DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **17**, Number **6**, September **2012**

doi:10.3934/dcdsb.2012.17.1889

pp. **1889–1902**

ERROR ESTIMATES FOR A BAR CODE RECONSTRUCTION METHOD

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Abstract. We analyze a variational method for reconstructing a bar code signal from a blurry and noisy measurement. The bar code is modeled as a binary function with a finite number of transitions and a parameter controlling minimal feature size. The measured signal is the convolution of this binary function with a Gaussian kernel. In this work, we assume that the blur kernel is known and establish conditions (involving noise level and variance of the convolution kernel) under which the variational method considered recovers essentially the correct bar code.

1. **Introduction.**

In this work, we study a method for decoding bar code signals from a laser-based scanner. This amounts to a deconvolution problem for a binary, 1D signal once the sensing process is modeled. The model and the analysis presented here can also be applied to camera-based bar code scanners that operate by taking a photograph of a bar code and reading the acquired image after some pattern detection. Indeed, there are several issues associated with camera-based scanners analogous to the laser-based scanners; chief among them are deblurring and super-resolution. Both of these signal processing problems may also be approached from the perspective of deconvolution for a binary image. Therefore the discussion in this note could potentially be applied to analysis of camera-based scanners with some modifications, including in decoding of 2D bar codes.

A typical 1D bar code consists of black bars over a white background. A scanner sends out a narrow laser beam which moves across the bar code. The scanner is equipped with a light detector. Therefore, when the laser beam is on the black part of the bar, little light is reflected, and so, little light is detected. When the beam is on the white part, more light is reflected and detected. As the beam moves across the bar code at a constant speed, the amount of light detected, in the form of a voltage reading, is a signal. The peaks of the signal occur when there is a lot of reflection (white parts of the bar code) and the valleys correspond to when there is

2000 *Mathematics Subject Classification.* Primary: 68U10, 65R20, 65R32; Secondary: 49K40, 49N45, 49N60.

Key words and phrases. Deconvolution, inverse problem, variational methods, calculus of variations, error estimates.

little reflection (black parts of the bar code). Thus the signal is an imprint of the bar code.

One of the challenges of bar code reconstruction from the measured voltage reading stems from the fact that the scanning laser beam is of finite width and has an intensity profile across the beam. Thus, the reading of the bar code by the laser is “non-local”. A simple model for the relationship between a bar code and its signal is a convolution.

We start by describing the bar code as a binary function, taking on values of 0 or 1

$$z(x) \in \{0,1\},\ a.e., \quad z \in BV.$$

The effect of the laser on the bar code is modeled by a point-spread function. For simplicity, we will take it to be a scaled Gaussian

$$g_{\alpha,t} = \frac{\alpha}{\sqrt{4\pi t}} e^{-x^2/4t}. \tag{1}$$

The parameter α captures how image intensity is converted to voltage, whereas the parameter t describes the ‘width’ of the laser beam – smaller t makes the beam have a smaller footprint while preserving the amount of energy. The signal recorded from a bar code $z(x)$ is

$$h(x) = \int g_{\alpha,t}(x-y)z(y) + r(x), \tag{2}$$

where $r(x)$ is the noise. We will assume that the noise is bounded in L^2. Note that in this signal model, we have reversed the roles of black and white bars – the signal is large when the laser is on the black bar instead. This is just for convenience. Such a signal may be obtained by subtracting an actual signal from a constant background signal.

For an illustration of our signal model, we refer to Figure 1. The top figure shows a bar code. The middle figure shows its representation as a binary function. The bottom figure displays a simulated signal detected by the scanner.

The bar code decoding problem is to determine $z(x)$, and the scanner parameters α and t given a signal $h(x)$ in (2). When α and t are known, the problem amounts to deblurring. Without using any a priori information on the unknown $z(x)$, the problem is ill-conditioned.

A careful modeling and mathematically rigorous study of the bar code inverse problem was provided in [4]. There, it was shown that under the above convolutional model, a bar code is uniquely determined by its signal. The bar code reconstruction problem was posed as a variational problem, and precise mathematical formulations admitting solutions were provided. The computational approach of the paper was based on the well-known phase-field approximation of Modica-Mortola [7]. The numerical results contained in the paper demonstrated the effectiveness of this approach for the specific problem of bar code reconstruction.

Analysis of conditions under which the reconstructions by models of [4] approximate the true bar code signal was carried out recently in [1] and [2]. In these works, discussed in more detail in the next section, the convolution kernel is assumed known and is taken to be a very specific function with small support and/or the noise in the observed signal is assumed to be absent. Our present work takes the point of view that convolution with wide kernels *and* the presence of noise are essential features of interest in applications. Our goal, motivated by the encouraging numerical evidence from [4], is to exhibit a range for the parameters in the

