and embedding dimension n, delay coordinates yield a faithful representation of the phase space in a sense we will explain. Delay coordinates have been employed in many applications [4,5]. Current theory for delay coordinates [2] applies perturbations to the observation function π. We consider the situation where the observation function is fixed as a linear projection and only the dynamical system $\frac{dx}{dt} = f(x)$ is perturbed.

Packard et al. [1] demonstrated that coordinate vectors such as $(\pi\phi_t(x), \frac{d}{dt}\pi\phi_t(x))$ give good representations of strange attractors. They noted that delay coordinate vectors would be equivalent to coordinate vectors formed using derivatives of the observed quantity.

A mathematical analysis of delay coordinates was undertaken in a famous paper by Takens [3] and independently by Aeyels [6]. In particular, Takens considered when $x \to \xi(x; \tau, n)$ is an embedding. Suppose M is a manifold of dimension m, $A \subset M$ a submanifold of M of dimension d, and $f : M \to N$ a continuous map from M to the manifold N. The restriction $f\big|_A$ is an embedding of A in N if the tangent map df has full rank at every point of A, $f\big|_A$ is injective, and $f\big|_A$ maps open sets in A to open sets in its range in the subspace topology [7,8]. For the definition to make sense, the manifolds and f must be at least C^1. More generally, the manifolds M, N and the map f may be assumed to be C^r with $r \geq 1$ or with $r = \infty$. Takens concluded that delay coordinates yield an embedding of compact manifolds without boundary if $n \geq 2m+1$, for *generic* observation functions π and *generic* vector fields f. A property is generic in the C^r topology if it holds for functions f or π belonging to a countable intersection of open and dense sets [9].

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Because the C^r spaces are Baire spaces [8], a countable intersection of open and dense sets is dense as well as uncountable.

The paper by Sauer et al. [2] marked a major advance in the theory of delay coordinates. The approach to embedding theorems outlined by Takens relied on parametric transversality. Parametric transversality arguments typically have a local part and a global part, and the transition from local arguments to a global theorem is made using partitions of unity [8].

Sauer et al. [2] sidestepped transversality theory almost entirely. Unlike in transversality theory, there is no explicitly local part in the arguments of Sauer et al. [2]. The local part of the argument comes down to a verification of Lifschitz continuity. The set being embedded is only assumed to have finite box counting dimension. The arguments are mostly probabilistic and the globalization step relies only on the finiteness of the box counting dimension. The only real analogy to differential topology appears to be to the proof of Sard's theorem [8], which is also proved using probabilistic arguments. Sauer et al. prove prevalence [10], which goes beyond genericity. A property is prevalent with respect to the observation function π, if the property holds when any given π is replaced by $\pi + \sum_{\alpha \in I_\alpha} c_\alpha p_\alpha$, with p_α being monomials indexed by the finite set I_α, for almost every choice of the coefficients c_α.

The embedding theorem of Sauer et al. [2] fixes the dynamical system and allows only the observation function π to be perturbed. The statements of genericity and prevalence are with regard to π, not the original dynamical system. If consideration is restricted to subsets A of box counting dimension d, Sauer et al. only require $n > 2d$. Thus, we could even have $n < m$.

As mentioned, we investigate embedding theorems in which the observation function is fixed. For example, π could be fixed as a linear projection that extracts some component of the state vector. We allow perturbations of the dynamical system only.

The motivation for considering such embedding theorems is as follows. First, on purely aesthetic grounds, it appears desirable to have an embedding theory that depends upon the dynamics and not the observation function. Second, in many applications the observation function is fixed, whereas the dynamical system itself is parametrized [4,5,11–13]. If π extracts a single component at a single point in the velocity field of a fluid, it is more pertinent to make the embedding theory depend upon the dynamics rather than upon the observation function.

Aeyels [6] stated that delay coordinates are injective for generic flows and a fixed observation function. In the context of applications, stronger theorems would be desirable as argued by Sauer et al. [2]. First, an open and dense set can have arbitrarily small measure implying that prevalence, which is stronger than genericity, is a more appropriate concept. Second, the dynamics may be confined to an attractor of dimension much smaller than that of the state vector of the flow. In such a situation, we would like the embedding dimension to be determined by the dimension of the attractor and not the dimension of the state vector of the flow.

In this article, we consider the second of these two directions. Obtaining an embedding dimension that depends on the dimension of the attractor and not the flow introduces new difficulties when the observation is fixed and the flow is parametrized. Current proofs [2,3] rely on perturbing the observation function to produce an embedding. When the observation function is fixed, the additional step of propagating perturbations to the flow to the observed delay coordinates will need to be handled. We need to understand how perturbing the flow perturbs the invariant set or attractor, which is assumed to persist, and how the perturbations to the invariant set or attractor propagate to delay coordinates. When the flow is fixed and the observation function is perturbed, the attractor to be embedded, which depends only upon the flow, is unchanged by the perturbations. In contrast, when the observation function is fixed and the flow is perturbed, the set to be embedded is altered by the perturbations.

To get a handle on such difficulties, we limit ourselves to hyperbolic periodic orbits and prove that they embed generically in $\mathbb{R}^3$. The techniques we use are those of transversality theory. Although periodic orbits are only a special case, they are an important special case and arise frequently in applications, for example [14,15].

To conclude this introduction, we mention some other extensions of delay coordinate embedding theory. Embedding theory has been considered for endomorphisms [16] as well as delay differential equations [17], for continuous but not necessarily smooth observation functions [18,19], and in concert with Kalman filtering [20]. The concept of determining modes and points in fluid mechanics and PDEs is related to embedding theory [13,21,22]. Delay coordinates have been used for noise reduction [23,24]. The embedding theory of Sauer et al. [2] has been generalized to PDEs by Robinson [13,22]. The current embedding theory for PDEs also relies on perturbing the observation function.

2. Embedding periodic signals in $\mathbb{R}^3$

In the next section, we consider periodic solutions of differential equations. In this section, we begin by considering periodic signals. A periodic signal is any function $o : \mathbb{R} \to \mathbb{R}$ with a period $T > 0$. Fig. 2.1 shows a periodic signal and its delay embedding in $\mathbb{R}^3$.

To make the definition of periodic signals more precise, let $\mathcal{O}^r$ be the set of C^r functions $o : [0, T] \to \mathbb{R}$ with period $T > 0$. Periodicity requires r derivatives of $o(t)$ to match at $t = 0$ and $t = T$. The domain of functions in $\mathcal{O}^r$, which we will write as $[0, T)$ for signals o of period T, is compact and homeomorphic to S^1. More precisely, the domain is the identification space obtained by identifying 0 and T in $[0, T]$. For convenience, we shall refer to it as $[0, T)$, with the understanding that when we refer to an interval (α, β) it can wrap around. The elements of $\mathcal{O}^r$ will be referred to as periodic signals. Even if $o \in \mathcal{O}^r$ is constant, it must be equipped with a period $T > 0$, and if T is chosen differently, we get a different element of $\mathcal{O}^r$.

For the periodic signal shown in Fig. 2.1, the map $t \to (o(t), o(t-\tau), o(t-2\tau))$ for $0 \le t < T$ results in an embedding of the circle. Each point of the circle $[0, T)$ maps to a distinct point in $\mathbb{R}^3$ so that the delay map is injective. The delay is also immersive because a small movement along the periodic signal maps to a small and nonzero movement in the embedding space $\mathbb{R}^3$. Because the delay map is both injective and immersive, it is an embedding.

Fig. 2.2 shows a situation in which the delay map is not injective. This example is in fact the same as in Fig. 2.1 but the period is taken to be double of what it is in Fig. 2.1. As a result, points which are separated by the fundamental period map to the same point in $\mathbb{R}^3$. As shown in Fig. 2.2, the signal may be modified so that the fundamental interval is not repeated and the delay map still fails to be injective. Later in this section, we will prove that signals whose delay maps embed the circle in $\mathbb{R}^3$ are more typical.

