Coarsening Rate of an Ill-Posed Diffusion Equation:
Applications to Image Processing, Granular Flow, and Chemotaxis

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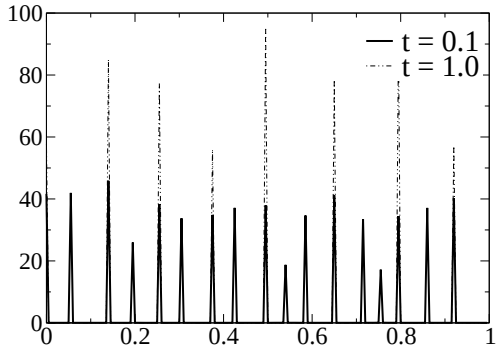
Ill-posed diffusion equations that can violate parabolicity arise in a number of applications. In image processing, the famous nonlinear image denoising model of Perona and Malik (PM) is based on one such PDE:

$$u_t = (R(u_x))_x + (R(u_y))_y$$

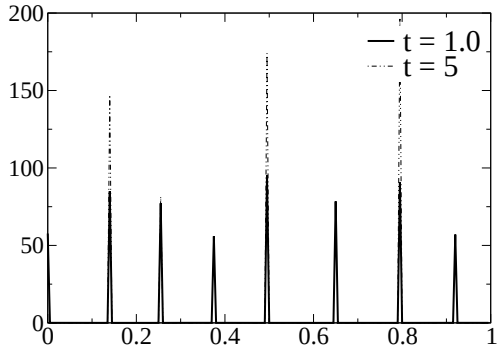
Here, R'(ξ) is positive for small |ξ|, and negative for large |ξ|; a typical choice is $R(\xi)=\xi/(1+\xi^2)$. The same PDE arises also in a 1D model due to Witelski, Shaeffer, and Shearer describing the formation of shear bands in granular medium under anti-plane shear. Also in 1D, the derivative $v=u_x$ of the PM equation satisfies the 1D version of:

$$v_t = (R(v))_{xx} + (R(v))_{yy}$$

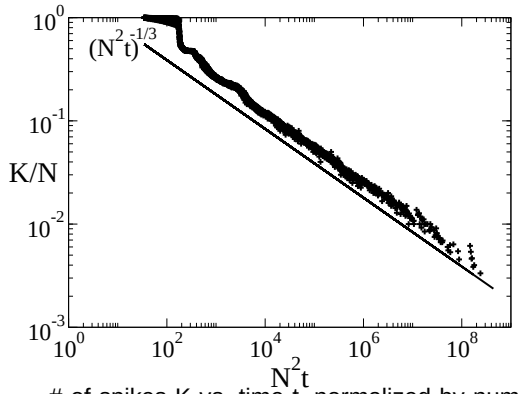
This variant arises in a model of chemotactic motion of bacteria due to Painter, Horstmann, and Othmer. Using a recent technique due to Kohn and Otto, we prove a rigorous upper bound on how fast spikes that form in the solution of this PDE can decrease in number as they merge to form bigger and fewer spikes. Our rigorous results, in 1D and 2D, have implications for all three applications, and verify the rate numerically observed by Witelski et. al.



Early stages of evolution.



Later stages of evolution.



# of spikes K vs. time t, normalized by number of grid points N. Our bound is the solid line.