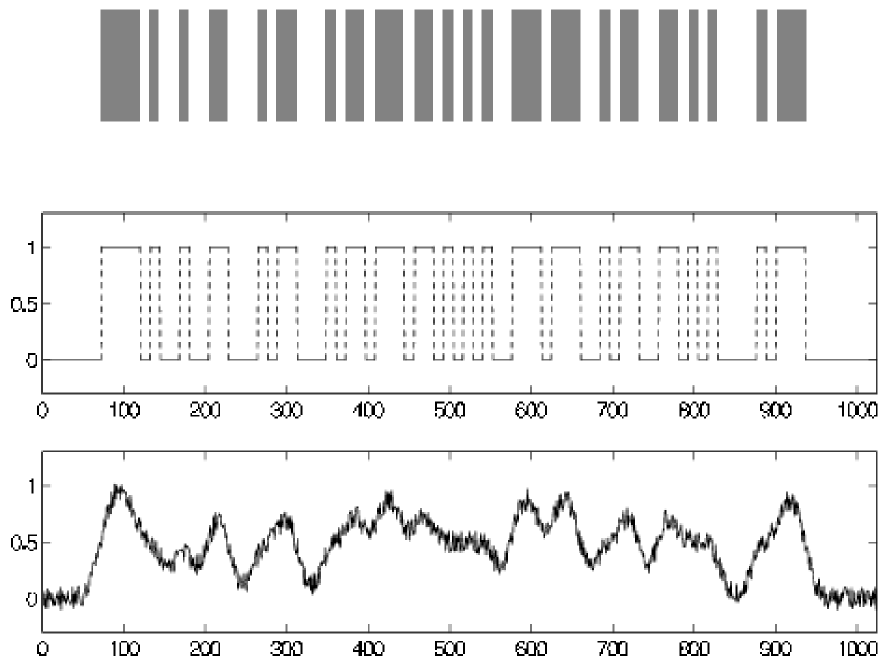

FIGURE 1. A barcode (top), its representation as a binary function (middle), and its corresponding simulated signal (bottom) according to the observation model (2).

models of [4] that lead to good reconstructions even when the convolution kernel, with known α and t, may be very wide compared to the features in the bar code *and* there is substantial noise present in the measurements. We are especially interested in the relationship between reconstruction error, the blur parameter t, and the noise level that can be tolerated.

The paper is organized as follows. In Section 2, we recall in detail the reconstruction method and its approximation from [4]. We also discuss at greater length the previous analytical results of Choksi et. al. that yielded explicit error estimates in special situations. Then, in Section 3, we present our error estimates that apply in the presence of noise and blur.

2. **The reconstruction method.**

The computational approach proposed in [4] is to estimate the bar code $z(x)$ and the parameters α and t as the minimizer of the following energy:

$$E_\varepsilon(u) = \int \left[\varepsilon(u')^2 + \frac{1}{\varepsilon}W(u)\right]\, dx + \lambda \|g_t * u - h\|^2. \tag{3}$$

Here $\varepsilon > 0$ is a small parameter to be chosen by the user, and λ is the *fidelity constant* which can, in principle, be related to the amplitude of noise expected. The function $W(u)$ is the standard double-well potential:

$$W(u) = u^2(1-u)^2, \tag{4}$$

which is displayed in Figure 2.

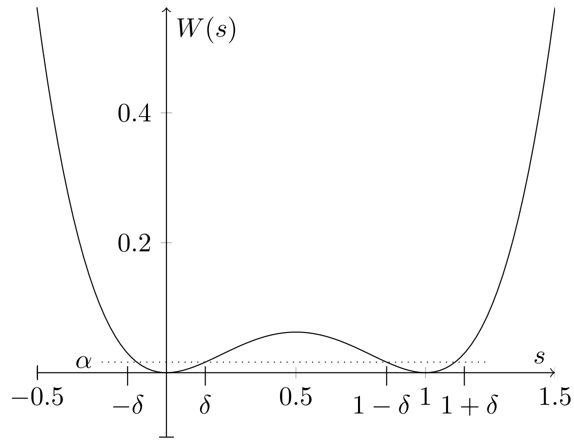

FIGURE 2. The graph of the double well $W(s)$.

In what follows, we will always assume that the Gaussian kernel is known and simply set α to be 1, concentrating on the role of t on the reconstruction. For simplicity of notation, we set $g_t = g_{1,t}$. Energy (3) is used in [4] as an approximation of the sharp interface model

$$\min_{u(x)\in\{0,1\}} c_0 \int |u'|\, dx + \lambda \|g_t * u - h\|^2_{L^2}. \tag{5}$$

which is nothing else than Rudin, Osher, and Fatemi's total variation model [8] in the deblurring setting, restricted to binary functions. The constant $c_0 > 0$ is given by the details of the specific choice of the double well potential W in (3) and depends on its profile between the wells; see [7].

In practice, variational method (3) works very well, and reconstructs the bar codes from very blurry and noisy observations; extensive numerical experiments are provided in [4], including for the case where the parameters α and t in (1) are part of the unknown. We recall from [4] that even when α and t are known, a regularization term that controls the number of transitions (e.g. the total variation) in u is essential. The constraint of $z(x)$ being binary is not enough to control spurious oscillations (interfaces).

It follows from results in standard references [7, 3] that (3) converges in the sense of Gamma convergence to (5) (up to a multiplicative constant in front of the regularization term) as $\varepsilon \to 0^+$. Such convergence results do not always come with explicit error estimates. In this note, our goal is to obtain guaranteed error bounds: identify conditions on the blur level t and noise level $\|r\|_{L^2}$ as well as the approximation parameter ε, that ensure accurate reconstruction of the original bar code $z(x)$. Although this can be approached in two separate steps, namely first studying distance of a minimizer of (5) to the true bar code and then studying convergence rate of (3) to (5), here we directly estimate the distance of the minimizer of (3) from $z(x)$ since this turns out to be more succinct.