2.1. Local argument for periodic signals

If $r \in \mathbb{Z}^+$ and $o, o' \in \mathcal{O}^r$ are two periodic signals, define

$$d_r(o, o') = \sup_{k=0,\dots,r} \sup_{0 \le s < 1} |o^{(k)}(sT) - o'^{(k)}(sT')| + |T - T'|. \tag{2.1}$$

The C^r topology on $\mathcal{O}^r$ is defined by this metric. The $\mathcal{O}^r$ norm of a periodic signal is $\|o\|_r = \sup_{k=0,\dots,r} \sup_{0\le t<T} |o^{(k)}(t)|$. By our definition, $\mathcal{O}^r$ is not a vector space because signals with different periods cannot be added. However, signals of a fixed period are a vector space and $\|\cdot\|_r$ is a norm over it. The C^∞ topology is the union of C^r topologies over $r \in \mathbb{Z}^+$ as explained in [8]. For concepts and results of differentiable topology, such as critical points, regular values, and Sard's theorem, our main reference is Hirsch [8]. The same topics are discussed from a dynamical point of view in [9,25].

Fig. 2.1. A periodic signal (only a single period is shown) and its delay embedding in $\mathbb{R}^3$ with delay τ. The points a, b, c map to A, B, C with delay coordinates.

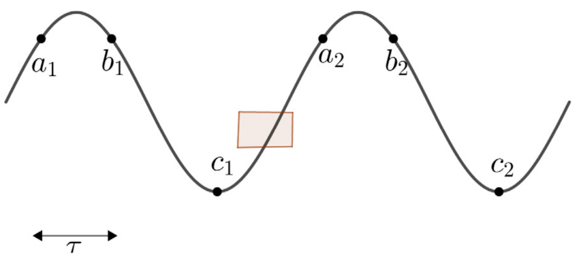

Fig. 2.2. The points a_1 and a_2, and likewise b_1, b_2 and c_1, c_2, map to the same point in $\mathbb{R}^3$ under delay embedding with the delay shown. The fundamental period of this signal is half of what is shown. However, by modifying the signal in the box shown, its fundamental period becomes equal to the interval shown and the delay map still fails to be injective because c_1 and c_2 map to the same point in $\mathbb{R}^3$.

Fig. 2.2 shows a signal which does not embed the circle in $\mathbb{R}^3$ under delay mapping. However, it is clear from observation that points that are nearby such as a_1 and b_1 map to distinct points in $\mathbb{R}^3$. In fact, quite generally, if the number of critical points in $[0, T)$ is finite, nearby points in the signal will map to distinct points in $\mathbb{R}^3$, as we later prove. We begin by considering whether any periodic signal may be perturbed slightly so that it has only finitely many critical points.

Lemma 1. *Let $o \in \mathcal{O}^r$, $r \geq 2$, be a periodic signal of period $T > 0$. If 0 is a regular value of do/dt, then the periodic signal $o(t)$ has finitely many critical points in $[0, T)$.*

Proof. Suppose $do/dt = 0$ at infinitely many points on the compact circle $[0, T)$. Let $p \in [0, T)$ be an accumulation point of the set of zeros. Then $d^2o(p)/dt^2 = 0$ and $do(p)/dt = 0$ implying that 0 is not a regular value of do/dt. □

The following lemma generates a periodic signal of period T whose derivative is $\frac{do}{dt} = \epsilon$ everywhere except over a given interval (α, β). Any function whose derivative is $\frac{do}{dt} = \epsilon, \epsilon \neq 0$, everywhere cannot be periodic. Therefore, the proof of the lemma comes down to modifying the derivative carefully in the interval (α, β).

Lemma 2. *Given $(\alpha, \beta) \subset [0, T)$ and $\delta > 0$, for all sufficiently small ϵ there exists an infinitely differentiable periodic signal o of period T such that $do(t)/dt = \epsilon$ for $t \notin (\alpha, \beta)$ and $|do(t)/dt| < \delta$ for $t \in (\alpha, \beta)$. In addition, for $r \in \mathbb{Z}^+$, $\left\|o\right\|_r \to 0$ as $\epsilon \to 0$.*

Proof. Let $\lambda(x)$ be an infinitely differentiable bump function with $\lambda(x) \in [0, 1]$ for $x \in [0, 1]$, $\lambda(x) = 1$ for $x \in [1/4, 3/4]$, and $\lambda(x) = 0$ for $x \in [0, 1/8]$ and $x \in [7/8, 1]$. If $\int_0^1 \lambda(x)\, dx = c$ then $1/2 < c < 1$. The bump function $\lambda(x)$ is used to modify do/dt in the interval (α, β).

Define $do(t)/dt = \epsilon$ for $t \notin (\alpha, \beta)$ and more generally

$$\frac{do(t)}{dt} = \epsilon - k\lambda((t-\alpha)/(\beta-\alpha))$$

for $t \in [0, T)$. The idea behind the construction is shown in Fig. 2.3: if the bump function is shifted to the interval (α, β) and a suitable multiple is subtracted, $\frac{do}{dt}$ may then be integrated to obtain a periodic function.

More precisely, it follows that $\int_0^T (do(t)/dt)\, dt = \epsilon T - k(\beta-\alpha)c$. The integral is zero if $k = \epsilon T/(\beta-\alpha)c$. For ϵ small, k is small as well. We may obtain $o(t)$ by integrating $do(t)/dt$, with $\left\|o\right\|_r$ proportional to ϵ. Thus, for any given $\delta > 0$, the lemma will hold for $0 < \epsilon < \epsilon_0$ and ϵ_0 small enough. □

The following lemma proves that any sufficiently smooth periodic signal can be perturbed to a nearby periodic signal with finitely many critical points.

Lemma 3. *If $o' \in \mathcal{O}^r$, $r \geq 2$, is a periodic signal, there exists another periodic signal o of the same period with $d_r(o, o')$ arbitrarily small and such that o has only finitely many critical points (including local maxima and minima) and 0 is a regular value of do/dt.*

Proof. If $o'(t)$ is constant we can perturb to $\epsilon \sin(tT/2\pi)$ for arbitrarily small ϵ and verify the theorem. We will assume that o' is not constant.

Consider $\frac{do'}{dt}(t)$ as a map from the circle $[0, T)$ to $\mathbb{R}$. If 0 is a regular value of this map, we are done by Lemma 1.

If not, there exists a regular value ϵ of do'/dt arbitrarily close to 0 by Sard's theorem (here $r \geq 2$ is needed). Suppose we look at $do'(t)/dt - \epsilon$. This function has a regular value at 0. However, the corresponding perturbation of o' is $o'(t) - t\epsilon$ and is not periodic.

Because $o'(t)$ is not constant, there exists an interval (α, β) in the circle $[0, T)$ over which $do'(t)/dt$ is nonzero. Without loss of generality, we assume $do'(t)/dt > \delta > 0$ in the interval (α, β) (consider $-o'(t)$ for the case where the derivative is negative). Using Lemma 2, we may find a periodic signal $p(t)$ such that $dp/dt = \epsilon$ for $t \notin (\alpha, \beta)$ and $|dp/dt| < \delta$ for $t \in (\alpha, \beta)$. Set $o(t) = o'(t) - p(t)$ to obtain a periodic signal with 0 being a regular value of do/dt to complete the proof. □

Remark. Lemma 1 is evidently true if we only assume the second derivative of the periodic signal $o(t)$ to exist and not necessarily continuous. In fact, Lemma 3 is also true under the same weaker assumption because, in one dimension, Sard's theorem requires only the existence of the derivative (see Exercise 1 of Section 3.1 of [8]).

The proof of Lemma 3 may be illustrated using Fig. 2.4. The figure shows a part of the graph of $f(x) = x^3 \sin(1/x)$ and its

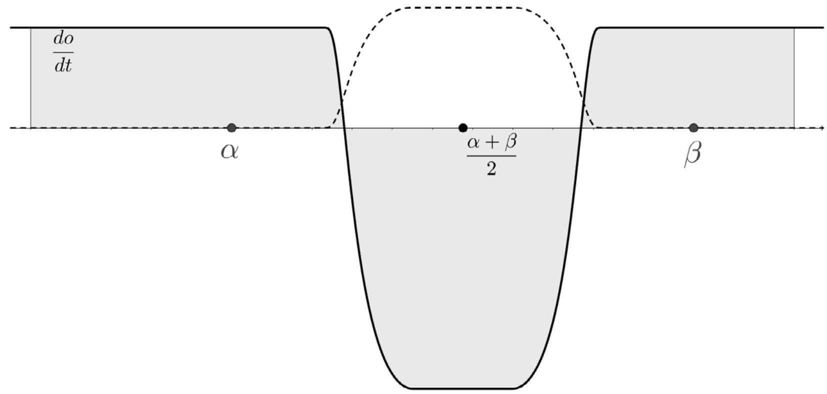

Fig. 2.3. An infinitely differentiable (bump) function (dashed line), which is zero outside (α, β) and 1 near the middle of that interval, subtracted from a constant value of $\frac{do}{dt}$. If the amount subtracted is adjusted, the integral of $\frac{do}{dt}$ over one full period becomes zero as shown.

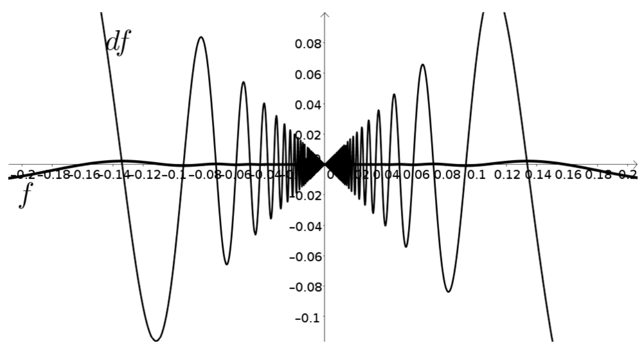

Fig. 2.4. The function $f(x) = x^3 \sin\left(\frac{1}{x}\right)$ and its derivative.

derivative $\dot{f}(x)$. It is evident that the critical points of $\dot{f}$, where $\ddot{f}(x) = 0$, accumulate at the origin. In fact, a small perturbation cannot eliminate the accumulation of critical points because $f(x)$ does not have a second derivative at $x = 0$. However, if $f(x) = x^5 \sin(1/x)$, a function whose second derivative looks like the derivative shown in Fig. 2.4, Sard's theorem may be used to obtain a small perturbation such that 0 is a regular value of the derivative of the perturbed function.

If o is a periodic signal with finitely many critical points, then its circular domain $[0, T)$ may be decomposed into finitely many intervals with local minima, maxima, or a critical point that is neither at either end. Let μ denote the minimum width among such intervals. Because $o(t)$ is monotonic in each interval, we refer to each such interval as an interval of strict monotonicity. If the delay is τ, we denote the point $(o(t), o(t-\tau), o(t-2\tau))$ by $o(t;\tau)$.

Lemma 4. *If $0 < |t_1 - t_2| \leq \mu/3$, where μ is the minimum length of an interval of strict monotonicity, and if the delay τ satisfies $0 < \tau \leq \mu/3$, then $o(t_1;\tau) \neq o(t_2;\tau)$. If 0 is a regular value of $\frac{do(t)}{dt}$, we also have $\frac{do(t;\tau)}{dt} \neq 0$ for all $t \in [0, T)$.*

Proof. Because $|t_1 - t_2| \leq \mu/3$, t_1 and t_2 lie in either the same interval of strict monotonicity of the periodic signal $o(t)$ or in neighboring intervals. If they lie in the same interval, we must have either $o(t_1) < o(t_2)$ or $o(t_2) < o(t_1)$ proving the lemma.

If t_1 and t_2 lie in neighboring intervals, we may assume $t_1 < t_2$ without loss of generality. If $o(t_1) \neq o(t_2)$, there is nothing to prove. So we assume $o(t_1) = o(t_2)$ in addition. Again without loss of generality, we assume that $o(t)$ first increases and then decreases as t increases from t_1 to t_2.

With these assumptions, t_1 and $t_1 - \tau$ must lie in the same interval of monotonicity because $\tau \leq \mu/3$, and therefore $o(t_1 - \tau) < o(t_1)$. Further $t_2 - \tau \in (t_1 - \tau, t_2)$ and the unique minimum of $o(t)$ for $t \in [t_1 - \tau, t_2]$ is attained when $t = t_1 - \tau$. Therefore $o(t_1 - \tau) < o(t_2 - \tau)$, and we once again have $o(t_1;\tau) \neq o(t_2;\tau)$.

For the claim about $\frac{do(t;\tau)}{dt} \neq 0$, we note that $\frac{do}{dt}$ cannot equal zero at both t and $t - \tau$, because $\tau < \mu$. $\square$

With Lemma 4, the local argument for embedding periodic signals is partly complete. Globalizing the argument will involve additional perturbations, which we now define.

Let λ be a C^∞ bump function with $\lambda(x) = 1$ for $|x| \leq 1/2$, $\lambda(x) = 0$ for $|x| \geq 1$, and $\lambda(x) \in [0, 1]$ for all $x \in \mathbb{R}$. Let $h = \tau/2$ and $j \in \mathbb{Z}$. Define

$$\lambda_j(t) = \lambda\left(\frac{t - jh}{h}\right)$$

for $j = 0, 1, \ldots, n$ and $n = \lfloor T/h \rfloor$. We interpret t modulo T and regard $\lambda_j(t)$ as a periodic signal with the circular domain $[0, T)$: a pulse of period T and width h centered at jh which is equal to 1 for $|t - jh| \leq h/2$. We now consider the perturbation

$$o_\epsilon(t) = o(t) + \epsilon_0\lambda_0(t) + \epsilon_1\lambda_1(t) + \cdots + \epsilon_n\lambda_n(t), \tag{2.2}$$

where $\epsilon = (\epsilon_0, \ldots, \epsilon_n) \in \mathbb{R}^{n+1}$. For any $t_0 \in [0, T)$, there exists a bump function $\lambda_j(t)$ with $0 \leq j \leq n$ such that $\lambda_j(t_0) = 1$ and therefore $\lambda_j(t) = 0$ if $|t - t_0| \geq \tau = 2h$.

Before we turn to the global argument, we must prove that the local structure asserted by Lemma 4 is preserved when o is perturbed to o_ϵ as in (2.2). The lemma below guarantees $o_\epsilon(t_1;\tau) \neq o_\epsilon(t_2;\tau)$ for $|t_1 - t_2| \leq 3\tau$. The bound 3τ ensures that $o_\epsilon(t_1;\tau) = o_\epsilon(t_2;\tau)$ can happen only when the intervals $[t_1 - 2\tau, t_1]$ and $[t_2 - 2\tau, t_2]$ do not overlap.

Lemma 5. *Let $o \in \mathcal{O}^r$, $r \geq 2$, be a periodic signal defined over the domain $[0, T)$ and with minimum interval of strict monotonicity equal to μ. Assume that 0 is a regular value of do/dt. There exists ϵ_0 such that if $\|\epsilon\| \leq \epsilon_0$, then for the perturbation defined by (2.2) and delay τ satisfying $0 < \tau < \mu/12$, we have $o_\epsilon(t_1;\tau) \neq o_\epsilon(t_2;\tau)$ for all (t_1, t_2) with $|t_1 - t_2| \leq 3\tau$. In addition, 0 remains a regular value of $\frac{do_\epsilon}{dt}$.*

Proof. By assumption the periodic signal $o(t)$ has finitely many critical points. Let $t_1 < t_2 < \cdots < t_k$ be the critical points in the circular interval $[0, T)$; at these points and only at these, we have $do/dt = 0$. Since 0 is a regular value of do/dt, we have $\frac{d^2o(t_j)}{dt^2} \neq 0$ for $j = 1, \ldots, k$.

In the circle $[0, T)$, choose compact intervals $K_i = [t_i - \delta, t_i + \delta]$, $i = 1, \ldots, k$, such that $\delta < \mu/4$ and $\frac{d^2o(t)}{dt^2} \neq 0$ for any $t \in K_i$. By continuity in the perturbing parameters ϵ_i, for sufficiently small

$\|\epsilon\|$ the perturbed periodic signal (2.2) also has nonzero second derivative on $\cup K_i$.

Define the interval K_i' to be $[t_i + \delta/2, t_{i+1} - \delta/2]$ (K_k' wraps around the circle). Each K_k' is an interval of strict monotonicity. By compactness, $|do/dt|$ attains a minimum strictly greater than 0 over $\cup K_i'$. Again by continuity, any perturbation of the form (2.2) with $\|\epsilon\|$ sufficiently small also has nonzero derivative over $\cup K_i'$.