In [1], the authors provide error estimates between the true bar code $z(x)$ and the estimated one obtained as the minimizer of energy (5). Their analysis is restricted to a very specific, compactly supported blurring kernel, and needs to assume that the support of the kernel is small compared to the minimal feature size (7) so that no serious blurring has in fact taken place. This strict assumption is relaxed in the more recent work [2] which also considers 2D signals; however, the specific models studied there allow no noise and focus on an unconstrained version of model (5) – i.e. the minimization is over all functions. However, as we show in this note, one can in fact obtain error estimates that are valid in the presence of noise and for a Gaussian with no restriction on its width. In particular, we show that for a given value of the parameter t describing the width of the Gaussian, there exists a (non-zero) noise level and corresponding choice of the parameter λ for which the minimizer of (3) essentially captures the true, uncorrupted signal $z(x)$ provided that ε is small enough. Indeed, owing to the very special form of bar code signals, one can write down a quite explicit error bound in this vein, as we now explore.

3. **Error estimate.** Let us be more precise about the binary functions that represent bar code images. In one dimension, a bar code is the characteristic function of a finite union of disjoint intervals. For convenience, we will assume that the binary function is supported in $(0, 2\pi)$ and has been extended periodically to $\mathbb{R}$. Let N denote the number of intervals present in the support of the bar code. Let $I_j = (b_j - d_j, b_j + d_j)$, with $b_j + d_j < b_{j+1} - d_{j+1}$ for $j = 1, 2, \ldots, N-1$, denote the j-th interval. The width of the interval $|I_j| = 2d_j$. A bar code $z(x)$ will then be represented as

$$z(x) = \sum_{j=1}^{N} \mathbf{1}_{I_j}(x). \tag{6}$$

The places where $z(x)$ is equal to 1 correspond to the black bars. A bar code starts and ends with a white bar, therefore, interlaced between the black bars are $N+1$ white bars. Let us denote the intervals where $z(x) = 0$ by L_j, with $j = 1, 2, \ldots, N+1$. Figure 3 illustrates the situation of the disjoint intervals I_j and L_j.

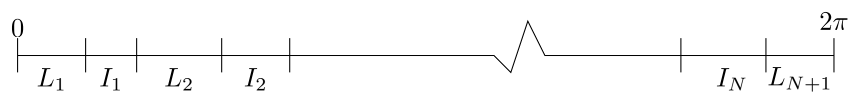

FIGURE 3. The arrangement of the subintervals I_j and L_j on the interval $[0, 2\pi]$.

An important parameter is the *minimal feature size* $\omega > 0$, which we define as

$$\omega = \min\left\{ \min_{j=1,2,\ldots,N} |I_j|, \min_{j=1,2,\ldots,N+1} |L_j| \right\}. \tag{7}$$

In words, ω represents the minimum size of the maximal intervals on which the binary function is constant. Let $B(N, \omega)$ denote the set of all bar codes with at most N intervals and minimal feature size at least ω.

The main result we prove is the following.

Theorem 3.1. *Fix $N \geq 1$, $\omega > 0$, and $t > 0$. Let u_ε be a minimizer of (3). Also, assume that $\varepsilon \in (0, 1)$ and $r \neq 0$ in (2). Then, there exist constants C_1 and C_2 depending only on N and ω such that for*

$$\lambda = \frac{1}{\|r\|_{L^2}^2} \tag{8}$$

we have the error estimate

$$\|u_\varepsilon - z\|_{L^2} \leq C_1 e^{N^2 t} \|r\|_{L^2}^{\frac{1}{4}} \tag{9}$$

whenever $z \in B(N, \omega)$ *and* $\varepsilon \leq \|r\|_{L^2}^2$. *Moreover,*

$$\|u_\varepsilon\|_{BV} \leq C_2 N. \tag{10}$$

The theorem is an error estimate on the reconstructions found by the variational model (3). It says that for the typical values of the parameter λ, the model gives minimizers whose number of transitions is comparable to that of the exact bar code, and the L^2 error of which can be estimated in terms of the noise level.

The proof of the theorem is given towards the end of this section. We start by establishing a number of results needed for the proof.

We first construct a sequence $\{p_\varepsilon(x)\}$ of *prototype* minimizers for energy (3), which is analogous to the recovery sequence in Gamma convergence arguments. Define the function

$$\psi(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ (x+1)/2 & \text{for } -1 < x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}.$$

A prototypical minimizer $p_\varepsilon(x)$ can then be defined as

$$p_\varepsilon(x) = \sum_{j=1}^{N} \left\{ \psi\left(\frac{x - b_j + d_j}{\varepsilon}\right) - \psi\left(\frac{x - b_j - d_j}{\varepsilon}\right) \right\}.$$