Thus, for $\|\epsilon\|$ sufficiently small, K_i' remain intervals of strict monotonicity for the perturbed periodic signal, and each K_i can contain at most one critical point of the perturbed periodic signal. The minimum interval of strict monotonicity is at least $\mu - \delta \geq 3\mu/4$. We now apply Lemma 4 to infer that $0 < \tau \leq \mu/4$ implies $o_\epsilon(t_1; \tau) \neq o_\epsilon(t_2; \tau)$ for $0 < |t_1 - t_2| \leq \mu/4$. We limit τ to the interval $(0, \mu/12)$ to complete the proof. □

2.2. Global argument for periodic signals

The global argument relies on the parametric transversality theorem [8,9].

Lemma 6. *Let $o \in \mathcal{O}^r$, $r \geq 2$, be a periodic signal defined over the circle $[0, T)$. There exists an arbitrarily small perturbation of the periodic signal o to o', with the same period, and a delay $\tau > 0$, such that $t \to o'(t; \tau)$ is an embedding, with 0 a regular value of do'/dt.*

Proof. By Lemma 3, we may make an initial perturbation to o if necessary and assume that o has finitely many critical points, that 0 is a regular value of do/dt, and that $\mu > 0$ is the minimum width of an interval of strict monotonicity.

Now consider perturbations of o to o_ϵ of the form (2.2). By Lemma 5, we may assume $o_\epsilon(t_1; \tau) \neq o_\epsilon(t_2; \tau)$ for $t_1 \neq t_2$ and $|t_1 - t_2| \leq 3\tau$ for $\tau < \mu/12$, *provided* $\|\epsilon\|$ is sufficiently small.

Consider the set

$$\mathcal{T} = \left\{(t_1, t_2) \middle| |t_1 - t_2| > 3\tau,\ t_1 \in [0, T),\ t_2 \in [0, T)\right\},$$

where $[0, T)$ is interpreted as the circle, as before. For the applicability of the parametric transversality theorem later in the proof, it is important to note that $\mathcal{T}$ is a manifold of dimension 2 *without* a boundary.

Consider $(o_\epsilon(t_1; \tau), o_\epsilon(t_2; \tau))$ as a function from the domain $\{(\epsilon_1, \dots \epsilon_n)\} \times \mathcal{T}$ to $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$. We will now verify that this function is transverse to the diagonal in $\mathbb{R}^3 \times \mathbb{R}^3$. If $o_\epsilon(t_1; \tau) \neq o_\epsilon(t_2; \tau)$ there is nothing to prove. Suppose $o_\epsilon(t_1; \tau) = o_\epsilon(t_2; \tau)$ and consider the point in $\mathbb{R}^6$ given by

$$(o_\epsilon(t_1), o_\epsilon(t_1 - \tau), o_\epsilon(t_1 - 2\tau), o_\epsilon(t_2), o_\epsilon(t_2 - \tau), o_\epsilon(t_2 - 2\tau))$$

The intervals $[t_1 - 2\tau, t_1]$ and $[t_2 - 2\tau, t_2]$ are disjoint because $|t_1 - t_2| > 3\tau$. By construction, there exist $i_1, i_2, i_3, i_4, i_5, i_6$ such that $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3}, \lambda_{i_4}, \lambda_{i_5}, \lambda_{i_6}$ are each equal to 1 at exactly one of the six points $t_1, t_1 - \tau, t_1 - 2\tau, t_2, t_2 - \tau, t_2 - 2\tau$ and zero at the others. If the tangent direction in the domain is taken to perturb ϵ_{i_j} for $j \in \{1, \dots, 6\}$, it maps to a perturbation of the jth coordinate in $\mathbb{R}^6$, more precisely the elementary vector $\boldsymbol{e}_j$. Therefore, the tangent map is surjective and transversality is verified.

By the parametric transversality theorem [Hirsch, Chapter 3, Theorem 2.7], we may choose ϵ arbitrarily small such that $(o_\epsilon(t_1; \tau), o_\epsilon(t_2; \tau))$ considered as a function from $\mathcal{T}$ to $\mathbb{R}^6$ is transverse to the diagonal of $\mathbb{R}^3 \times \mathbb{R}^3$. Since $\mathcal{T}$ is of dimension 2, that can only happen if $o_\epsilon(t_1; \tau) \neq o(t_2; \tau)$ for $(t_1, t_2) \in \mathcal{T}$.

To complete the proof, we only need to check the smoothness/dimension condition in the parametric transversality theorem. The dimension of $\mathcal{T}$ is 2 and the codimension of the diagonal in $\mathbb{R}^6$ is 3. Thus, it is sufficient if the map from $\{(\epsilon_1, \dots \epsilon_n)\} \times \mathcal{T}$ to $\mathbb{R}^6$ is C^1 which it is. □

Lemma 7. *Let $o \in \mathcal{O}^r$, $r \geq 2$, be a periodic signal such that $t \to o(t; \tau)$ is an embedding of the circle $[0, T)$ in $\mathbb{R}^3$ for delay $\tau > 0$. There exists $\epsilon_0 > 0$ such that $d_r(o, o') < \epsilon_0$ and $T = T'$ (perturbation has same period) imply that $t \to o'(t; \tau)$ is also an embedding of the circle $[0, T)$.*

Proof. By the inverse function theorem (see [8, Appendix]), there exists $\epsilon_0 > 0$ such that for every $\tilde{t} \in [0, T)$ there exists a neighborhood of $\tilde{t}$ over which $t \to o'(t; \tau)$ is an injection if $d_r(o', o) < \epsilon_0$ and $T = T'$. Using a Lebesgue-δ argument we may assume that $o'(t_1; \tau) \neq o'(t_2; \tau)$ for $0 < |t_1 - t_2| < \epsilon_0$, making ϵ_0 smaller if necessary.

Although arguments like the one above are common in differential topology, we state the version of the inverse function theorem invoked for clarity. The version used is as follows. Suppose f is a C^r map from U, an open subset of $\mathbb{R}^m$ to V, an open subset of $\mathbb{R}^n$ with $m < n$. Suppose $f(x) = y$ and that the tangent map $\frac{\partial f}{\partial x}$ is injective at x. Then there exists a neighborhood $\mathcal{N}$ of f in the weak C^r topology ($r \geq 1$), a neighborhood U' of x, V' of y, and W' of $0 \in \mathbb{R}^{n-m}$, such that for every $g \in \mathcal{N}$ there exists a diffeomorphism $G : V' \to U' \times W'$ with G^{-1} restricted to $U' \times 0$ coinciding with g. This theorem is applied with $m = 1$ and $n = 3$.

The rest of the proof is a standard compactness argument. Let

$$\min_{|t_1 - t_2| \geq \epsilon_0} |o(t_1; \tau) - o(t_2; \tau)| = \delta > 0,$$

where the minimum exists because of compactness and is greater than 0 because $t \to o(t; \tau)$ is an embedding. By continuity, the minimum must be positive for o' sufficiently close to o. Similarly, immersivity of o' sufficiently close to o is a direct consequence of compactness of the circle. Thus, $t \to o'(t; \tau)$ is also an embedding. □

Theorem 8. *The set of periodic signals $o \in \mathcal{O}^r$, of period T and with $r \geq 2$, for which there exists a delay $\tau > 0$ such that $t \to o(t; \tau)$, $0 \leq t < T$, is an embedding of the circle in $\mathbb{R}^3$ is open and dense in $\mathcal{O}^r$.*

Proof. By Lemma 6, there exists an arbitrarily small perturbation to o' such that $t \to o'(t; \tau)$ is an embedding for $0 < \tau < \tau_0$ and with 0 a regular value of do'/dt. Thus the set of periodic signals with a delay embedding and with 0 a regular value of do/dt is dense. We only have to prove that the set is open.

Given periodic signal o with $t \to o(t; \tau)$ an embedding, Lemma 7 shows that $t \to o'(t; \tau)$ remains an embedding for $d_r(o, o')$ sufficiently small if $T = T'$. If $T \neq T'$, we may still apply Lemma 7, by defining $o''(t) = o'(tT'/T)$ which is a periodic signal of period T. If $d_r(o, o') \to 0$,then $d_r(o, o'') \to 0$. Finally, $t \to o''(t; \tau)$ is an embedding implies that $t \to o'(t; \tilde{\tau})$ is an embedding with $\tilde{\tau} = \tau T'/T$. □

Remark. A reviewer has noted that Theorem 8 may be obtained directly from the results of [2]. In outline, suppose $o(t) = \pi_1 \mathbf{p}(t)$, where π_1 is the projection to the first coordinate, and $\mathbf{p}$ is a periodic solution of $dx/dt = f(x)$. The theory of [2] implies that a perturbation of the observation function π_1 will produce a periodic signal whose delay map is an embedding. In principle, this argument allows the delay τ to be arbitrary. There are two difficulties to be overcome, however. First, the theory of [2] must be improved as we point out in [26]. Second, an argument for producing an $f(x)$ with a periodic solution $\mathbf{p}(t)$ such that $o(t) = \pi_1 \mathbf{p}(t)$ must be included. However, when Theorem 8 is applied later it is in a context where $o(t)$ arises as $\pi_1 \mathbf{p}(t)$, and these difficulties can be easily dealt with.