One can view $p_\varepsilon(x)$ as a "smoothed" version of the bar code $z(x)$. It is important to keep the bars in the smoothed bar code well separated so we set $\omega > 2\varepsilon$. Define

$$\sigma(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ (x+1)/2 & \text{for } -1 < x < 0 \\ (-x+1)/2 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } x \geq 1 \end{cases},$$

that is, $\sigma(x)$ is a hat function centered at 0 with height 1/2 and width 2. Then

$$z(x) - p_\varepsilon(x) = \sum_{j=1}^{N} \left\{ \sigma\left(\frac{x - b_j + d_j}{\varepsilon}\right) - \sigma\left(\frac{x - b_j - d_j}{\varepsilon}\right) \right\}.$$

We can show by direct calculation that

$$\|p_\varepsilon - z_\varepsilon\|_{L^2}^2 = \frac{N}{3}\varepsilon. \tag{11}$$

Consider the energy in (3) corresponding to $p_\varepsilon(x)$

$$E_\varepsilon(p_\varepsilon) = \int \left[\varepsilon (p_\varepsilon')^2 + \frac{1}{\varepsilon} W(p_\varepsilon) \right] dx + \lambda \|g_t * p_\varepsilon - h\|_{L^2}^2.$$

The contribution from the first term on the right-hand side is easy to calculate and is equal to N. The second term can also be evaluated explicitly, giving $2N/15$. Thus we have

$$E_\varepsilon(p_\varepsilon) \leq \frac{17}{15} N + \lambda \|g_t * p_\varepsilon - h\|_{L^2}^2. \tag{12}$$

Now we use the observation model (2) to get

$$g_t * p_\varepsilon - h = g_t * (p_\varepsilon - z) - r.$$

Since g_t is a Gaussian kernel, the convolution operator associated with it is bounded. Hence

$$\|g_t * (p_\varepsilon - z) - r\|_{L^2}^2 \le C\|p_\varepsilon - z\|_{L^2}^2 + \|r\|_{L^2}^2 \le \frac{CN\varepsilon}{3} + \|r\|_{L^2}^2,$$

upon using (11). Thus from (12) we arrive at the estimate

$$E_\varepsilon(p_\varepsilon) \le 2C_1 N + \lambda\left(C_2 N\varepsilon + \|r\|_{L^2}^2\right), \tag{13}$$

where C_1 and C_2 are constants independent of ε.

Let $u_\varepsilon(x)$ denote a global minimizer of energy (3):

$$u_\varepsilon = \arg\min_{u\in H^1} \int \left[\varepsilon(u')^2 + \frac{1}{\varepsilon}W(u)\right] dx + \lambda\|g_t * u - h\|_{L^2}^2 \tag{14}$$

Also, denote the minimum energy by

$$m_\varepsilon = E_\varepsilon(u_\varepsilon). \tag{15}$$

We will be using well-known ideas from [7] for the next result. Indeed the theory of Modica-Mortola approximation is well developed. However, none of the known results in the literature are in the form that we can directly use. So, instead of starting with known results and modifying them to suit our purpose, we choose to derive the results we need to make the exposition self-contained.

Lemma 3.2. *Let u_ε be a minimizer of (3). Then, there exists a $\gamma \in (0,1)$ such that the binary image*

$$b_\varepsilon(x) = \mathbf{1}_{\{x\,:\,u_\varepsilon > \gamma\}}(x)$$

satisfies

$$\|b_\varepsilon - u_\varepsilon\|_{L^2}^2 \le C_1\varepsilon, \tag{16}$$

and

$$\|b_\varepsilon\|_{BV} \le C_2 m_\varepsilon, \tag{17}$$

where C_1 and C_2 are independent of ε.

Proof. We note that $u_\varepsilon(x)$ may take on values below 0 and above 1, and possibly have oscillations, as shown in the cartoon in Figure 4. Let $\delta > 0$. We partition the domain for u_ε into

$$S_\varepsilon = \{x\,:\,|u_\varepsilon(x)| < \delta \text{ or } |u_\varepsilon(x) - 1| < \delta\},$$

and its complement

$$S_\varepsilon^c = \{x\,:\,|u_\varepsilon(x)| \ge \delta \text{ and } |u_\varepsilon(x) - 1| \ge \delta\}.$$

The set S_ε is the "good set" where u_ε is either near 0 or near 1. Its complement is where u_ε makes rapid transitions or where it undershoots or overshoots. Again, refer to Figure 4.

We first establish that S_ε^c is bounded. Consider the double well function $W(u)$ in (4) which is displayed in Figure 2. If $x \in S_\varepsilon^c$, then $W(u_\varepsilon(x)) \ge \alpha > 0$. Since

$$m_\varepsilon := E_\varepsilon(u_\varepsilon) \ge \int \frac{1}{\varepsilon}W(u_\varepsilon(x))dx \ge \int_{S_\varepsilon^c} \frac{1}{\varepsilon}W(u_\varepsilon(x))dx \ge \frac{\alpha}{\varepsilon}\left|S_\varepsilon^c\right|.$$

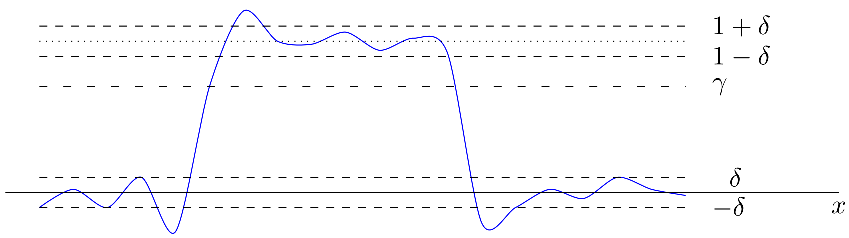

FIGURE 4. A cartoon of a typical section of the minimizer $u_\varepsilon(x)$.

Therefore, we have

$$|S_\varepsilon^c| \le \frac{\varepsilon}{\alpha} m_\varepsilon. \tag{18}$$

Let $b_\varepsilon(x)$ as in Lemma 3.2 with some $\gamma \in (\delta, 1-\delta)$. We split S_ε into two parts:

$$S_\varepsilon^{(1)} = \{x : |u_\varepsilon| < \delta\}, \text{ and } S_\varepsilon^{(2)} = \{x : |u_\varepsilon - 1| < \delta\}.$$

On $S_\varepsilon^{(1)}$, because $\gamma > \delta$, $b_\varepsilon(x) = 0$. Similarly, on $S_\varepsilon^{(2)}$, we have $b_\varepsilon(x) = 1$. Therefore

$$\int_{S_\varepsilon^{(1)}} (u_\varepsilon - b_\varepsilon)^2 dx = \int_{S_\varepsilon^{(1)}} u_\varepsilon^2 dx, \tag{19}$$

and

$$\int_{S_\varepsilon^{(2)}} (u_\varepsilon - b_\varepsilon)^2 dx = \int_{S_\varepsilon^{(1)}} (u_\varepsilon - 1)^2 dx. \tag{20}$$

Because of the specific form of $W(s)$, we have

$$W(s) \ge w_0 s^2 \ \text{ for } -\delta \le s \le \delta,$$
$$W(s) \ge w_0 (s-1)^2 \ \text{ for } 1-\delta \le s \le 1+\delta,$$

for some $w_0 \ge 0$. Using the above in (19), we get

$$\int_{S_\varepsilon^{(1)}} (u_\varepsilon - b_\varepsilon)^2 dx \le \frac{1}{w_0} \int_{S_\varepsilon^{(1)}} W(u_\varepsilon) dx \le \frac{1}{w_0} \int W(u_\varepsilon) dx \le \frac{1}{w_0} \varepsilon m_\varepsilon.$$

Similarly, from(20), we have

$$\int_{S_\varepsilon^{(2)}} (u_\varepsilon - b_\varepsilon)^2 dx \le \frac{1}{w_0} \varepsilon m_\varepsilon.$$

Therefore, we can conclude that

$$\int_{S_\varepsilon} (u_\varepsilon - b_\varepsilon)^2 dx \le \frac{2}{w_0} \varepsilon m_\varepsilon. \tag{21}$$

Next we deal with contributions to the norm of the difference between u_ε and b_ε from the set S_ε^c. We split this set into three parts:

$$S_\varepsilon^{c(1)} = \{x : u_\varepsilon(x) \le -\delta\}, \quad S_\varepsilon^{c(3)} = \{x : u_\varepsilon(x) \ge 1+\delta\},$$

and $S_\varepsilon^{c(2)}$, which is the remainder of the set. For $x \in S_\varepsilon^{c(1)}$, $b_\varepsilon(x) = 0$, and so, we can use the previous argument to get

$$\int_{S_\varepsilon^{c(1)}} (u_\varepsilon - b_\varepsilon)^2 dx \le \frac{1}{w_o} \varepsilon m_\varepsilon.$$

For $x \in S_\varepsilon^{c(3)}$, $b_\varepsilon = 1$, and we get

$$\int_{S_\varepsilon^{c(3)}} (u_\varepsilon - b_\varepsilon)^2 dx \leq \frac{1}{w_o} \varepsilon m_\varepsilon.$$

On $S_\varepsilon^{c(2)}$, $u_\varepsilon(x)$ is going through rapid transitions while b_ε is either 0 or 1. Since $u_\varepsilon(x)$ remains bounded between δ and $1-\delta$, we see that

$$(u_\varepsilon - b_\varepsilon)^2 \leq 1,$$

and therefore we can conclude that

$$\int_{S_\varepsilon^{c(2)}} (u_\varepsilon - b_\varepsilon)^2 dx \leq |S_\varepsilon^{c(2)}| \leq |S_\varepsilon^c| \leq \frac{\varepsilon}{\alpha} m_\varepsilon,$$

by (18). The bounds on the difference over these three parts imply that

$$\int_{S_\varepsilon^c} (u_\varepsilon - b_\varepsilon)^2 dx \leq \left(\frac{2}{w_0} + \frac{1}{\alpha} \right) \varepsilon,$$

which together with (21) proves the estimate (16).