Theorem 9. *Suppose that $o \in \mathcal{O}^r$, $r \geq 2$, and that $t \to o(t; \tau)$ is an embedding of the circle for some delay $\tau > 0$. Then $t \to o(t; \tau')$ remains an embedding if τ' is close enough to τ.*

Proof. The arguments used in Lemma 7 and Theorem 8 apply with little change. □

3. Embedding periodic orbits in $\mathbb{R}^3$

Fig. 3.1 shows a periodic orbit of the classical Lorenz system given by $dx/dt = 10(y-x)$, $dy/dt = -y - xz + 28x$, $dz/dt = -8z/3 + xy$.[1] The signal extracted from that orbit is nearly flat for a significant duration when the origin is approached.

In this section, we will prove that "typical" periodic orbits (in a sense that will be made precise) yield signals that result in embeddings of the circle. The following proposition proves that an embedding using delay coordinates persists when the vector field is perturbed slightly. It is the easier half of the argument.

Proposition 10. *Let $\frac{dx}{dt} = f(x)$, where $x \in \mathbb{R}^d$, $f : U \to \mathbb{R}^d$, and U an open subset of $\mathbb{R}^d$, be a dynamical system with f a C^{r-1} vector field, $r \geq 2$. Let $\mathbf{p} : [0,T) \to U$ be a hyperbolic periodic solution of period $T > 0$. Let $\mathbf{a} \in \mathbb{R}^d$ and $\mathbf{a} \neq 0$. Assume that $t \to (\mathbf{a}\cdot\mathbf{p}(t), \mathbf{a}\cdot\mathbf{p}(t-\tau), \mathbf{a}\cdot\mathbf{p}(t-2\tau))$ be an embedding of the circle $[0,T)$ in $\mathbb{R}^3$. There exists an open neighborhood of f in the C^{r-1} topology such that for each g in that neighborhood, there exists a C^r-close hyperbolic periodic solution $\mathbf{p}'(t)$ of period T' of $\frac{dx}{dt} = g(x)$ and a τ' close to τ such that $t \to (\mathbf{a}\cdot\mathbf{p}'(t), \mathbf{a}\cdot\mathbf{p}'(t-\tau'), \mathbf{a}\cdot\mathbf{p}'(t-2\tau'))$ is an embedding of the circle $[0,T')$ in $\mathbb{R}^3$.*

Proof. The fact that a hyperbolic periodic solution such as $\mathbf{p}$ perturbs to a nearby hyperbolic solution $\mathbf{p}'$ in a small enough open neighborhood of f is a standard result [13, Chapter 5]. If the signal $o(t) = \mathbf{a}\cdot\mathbf{p}(t)$ is such that $t \to o(t;\tau)$ is an embedding of the circle, then $t \to o'(t;\tau')$ is also an embedding for $o'(t) = \mathbf{a}\cdot\mathbf{p}'(t;\tau')$ by Theorem 8. The proof of Theorem 8 uses the choice $\tau' = \tau T'/T$. □

Suppose that the delay map of a signal obtained by projecting the first component of a periodic orbit does not embed in $\mathbb{R}^3$. We will show that the differential equation $\frac{dx}{dt} = f(x)$, $x \in \mathbb{R}^d$, can be perturbed ever so slightly such that a nearby periodic orbit of the perturbed equation results in an embedding of the circle. The proof relies on constructing a tube around the periodic orbit. A tube around a periodic orbit is illustrated in Fig. 3.2.

To construct a tube around any periodic orbit in $\mathbb{R}^d$, we begin by defining $\mathcal{P}^r$ in analogy to $\mathcal{O}^r$. Let $\mathcal{P}^r$ be the set of periodic orbits $\mathbf{p} : [0,T) \to \mathbb{R}^d$ that are r times continuously differentiable. As before, we assume that $[0,T)$ is a parametrization of S^1 and $T > 0$ for the period. As a part of the definition of $\mathcal{P}$, we require $\frac{d\mathbf{p}}{dt} \neq 0$ for $t \in [0,T)$. The set $\mathcal{P}^r$ is endowed with a topology by defining the metric d_r in analogy with (2.1):

$$d_r(\mathbf{p},\mathbf{p}') = \sup_{k=0,\ldots,r}\ \sup_{0\leq s<1} \left\| \mathbf{p}^{(k)}(sT) - \mathbf{p}'^{(k)}(sT') \right\| + |T - T'|.$$

The norm over $\mathbb{R}^d$ is the 2-norm. The kth derivative of $\mathbf{p}$ is denoted by $\mathbf{p}^{(k)}$. For convenience, $\frac{d\mathbf{p}}{dt}$ and $\frac{d^2\mathbf{p}}{dt^2}$ are also denoted as $\dot{\mathbf{p}}$ and $\ddot{\mathbf{p}}$, respectively. The tangent vector at t is defined as $\mathbf{s}(t) = \dot{\mathbf{p}}(t)/\left\|\dot{\mathbf{p}}(t)\right\|$.

We denote the projection from $\mathbb{R}^d$ to the first coordinate by π_1. If $\mathbf{p}$ is a solution of the dynamical system $\frac{dx}{dt} = f(x)$, we wish to show that either $o(t) = \pi_1\mathbf{p}(t)$ is such that $t \to o(t;\tau)$ is an embedding of the circle $[0,T)$ for some delay $\tau > 0$, or that there exists an arbitrarily close perturbed dynamical system $\frac{dx}{dt} = f'(x)$ with a nearby periodic orbit $\mathbf{p}'$ such that $t \to o'(t;\tau)$ is an embedding of the circle, if $o' = \pi_1 \circ \mathbf{p}'$.

To begin with, the signal $o(t)$ may even be identically zero. In our proof, we use the results of the previous section to perturb it to $o'(t)$ such that $t \to o'(t;\tau)$ is an embedding and then show how to perturb the flow to realize $o'(t)$ as $\pi_1 \circ \mathbf{p}'$.

The next lemma constructs a tube around the periodic orbit $\mathbf{p}$ in $\mathbb{R}^d$ (see Fig. 3.2). That tube will be used to perturb f to f'. Known results in differential geometry [28,29] may be used to assert the existence of a tube. However, uniformity and smoothness guarantees that we need could not be found in the literature. Therefore, an elementary proof of the lemma is included. The proof will later be modified to deduce the existence of a tube whose radius is uniform in a neighborhood of $\mathbf{p}$. In the following lemma, δ may be thought of as the radius of a tube around $\mathbf{p}$.

Lemma 11. *Suppose $\mathbf{p} \in \mathcal{P}^r$, $r \geq 2$, and that its period is $T > 0$. Then there exists $\delta > 0$ such that*

- $\left\|\dot{\mathbf{p}}(t)\right\|^2 - \delta\left\|\ddot{\mathbf{p}}\right\| > \delta$ *for* $t \in [0,T)$,
- *if $x \in \mathbb{R}^d$ and $dist(x,\mathbf{p}) \leq \delta$, there exists a unique $t \in [0,T)$ such that $dist(x,\mathbf{p}) = \left\|x - \mathbf{p}(t)\right\|$.*

Proof. The proof is organized so as to be easy to uniformize in the next lemma.