To prove the next part of the lemma, we note that [7]

$$|u_\varepsilon'| \left| W(u_\varepsilon)^{1/2} \right| \leq \varepsilon (u_\varepsilon')^2 + \frac{1}{\varepsilon} W(u_\varepsilon),$$

so that

$$\int |u_\varepsilon'| \left| W(u_\varepsilon)^{1/2} \right| dx \leq m_\varepsilon.$$

Now, letting

$$\Psi(s) = \int_0^2 W(\xi)^{1/2} d\xi,$$

we see that

$$\left|(\Psi(u_\varepsilon))'\right| = |u_\varepsilon'| W(u_\varepsilon)^{1/2}.$$

Therefore, we can conclude that

$$\int \left|(\Psi(u_\varepsilon))'\right| dx \leq m_\varepsilon.$$

From the co-area formula [5] we have

$$\int_{-\infty}^{\infty} \mathrm{Per}\,(\Psi(u_\varepsilon) > \mu)\, d\mu = \int \left|(\Psi(u_\varepsilon))'\right| dx \leq m_\varepsilon, \tag{22}$$

where Per() refers to the perimeter of the set (). In one dimension

$$\mathrm{Per}(\Psi(u_\varepsilon) > \mu) = \{x \,:\, \Psi(u_\varepsilon(x)) = \mu\}.$$

Noting that $\mathrm{Per}(\Psi(u_\varepsilon) > \mu) = \mathrm{Per}(u_\varepsilon > \Psi^{-1}(\mu))$, let

$$I = \int_{-\infty}^{\infty} \mathrm{Per}(u_\varepsilon > \Psi^{-1}(\mu)) d\mu.$$

Then,

$$\begin{aligned} I &\geq \int_{\{\mu \,:\, \mathrm{Per}(u_\varepsilon > \Psi^{-1}(\mu) > M\}} \mathrm{Per}(u_\varepsilon > \Psi^{-1}(\mu))\} d\mu \\ &\geq M \int_{\{\mu \,:\, \mathrm{Per}(u_\varepsilon > \Psi^{-1}(\mu) > M\}} d\mu \\ &\geq M \left| \{\mu \,:\, \mathrm{Per}(u_\varepsilon > \Psi^{-1}(\mu)) > M\} \right|. \end{aligned}$$

Therefore, we can conclude from (22) that

$$\big|\{\mu \,:\, \text{Per}(u_\varepsilon > \Psi^{-1}(\mu)) > M\}\big| \le \frac{m_\varepsilon}{M}$$

Now choose

$$M = \frac{2m_\varepsilon}{\Psi(3/4) - \Psi(1/4)},$$

so that we have

$$\big|\{\mu \,:\, \text{Per}(u_\varepsilon > \Psi^{-1}(\mu)) > M\}\big| \le \frac{\Psi(3/4) - \Psi(1/4)}{2}.$$

Then, it is clear that over the interval $\mu \in [\Psi(1/4), \Psi(3/4)]$, there exists μ^* such that

$$\text{Per}(u_\varepsilon > \Psi^{-1}(\mu)) < M.$$

Thus, letting $\gamma = \Psi^{-1}(\mu^*)$, we can conclude that

$$\text{Per}(u_\varepsilon > \gamma) < M = \frac{2m_\varepsilon}{\Psi(3/4) - \Psi(1/4)}.$$

Recalling now that $\|b_\varepsilon\|_{BV} = \text{Per}(u_\varepsilon > \gamma)$, we have demonstrated (17) and the proof of Lemma 3.2 is complete. □

Remark 1. We provide a little intuition into the results of Lemma 3.2. It states that one can create a binary function $b_\varepsilon(x)$ from the minimizer $u_\varepsilon(x)$ satisfying (16). Moreover, the number of jumps in the binary function $b_\varepsilon(x)$ is proportional to the minimum energy m_ε.

The following lemma gives a lower bound on binary functions that are smoothed by a Gaussian kernel and can be interpreted as how much information is lost from a bar code function $z(x)$ when it is convolved with the kernel g_t in (1):

Lemma 3.3. *For each $N \ge 1$ and $\omega > 0$ there exists a constant $C > 0$ depending only on N and ω such that*

$$\|g_t * (z - y)\|_{L^2} \ge C \exp\big(-\frac{N^2 t}{2}\big)\|z - y\|_{L^2}^4 \tag{23}$$

for any $z \in B(N, \omega)$ and **any other** *binary function y.*

Proof. Fix $N \ge 1$ and $\omega > 0$ and take any $z \in B(N, \omega)$. Recall that z then has the form in (6); it has $2N$ discontinuities (transitions), located at $b_j \pm d_j$ for $j = 1, 2, \ldots, N$.

Let $p(x)$ denote the unique (up to sign) 2π-periodic real trigonometric polynomial of degree $2N$ that has its roots at the transition points $b_j \pm d_j$ and that has unit L^2 norm. Fix the sign of $p(x)$ by demanding that $p(b_1) > 0$. Recall (e.g. [6]) that the trigonometric interpolant $p(x)$ is by definition a function of the form

$$p(x) = a_0 + \sum_{j=1}^{N} a_j \cos(jx) + \sum_{j=1}^{N} a_{N+j} \sin(jx) \tag{24}$$

satisfying the $2N+1$ interpolation conditions listed above for an appropriate choice of the constants $a_0, a_1, \ldots, a_{2N}$. It can be expressed alternatively in the following Lagrange form

$$p(x) = \theta \prod_{j=1}^{N} \left[\sin\frac{1}{2}(x - b_j - d_j)\, \sin\frac{1}{2}(x - b_j + d_j)\right] \tag{25}$$

for some constant $\theta \neq 0$. Using (25) and the requirement $\|p\|_{L^2} = 1$, the constant θ can be estimated as

$$0 < c_1 \leq \theta \leq c_2$$

where c_1 and c_2 depend only on N and ω. Also as a consequence of $\|p\|_{L^2} = 1$, we have the simple bound $|a_j| \leq 1$ for all $j = 0, 1, \ldots, 2N$, on the coefficients appearing in the standard from (24) of $p(x)$.

From representation (25), it also follows easily that there exist constants $c_3 > 0$ and c_4 depending only on N and ω such that

$$|p'(b_j \pm r_j)| > c_3 \text{ for all } j, \tag{26}$$

and

$$\|p''\|_{L^\infty} \leq c_4. \tag{27}$$

Additionally, it can be seen, e.g. from (25), that $p'(x)$ also has exactly $2N$ simple roots that must therefore interlace those of $p(x)$. Combined with the bounds (26) and (27), this means that on the intervals I_j, for $j = 1, 2, \ldots, N$, and L_j, for $j = 2, 3, \ldots, N$

$$|p(x)| \geq c_5 \min\{|x - (b_j - d_j)|, |x - (b_j + d_j)|\}, \tag{28}$$

where $c_5 > 0$ depends only on N and ω. The inequality extends to cover the L_1 and L_{N+1} where we use periodicity to say that on these intervals

$$|p(x)| \geq c_5 \min\{|x - b_N - d_N + 2\pi|, |x - b_1 + d_1|\},$$

for $b_N + d_N - 2\pi < x < b_1 - d_1$.

It is now crucial to observe that for any binary function $y(x)$

$$z(x) - y(x) \geq 0 \text{ for } x \in \bigcup_{j=1}^{N} I_j, \tag{29}$$

and

$$z(x) - y(x) \leq 0 \text{ for } x \in \bigcup_{j=1}^{N+1} L_j. \tag{30}$$

Let us write

$$\|z - y\|^2 = \sum_{j=1}^{n} \int_{I_j} |z - y|^2 \, dx + \sum_{j=1}^{N+1} \int_{L_j} |z - y|^2 \, dx.$$

The average contribution from the individual terms on the right-hand side is $\|z - y\|^2/(2N+1)$. Therefore, there must be an interval I_* from the subintervals $\{I_j\}_{j=1}^N$ and $\{L_j\}_{j=1}^{N+1}$ such that

$$\int_{I_*} (z - y)^2 \, dx = |\{x \in I_* : |z - y| = 1\}| \geq \frac{1}{N} \|z - y\|_{L^2}^2. \tag{31}$$

Let

$$\delta := |\{x \in I_* : |z - y| = 1\}|. \tag{32}$$

so that $\delta > \frac{1}{N} \|z - y\|_{L^2}^2$ by (31).

Equations (29) and (30), together with our construction of $p(x)$ with the property that $p(b_1) > 0$ implies that

$$\int (z(x) - y(x)) p(x) \geq 0.$$

Moreover, we have

$$\begin{aligned}\int (z-y)p\,dx &\geq \int_{I_*}(z-y)p\,dx \\ &= \int_{\{x\in I_*\,:\,|z-y|=1\}} |p(x)|\,dx.\end{aligned}$$

Consider $p(x)$ on the interval I_*. Denote the interval by its end points as $I_* = (a_*, a^*)$. Then by (28) In this interval

$$p(x) \geq c_5 \min\{|x-a_*|, |x-a^*|\}.$$

Then it is easy to see that

$$\begin{aligned}\int_{\{x\in I_*\,:\,|z-y|=1\}} |p(x)|\,dx &\geq 2\int_0^{\delta/2} c_5 x\,dx \\ &= \frac{1}{4}c_5\delta^2 \\ &\geq \frac{1}{4}c_6\|z-y\|_{L^2}^4. \end{aligned} \tag{33}$$

where $c_6 > 0$ depends only on N and ω.

Inequality (33) implies that at least for one $k \in \{1, 2, \ldots, N\}$ we have

$$\int (z-y)\cos(kx)\,dx \geq \frac{C}{2N+1}\|z-y\|_{L^2}^4 \text{ or } \int (z-y)\sin(kx)\,dx \geq \frac{C}{2N+1}\|z-y\|_{L^2}^4.$$

Assuming with no loss of generality the former, we get

$$\|g_t*(z-y)\|_{L^2}^2 \geq C\|z-y\|_{L^2}^8\,\|g_t*\cos(kx)\|_{L^2}^2 \leq C\|z-y\|_{L^2}^8 \exp\big(-N^2 t\big)$$

where the constant $C > 0$ depends only on N and ω, completing the proof. □

We can now proceed to prove Theorem 3.1.