1. *Choice of* $\mathfrak{m}$ *and* $\mathfrak{m}^*$. Let $2\mathfrak{m} = \min_{t\in[0,T)} \left\|\dot{\mathbf{p}}(t)\right\| > 0$ and $\mathfrak{m}^* = \max_{t\in[0,T)} \left\|\ddot{\mathbf{p}}(t)\right\|$. The first part of the lemma would be satisfied if $4\mathfrak{m}^2 - \delta\mathfrak{m}^* > \delta$, or if $\delta < \frac{4\mathfrak{m}^2}{1+\mathfrak{m}^*}$.
2. *Choice of* $\mathfrak{M}$ *and* $\mathfrak{r}$. First, we introduce the notation

$$\left.\frac{d\mathbf{p}}{dt}\right|_{[t_1,t_2]}$$

for a vector each of whose components is the corresponding component of $\dot{\mathbf{p}}$ evaluated at some $t \in [t_1,t_2]$. Crucially, each component may chose a different t. This notation will facilitate application of the mean value theorem. The interval $[t_1,t_2]$ may wrap around $[0,T)$, in which case the interval width must be taken to be $T + t_2 - t_1$ and not $t_2 - t_1$. We ignore such wrap-arounds from this point onwards.
Suppose $t_1 < t_2$ and $t_m = \frac{t_1+t_2}{2}$. Then

$$\left\| \dot{\mathbf{p}}(t_m) - \left.\frac{d\mathbf{p}}{dt}\right|_{[t_1,t_2]} \right\| \leq \max_{t\in[0,T)} \left\|\ddot{\mathbf{p}}\right\|_\infty \sqrt{d}(t_2 - t_1).$$

The $\sqrt{d}$ factor here arises in converting a componentwise bound using the ∞-norm to a bound on the 2-norm. Evidently, if we choose $\mathfrak{M} = \max_{t\in[0,T)} \left\|\ddot{\mathbf{p}}\right\|_\infty \times \sqrt{d}$ and $\mathfrak{r} = \frac{\mathfrak{m}}{\mathfrak{M}}$, we may assert that

$$\left\| \dot{\mathbf{p}}(t_m) - \left.\frac{d\mathbf{p}}{dt}\right|_{[t_1,t_2]} \right\| \leq \mathfrak{m} \tag{3.1}$$

for $t_1 < t_2$ and $t_2 - t_1 \leq \mathfrak{r}$.
If $\mathbf{s}(t_m)$ is the unit tangent vector to $\mathbf{p}$ at t_m, we have

$$\begin{aligned}\mathbf{s}(t_m)\cdot(\mathbf{p}(t_2) - \mathbf{p}(t_1)) &= \mathbf{s}(t_m)\cdot\left(\left.\frac{d\mathbf{p}}{dt}\right|_{[t_1,t_2]}(t_2 - t_1)\right)\\ &= \mathbf{s}(t_m)\cdot\dot{\mathbf{p}}(t_m)(t_2 - t_1) + \mathbf{s}(t_m)\\ &\quad\cdot\left(\left.\frac{d\mathbf{p}}{dt}\right|_{[t_1,t_2]} - \dot{\mathbf{p}}(t_m)\right)(t_2 - t_1),\end{aligned}$$

where the first equality is obtained by applying the mean value theorem to each component of $\mathbf{p}(t_2) - \mathbf{p}(t_1)$. Now, $\mathbf{s}(t_m)\cdot\dot{\mathbf{p}}(t_m) = \left\|\dot{\mathbf{p}}(t_m)\right\| \geq 2\mathfrak{m}$ by choice of $\mathfrak{m}$. By (3.1), the second term in the display above is at most $\mathfrak{m}(t_2 - t_1)$ in magnitude. Therefore,

$$|\mathbf{s}(t_m)\cdot(\mathbf{p}(t_2) - \mathbf{p}(t_1))| \geq \mathfrak{m}(t_2 - t_1)$$

for $t_1 < t_2$ and $t_2 - t_1 \leq \mathfrak{r}$.

[1] The periodic orbit of Fig. 3.1 in [27] could not be computed using the techniques of [27]. It was computed some years later using an initial guess that was constructed from the periodic orbit $A^{25}B^{25}$.

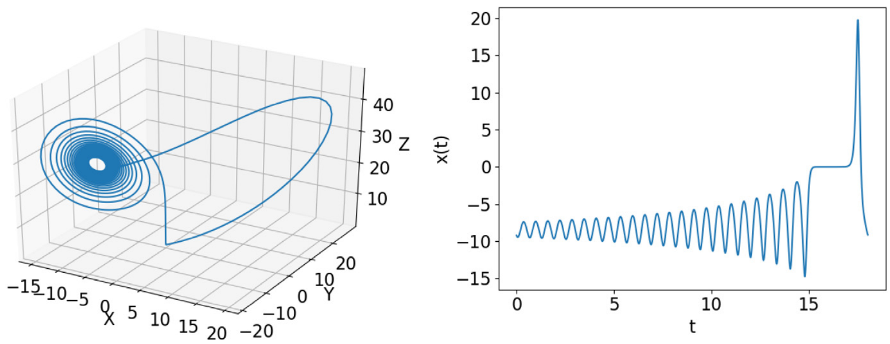

Fig. 3.1. A periodic orbit of the classical Lorenz system and its x-coordinate as a function of time (over a single period). The periodic orbit shown is $A^{24}B$ in the nomenclature of [27].

3. *Choice of* $\mathfrak{M}^*$. Suppose $\mathbf{w}_1$ is a vector orthogonal to $\mathbf{s}(t_1)$ and $t_1 < t_2$ with $t_m = \frac{t_1+t_2}{2}$ as before. Then, we have $\mathbf{s}(t_m)\cdot\mathbf{w}_1 = (\mathbf{s}(t_m) - \mathbf{s}(t_1))\cdot\mathbf{w}_1$, which implies
$$\begin{aligned}|\mathbf{s}(t_m)\cdot\mathbf{w}_1| &\le \left\|\mathbf{s}(t_m) - \mathbf{s}(t_1)\right\| \left\|\mathbf{w}_1\right\| \\ &\le \sqrt{d}\max_{t\in[0,T)} \left\|\dot{\mathbf{s}}(t)\right\|_\infty (t_m - t_1)\left\|\mathbf{w}_1\right\|,\end{aligned}$$
where the $\sqrt{d}$ factor arises in converting a componentwise bound to a bound on the 2-norm. An explicit formula for $\dot{\mathbf{s}}$, the time derivative of the unit tangent, will be given in the next proof. If we choose $\mathfrak{M}^* = \sqrt{d}\max_{t\in[0,T)}\left\|\dot{\mathbf{s}}(t)\right\|_\infty$, we may replicate the argument given using $\mathbf{w}_1, t_1$ with $\mathbf{w}_2, t_2$ and assert
$$\begin{aligned}|\mathbf{s}(t_m)\cdot\mathbf{w}_1| &< \mathfrak{M}^*\left\|\mathbf{w}_1\right\|(t_2 - t_1) \quad \text{and} \quad |\mathbf{s}(t_m)\cdot\mathbf{w}_2| \\ &< \mathfrak{M}^*\left\|\mathbf{w}_2\right\|(t_2 - t_1).\end{aligned}$$
4. *Choice of* Δ. We define $\Delta = \min_{|t_2-t_1|\ge\mathfrak{r}}\left\|\mathbf{p}(t_2) - \mathbf{p}(t_1)\right\|$. Because a periodic orbit cannot self-intersect, we must have $\Delta > 0$.

We will choose δ to be smaller than the least of
$$\frac{4\mathfrak{m}^2}{1+\mathfrak{m}^*}, \frac{\mathfrak{m}}{2\mathfrak{M}^*}, \frac{\Delta}{2}.$$
The first part of the lemma follows immediately. Now suppose $x\in\mathbb{R}^d$ and $\text{dist}(x,\mathbf{p}) \le \delta$. Suppose $\text{dist}(x,\mathbf{p})$ is equal to $\left\|x - \mathbf{p}(t_1)\right\|$ as well as $\left\|x - \mathbf{p}(t_2)\right\|$ for $t_1 < t_2$. By item 4 above, we must have $t_2 - t_1 < \mathfrak{r}$, which we will now assume.

Because $t = t_1$ minimizes $(x - \mathbf{p}(t))\cdot(x - \mathbf{p}(t))$, we may differentiate and deduce $(x - \mathbf{p}(t_1))\cdot\dot{\mathbf{p}}(t_1) = 0$. Equivalently $(x - \mathbf{p}(t_1)).\mathbf{s}(t_1) = 0$. Thus, we may write $x = \mathbf{p}(t_1) + \mathbf{w}_1$, with $\mathbf{w}_1$ orthogonal to the tangent $\mathbf{s}(t_1)$ and $\text{dist}(x,\mathbf{p}) = \left\|\mathbf{w}_1\right\|$. Likewise, we may write $x = \mathbf{p}(t_2) + \mathbf{w}_2$, with $\mathbf{w}_2$ orthogonal to the tangent $\mathbf{s}(t_2)$ and $\text{dist}(x,\mathbf{p}) = \left\|\mathbf{w}_2\right\|$.

From $\mathbf{p}(t_1) + \mathbf{w}_1 = \mathbf{p}(t_2) + \mathbf{w}_2$, we obtain
$$\mathbf{s}(t_m)\cdot(\mathbf{p}(t_2) - \mathbf{p}(t_1)) = \mathbf{s}(t_m)\cdot(\mathbf{w}_1 - \mathbf{w}_2)\,.$$
Taking absolute values, applying item 2 above to the left hand side, and item 3 above to the right hand side, we get
$$\mathfrak{m}(t_2 - t_1) < \mathfrak{M}^*\left(\left\|\mathbf{w}_1\right\| + \left\|\mathbf{w}_2\right\|\right)(t_2 - t_1),$$
or $\text{dist}(x,\mathbf{p}) > \frac{\mathfrak{m}}{2\mathfrak{M}^*} \ge \delta$, contradicting our hypothesis about x. Thus, the assumption $t_1 < t_2$ is mistaken, and we can only have $t_1 = t_2$ proving the second part of the lemma. □

The following lemma is a uniform version of the preceding Lemma 11. The lemma allows us to construct a tube of radius δ around all periodic orbits of period T that are within a distance ϵ of $\mathbf{p}$. Its proof is a minor modification of the preceding proof.

Lemma 12. *Suppose* $\mathbf{p}\in\mathcal{P}^r$, $r \ge 2$, *and that its period is* $T > 0$. *Then there exist* $\epsilon > 0$ *and* $\delta > 0$ *such that* $\mathbf{p}' \in \mathcal{P}^r$, *with the same period as* $\mathbf{p}$, *and* $d_r(\mathbf{p},\mathbf{p}') \le \epsilon$ *imply that*

- $\left\|\dot{\mathbf{p}}'(t)\right\|^2 - \delta\left\|\ddot{\mathbf{p}}'\right\| > \delta$ *for* $t\in[0,T)$,
- *if* $x\in\mathbb{R}^d$ *and* $\text{dist}(x,\mathbf{p}') \le \delta$, *then there exists a unique* $t\in[0,T)$ *such that* $\text{dist}(x,\mathbf{p}') = \left\|x - \mathbf{p}'(t)\right\|$.

Proof. In the previous proof, we demonstrated the existence of a δ that works for $\mathbf{p}$. This proof comes down to choosing ϵ so that $\mathfrak{m}, \mathfrak{m}^*, \mathfrak{M}, \mathfrak{r}, \mathfrak{M}^*$, and Δ work for all $\mathbf{p}'$ with the same period as $\mathbf{p}$ and satisfying $d_r(\mathbf{p},\mathbf{p}') \le \epsilon$.

The quantity $\mathfrak{m}$ is a lower bound on $\left\|\dot{\mathbf{p}}(t)\right\|$. Because ϵ controls $\left\|\dot{\mathbf{p}}(t) - \dot{\mathbf{p}}'(t)\right\|$ over $t\in[0,T)$, we may assume ϵ small enough and replace $\mathfrak{m}$ by $\mathfrak{m}/2$ to make it work for $\mathbf{p}'$.

The quantity $\mathfrak{m}^*$ is an upper bound on $\left\|\ddot{\mathbf{p}}(t)\right\|$. Because ϵ controls $\left\|\ddot{\mathbf{p}}(t) - \ddot{\mathbf{p}}'(t)\right\|$ over $t\in[0,T)$, we may assume ϵ small enough and replace $\mathfrak{m}^*$ by $2\mathfrak{m}^*$ to make it work for $\mathbf{p}'$.

The quantity $\mathfrak{M}$ is essentially an upper bound on $\left\|\ddot{\mathbf{p}}(t)\right\|_\infty$. Because ϵ controls $\left\|\ddot{\mathbf{p}}(t) - \ddot{\mathbf{p}}'(t)\right\|$ over $t\in[0,T)$, we may assume ϵ small enough and replace $\mathfrak{M}$ by $2\mathfrak{M}$ to make it work for $\mathbf{p}'$.

We may use the same definition of $\mathfrak{r} = \frac{\mathfrak{m}}{\mathfrak{M}}$ after modifying $\mathfrak{m}$ and $\mathfrak{M}$ as above.

The quantity $\mathfrak{M}^*$ is essentially an upper bound on $\left\|\dot{\mathbf{s}}(t)\right\|_\infty$. The unit tangent vector $\mathbf{s}$ is given by $\mathbf{s} = \dot{\mathbf{p}}/(\dot{\mathbf{p}}\cdot\dot{\mathbf{p}})^{1/2}$. Differentiating, we obtain
$$\dot{\mathbf{s}} = \frac{\ddot{\mathbf{p}}}{(\dot{\mathbf{p}}\cdot\dot{\mathbf{p}})^{1/2}} - \frac{\dot{\mathbf{p}}\,(\ddot{\mathbf{p}}\cdot\dot{\mathbf{p}})}{(\dot{\mathbf{p}}\cdot\dot{\mathbf{p}})^{3/2}}.$$
Because $r \ge 2$, we may control the variation in $\mathbf{p}$, $\dot{\mathbf{p}}$, and $\ddot{\mathbf{p}}$ by making ϵ small. Thus, we may assume ϵ small enough and replace $\mathfrak{M}^*$ by $2\mathfrak{M}^*$ to make it work for $\mathbf{p}'$.

We begin by defining $\Delta = \min_{|t_2-t_1|\ge\mathfrak{r}}\left\|\mathbf{p}(t_2) - \mathbf{p}(t_1)\right\|$ as before. By assuming ϵ small enough and replacing Δ by $\Delta/2$, we may assume Δ to work for all $\mathbf{p}'$.

The rest of the proof of the previous lemma works without change. □

Half of the smoothness lemma that follows is a special case of the main theorem in [28]. Given a periodic orbit and a tube around it, the lemma shows that each point in the tube can be expressed as a sum of a point on the periodic orbit and a vector orthogonal to the tangent at that point. Additionally, the lemma provides smoothness and uniformity guarantees.

Lemma 13. *Assume the same setting as in Lemma 12. Given* $\mathbf{p}'$ *with* $d_r(\mathbf{p},\mathbf{p}') \le \epsilon$ *and a point* $x_0 \in \mathbb{R}^d$ *with* $\text{dist}(x_0,\mathbf{p}') \le \delta$, *we may send* $x_0 \to t_0$, *where* $\mathbf{p}'(t_0)$ *is the unique point on* $\mathbf{p}'$ *closest to* x_0, *and* $x_0 \to \mathbf{w}_0$, *where* $\mathbf{w}_0 = x_0 - \mathbf{p}'(t_0)$. *The functions* $t_0(x_0)$ *and* $\mathbf{w}_0(x_0)$ *are* C^{r-1}. *In addition, the magnitudes of all derivatives of order* $r-1$ *or less have upper bounds that depend only on* $\mathbf{p}$ *and* δ.

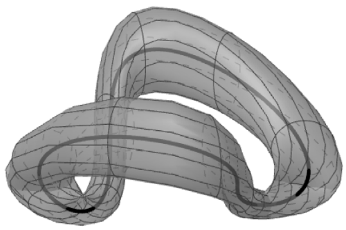

Fig. 3.2. A periodic orbit with a tube around it.

Proof. It is sufficient to prove the lemma for $t_0(x_0)$. The assertions about $\mathbf{w}_0(x_0)$ follow easily from that point.

The function $(x_0 - \mathbf{p}'(t)) \cdot (x_0 - \mathbf{p}'(t))$ has a unique minimum at $t = t_0$. By differentiating, we get the equation $(x_0 - \mathbf{p}'(t_0)).\frac{d\mathbf{p}'(t_0)}{dt} = 0$. If we define

$$\mathfrak{f}(x_0, t_0) = (x_0 - \mathbf{p}'(t_0)).\frac{d\mathbf{p}'(t_0)}{dt}$$

We may think of the equation $\mathfrak{f}(x_0, t_0) = 0$ as implicitly defining $t_0(x_0)$ as a function of x_0. We have

$$\frac{\partial \mathfrak{f}}{\partial t_0} = \ddot{\mathbf{p}}'(t_0) \cdot \left(x_0 - \mathbf{p}'(t_0)\right) - \frac{d\mathbf{p}'(t_0)}{dt} \cdot \frac{d\mathbf{p}'(t_0)}{dt}.$$

Here $\left\|x_0 - \mathbf{p}'(t_0)\right\| = \text{dist}(x_0, \mathbf{p}') \leq \delta$. We may use the first part of Lemma 12 and conclude that the partial derivative $\partial\mathfrak{f}/\partial t_0$ is greater than δ in magnitude.