Proof. We have

$$\begin{aligned}E_\varepsilon(p_\varepsilon) &\geq E_\varepsilon(u_\varepsilon) \\ &\geq \lambda\|g_t*u_\varepsilon - h\|_{L^2}^2 \\ &= \lambda\|g_t*(u_\varepsilon-b_\varepsilon)+(g_t*b_\varepsilon-h)\|_{L^2}^2,\end{aligned} \tag{34}$$

where b_ε is as in Lemma 1. Note the inequality

$$\|a+b\|_{L^2}^2 \geq \frac{1}{2}\|a\|_{L^2}^2 - 3\|b\|_{L^2}^2. \tag{35}$$

Using (35) in (34), we have

$$E_\varepsilon(p_\varepsilon) \geq \frac{\lambda}{2}\|g_t*b_\varepsilon-h\|_{L^2}^2 - 3\lambda\|g_t*(u_\varepsilon-b_\varepsilon)\|_{L^2}^2 \tag{36}$$

To obtain a bound on the second term in the right-hand side of (36), we use the fact that the convolution with g_t is bounded independent of $t > 0$, so that

$$\|g_t*(u_\varepsilon-b_\varepsilon)\|_{L^2}^2 \leq C\|u_\varepsilon-b_\varepsilon\|_{L^2}^2.$$

Now we use Lemma 3.2 to obtain

$$\|g_t*(u_\varepsilon-b_\varepsilon)\|_{L^2}^2 \leq C\varepsilon. \tag{37}$$

For the first term in (36), we have

$$\|g_t*b_\varepsilon-h\|_{L^2}^2 = \|g_t*b_\varepsilon - g_t*z - r\|_{L^2}^2,$$

using our measurement model. Using (35), we get

$$\|g_t * b_\varepsilon - h\|_{L^2}^2 \le \frac{1}{2}\|g_t * (b_\varepsilon - z)\|_{L^2}^2 - 3\|r\|_{L^2}^2. \tag{38}$$

Now using (37) and (38) in (36), we arrive at

$$E_\varepsilon(p_\varepsilon) \ge \frac{\lambda}{4}\|g_t * (b_\varepsilon - z)\|_{L^2}^2 - \frac{3\lambda}{2}\|r\|_{L^2}^2 - 3C\lambda\varepsilon. \tag{39}$$

Putting (13) and (39) together, we find

$$\frac{\lambda}{4}\|g_t * (b_\varepsilon - z)\|_{L^2}^2 - \frac{3}{2}\lambda\|r\|_{L^2}^2 - 3\lambda C_0\varepsilon \le 2C_1N + \lambda\left(C_2N\varepsilon + \|r\|_{L^2}^2\right),$$

which simplifies to

$$\frac{\lambda}{4}\|g_t * (b_\varepsilon - z)\|_{L^2}^2 \le 2C_1N + \lambda(3C_0 + C_2N)\varepsilon + \frac{5\lambda}{2}\|r\|_{L^2}^2, \tag{40}$$

In practice, λ in (3) is chosen low enough so that the number of transitions in (i.e. the total variation of) the minimizer u_ε is limited (comparable to the expected total variation of the exact solution). It is easy to estimate how large λ can be chosen while still keeping total variation of u_ε comparable to N. We have:

$$m_\varepsilon := E_\varepsilon(u_\varepsilon) \le E_\varepsilon(p_\varepsilon) \le 2N + \lambda\Big(CN\varepsilon + 2\|r\|_{L^2}^2\Big)$$

according to (13). We see that, for example, as long as

$$\lambda < \frac{2}{\|r\|_{L^2}^2} \text{ and } \varepsilon < \|r\|_{L^2}^2 \tag{41}$$

we have

$$\|u_\varepsilon\|_{BV} \le m_\varepsilon \le CN.$$

Hence, choosing for example $\lambda = \frac{1}{\|r\|_{L^2}^2}$, bound (40) has the form

$$\|g_t * (b_\varepsilon - z)\|_{L^2}^2 \le C\|r\|_{L^2}^2 \tag{42}$$

where the constant C depends only on N and ω. Using now Lemma 3.3 in (42), we get

$$\|b_\varepsilon - z\|_{L^2}^2 \le Ce^{N^2t}\|r\|_{L^2}^{\frac{1}{8}}. \tag{43}$$

Combining (43) with (16) from Lemma 3.2, we end up with

$$\|u_\varepsilon - z\|_{L^2} \le Ce^{N^2t}\|r\|_{L^2}^{\frac{1}{4}}$$

which establishes the claim. □

Remark 2. We can provide an intuitive interpretation of the statement in Theorem 3.1. It states that the reconstruction error scales like $\|r\|^{1/4}$ where recall that $r(x)$ is the noise in the signal. The constant multiplying the noise term is exponential in the product of N and t, where recall that N is the number of bars, and t is the width of the Gaussian blur kernel. Since the number of bars in a typical bar code is bounded, the theorem provides some insight to how much blur the method in [4] can tolerate.

Remark 3. A reviewer of this paper pointed out that the error estimate in Theorem 3.1 involves a strong (L^2) norm, and suggests that estimates involving a weaker norm, such as H^{-1}, may be even more useful. We agree with this assessment but we do not see a clear way to obtain error estimates involving a weaker norm using the techniques employed in this work.

Acknowledgments. Santosa dedicates this work to Avner Friedman who has been his teacher, mentor, and friend. Much of this work was carried out while the first author was visiting the Institute for Mathematics and its Applications. He thanks the IMA for the support and hospitality. The research of Esedoglu is supported in part by the National Science Foundation under award DMS 0748333 and the Sloan Foundation. Santosa gratefully acknowledges the support of the National Science Foundation through the award NSF DMS 0807856.

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Received August 2011; revised February 2012.

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