Thus, the C^{r-1} smoothness of $t_0(x_0)$ follows by the implicit function theorem. To upper bound the magnitudes of the derivatives, we simply have to use chain rule and implicit differentiation. For example, if $x_0 = (\xi_1, \ldots, \xi_d)$, we have

$$\frac{\partial t_0}{\partial \xi_1} = -\frac{\mathbf{e}_1 \cdot \frac{d\mathbf{p}'(t_0)}{dt}}{\frac{\partial \mathfrak{f}}{\partial t_0}}, \tag{3.2}$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0)$. Now the denominator is δ or more in magnitude and the magnitude of the numerator has an upper bound that depends only on $\mathbf{p}'$.

To obtain bounds for derivatives of $t_0(x_0)$ of order $r - 1$ or less, we may repeatedly differentiate (3.2). The bounds on the derivatives obtained in this manner depend only on the first r derivatives of $\mathbf{p}'$ and δ. If we assume $\epsilon < 1$, we may bound the first r derivatives of $\mathbf{p}'$ in terms of the derivatives of $\mathbf{p}$. Thus, the magnitudes of all derivatives of order $r - 1$ or less have upper bounds that depend only on $\mathbf{p}$ and δ. □

Theorem 14. *Let $\mathbf{p}(t)$ be a periodic solution of the dynamical system $d\mathbf{x}/dt = f(\mathbf{x})$, where f is C^{r-1}. If $o(t) = \pi_1\mathbf{p}(t)$ is a periodic signal, there exists either a delay $\tau > 0$ such that $t \to o(t; \tau)$, $0 \leq t < T$, is an embedding of the circle $[0, T)$ or another vector field f', arbitrarily close to f in the C^{r-1} topology, with a periodic solution $\mathbf{p}'(t)$ arbitrarily close to $\mathbf{p}(t)$ in $\mathcal{P}^r$ and of the same period such that $t \to \pi_1\mathbf{p}'(t; \tau)$ is an embedding of the circle $[0, T)$ for some $\tau > 0$.*

Proof. Let $o(t) = \pi_1\mathbf{p}(t)$ and assume that there is no delay $\tau > 0$ such that $t \to o(t; \tau)$ is an embedding. By Lemma 6, we can find a periodic signal $o'(t)$ of period T, and arbitrarily close to $o(t)$ in $\mathcal{O}^r$, such that $t \to o'(t; \tau)$ for some $\tau > 0$. Define

$$\mathbf{p}'(t) = \mathbf{p}(t) + \begin{pmatrix} o'(t) - o(t) \\ 0 \\ \vdots \end{pmatrix}. \tag{3.3}$$

It suffices to construct a vector field f' such that $\mathbf{p}'(t)$ is a periodic solution of $\frac{dx}{dt} = f'(x)$ and $f' \to f$ as $\mathbf{p}' \to \mathbf{p}$.

Using Lemmas 11 and 12, find an $\epsilon > 0$ and a $\delta > 0$, such that a δ-tube may be constructed as in the lemma for all periodic orbits $\mathbf{p}'$ of the same period as $\mathbf{p}$ satisfying $d_r(\mathbf{p}, \mathbf{p}') < \epsilon$. In addition, by taking o' close enough to o, we may assume that $d_r(\mathbf{p}, \mathbf{p}') < \epsilon$.

The following calculation is the heart of the proof:

$$\begin{aligned} \frac{d\mathbf{p}'(t)}{dt} &= \frac{d\mathbf{p}(t)}{dt} + \epsilon_1(t) \\ &= f(\mathbf{p}(t)) + \epsilon_1(t) \\ &= f(\mathbf{p}'(t)) + \epsilon_1(t) + \epsilon_2(t), \end{aligned}$$

where

$$\epsilon_1(t) = \begin{pmatrix} \dfrac{d(o'(t) - o(t))}{dt} \\ 0 \\ \vdots \end{pmatrix}$$

and $\epsilon_2(t) = f(\mathbf{p}(t)) - f(\mathbf{p}'(t))$. Evidently, as $o' \to o$ in $\mathcal{O}^r$, the periodic signals $\epsilon_1(t)$ and $\epsilon_2(t)$ go to 0 in $\mathcal{O}^{r-1}$.

Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a C^∞ bump function with $\lambda(x) = 1$ for $|x| \leq 1/2$ and $\lambda(x) = 0$ for $|x| \geq 3/4$. Suppose x_0 is a point in the δ-tube around $\mathbf{p}'$. Then Lemma 13, allows us to write x_0 as $x_0 = \mathbf{p}'(t_0(x_0)) + \mathbf{w}_0(x_0)$. The perturbation $\delta f : \mathbb{R}^d \to \mathbb{R}^d$ is defined as

$$\delta f(x_0) = (\epsilon_1(t_0(x_0)) + \epsilon_2(t_0(x_0)))\lambda\left(\frac{\mathbf{w}_0(x_0).\mathbf{w}_0(x_0)}{\delta^2}\right)$$

for x_0 in the δ-tube around $\mathbf{p}'$, and zero otherwise. As a consequence of Lemma 13, $\delta f \to 0$ in the C^{r-1} sense as $o' \to o$.

By construction, $\mathbf{p}'(t)$ is a periodic solution of the dynamical system $dx/dt = f'(x)$, with $f' = f + \delta f$. □

Finally, as a consequence of Proposition 10 and Theorem 14, we have the following theorem.

Theorem 15. *Let $\frac{dx}{dt} = f(x)$, where $x \in \mathbb{R}^d$, $f : U \to \mathbb{R}^d$, and U an open subset of $\mathbb{R}^d$, be a C^r, $r \geq 2$, dynamical system. Let $\mathbf{a} \in \mathbb{R}^d$ be a nonzero vector. Let $\mathbf{p} : [0, T) \to U$ be a hyperbolic periodic solution of period $T > 0$. There exists an open neighborhood of f in the C^{r-1} topology such that an open and dense set of g in that neighborhood admit a nearby hyperbolic periodic solution $\mathbf{p}'(t)$ of $dx'/dt = g(x')$ of period T' and a delay $\tau' > 0$ such that the delay map $t \to (\mathbf{a} \cdot \mathbf{p}'(t), \mathbf{a} \cdot \mathbf{p}'(t - \tau'), \mathbf{a} \cdot \mathbf{p}'(t - 2\tau'))$ is an embedding of the circle $[0, T')$ in $\mathbb{R}^3$.*

Proof. Proposition 10 and Theorem 14 imply Theorem 15 with $\mathbf{a} = (1, 0, \ldots, 0)$. The theorem may be reduced to that case for any $\mathbf{a} \neq 0$ by a linear change of variables. □

The theorem does not assert that periodic orbits can be embedded in $\mathbb{R}^3$ for an open and dense set of C^r vector fields g. Instead, the theorem limits itself to a neighborhood of a vector field f which is known to admit a hyperbolic periodic orbit. Such a restriction is essential because there exist open sets of vector fields none of which admit any periodic solution.

4. Discussion

In this paper, we have considered an extension of the delay coordinate embedding theory. The current embedding theory of Sauer et al. [2] is based on fixing the dynamical system and perturbing the observation function. We have obtained an embedding theorem for periodic orbits that fixes the observation function but perturbs the dynamical system.

Periodic solutions are a special case that arise in applications [14,15]. However, a generalization to a broader setting is desirable both from the theoretical point of view as well as for wider applicability.

Our approach in this paper relies heavily on the periodicity of signals. Yet some differences between our approach and that of Sauer et al. may be pertinent to more general settings. The approach of Sauer et al. is able to handle aspects of the embedding result, such as injectivity, immersivity, and distinct points on the same periodic orbit, relatively independently. Our argument is more layered. A global argument is structured above a local argument, and the argument for periodic orbits relies on the argument for periodic signals.

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