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Fibered Quadratic Hopf Algebras Related to Schubert Calculus

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We introduce and study certain quadratic Hopf algebras related to Schubert calculus of the flag manifold. © 2000 Academic Press

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INTRODUCTION

This paper contributes to the study of a particular family of quadratic associative algebras $\mathcal{E}_n$ that were introduced and studied in [FK] in connection with their role in the Schubert calculus of the flag manifold.

We introduce new Hopf-algebraic tools for the study of the algebras $\mathcal{E}_n$. Specifically, we define a Hopf algebra structure on the twisted group algebra $\mathcal{E}_n\{\mathcal{S}_n\}$, where $\mathcal{S}_n$ is the symmetric group. We then use this Hopf

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algebra structure to obtain a tensor product decomposition of $\mathcal{E}_n$, which in particular implies a Hilbert series factorization conjectured by Kirillov [K].

QUADRATIC ALGEBRAS

Fix a positive integer n. Let $\mathcal{F}_n$ denote the free associative algebra generated by the symbols $[ij]$, for all $i, j \in \{1, \ldots, n\}$, $i \neq j$, subject to the relations

$$[ij] + [ji] = 0. \tag{1}$$

We will use the convention $[ii] = 0$.

Let $\mathcal{E}_n$ be the quotient of $\mathcal{F}_n$ modulo the ideal generated by the left-hand sides of the relations

$$[ij]^2 = 0, \tag{2}$$

$$[ij][jk] + [jk][ki] + [ki][ij] = 0, \quad i, j, k \text{ distinct}, \tag{3}$$

$$[ij][kl] - [kl][ij] = 0, \quad i, j, k, l \text{ distinct}. \tag{4}$$

The quadratic algebras $\mathcal{E}_n$ were first introduced and studied in [FK] because of the role they play in the Schubert calculus of the flag manifolds. A tantalizing question posed in [FK] asks whether $\mathcal{E}_n$ is generally finite-dimensional. The answer to this question is currently unknown.

In this paper, we prove that $\mathcal{E}_n$ can be decomposed (as a graded module) into a tensor product, one of the tensor factors being $\mathcal{E}_{n-1}$. We describe the second factor implicitly, in terms of a certain family of twisted derivations.

The algebras $\mathcal{E}_n$ are naturally graded; the formulas for their Hilbert polynomials, for $n \leq 5$, can be found in [FK]. Our main result implies that the Hilbert polynomial (or series?) of $\mathcal{E}_{n-1}$ divides that of $\mathcal{E}_n$ (and the ratio has non-negative coefficients), proving a conjecture stated in [K].

The algebras $\mathcal{E}_n$ are not Koszul for $n \geq 3$ (proved by Roos [R]).

To provide some motivation, let us briefly explain the nature of the connection between the algebras $\mathcal{E}_n$ and Schubert calculus, although this connection itself will not be a subject of our purely algebraic studies. The algebra $\mathcal{E}_n$ contains a commutative subalgebra generated by the "Dunkl elements" $\theta_j = -\sum_{j<j}[ij] + \sum_{j<k}[jk]$. This subalgebra was shown in [FK] to be canonically isomorphic to the cohomology ring of the flag manifold. In other words, the algebra generated by the θ_j is canonically the quotient of the polynomial algebra by the ideal generated by the symmetric polynomials in the θ_j without constant term. Furthermore, the structure constants of the cohomology ring, with respect to the basis of Schubert cycles, can be interpreted via certain combinatorial action of $\mathcal{E}_n$ on the group algebra of $\mathcal{S}_n$ (see [FK]).

We will study $\mathcal{E}_n$ along with the algebra $\mathcal{G}_n$ (first considered in [K]) defined as the quotient of $\mathcal{F}_n$ modulo (3)–(4); in other words, in $\mathcal{G}_n$ we no longer require $[ij]^2 = 0$.

Twisted Group Algebras

The symmetric group $\mathcal{S}_n$ acts on $\mathcal{F}_n$ by $\sigma([ij]) = [\sigma(i)\sigma(j)]$. Since the relations (1)–(4) are $\mathcal{S}_n$-stable, we also have an $\mathcal{S}_n$-action on $\mathcal{E}_n$ and $\mathcal{G}_n$.

For a group W acting on an algebra A (by algebra endomorphisms), let $A\{W\}$ denote the *twisted group algebra*, i.e., the algebra of linear combinations

$$\sum_{w\in W} a_w w, \qquad a_w \in A,$$

subject to the commutation rules $wa = w(a)w$, for any $w \in W$, $a \in A$. We remark that A is naturally an $A\{W\}$-module.

Let us denote

$$\tilde{\mathcal{F}}_n \stackrel{\text{def}}{=} \mathcal{F}_n\{\mathcal{S}_n\}, \qquad \tilde{\mathcal{E}}_n \stackrel{\text{def}}{=} \mathcal{E}_n\{\mathcal{S}_n\}, \qquad \tilde{\mathcal{G}}_n \stackrel{\text{def}}{=} \mathcal{G}_n\{\mathcal{S}_n\}.$$

Notice that $\tilde{\mathcal{E}}_n$ and $\tilde{\mathcal{G}}_n$ are the quotients of $\tilde{\mathcal{F}}_n$ modulo the relations (2)–(4) and (3)–(4), respectively. In turn, $\tilde{\mathcal{F}}_n$ is generated by $\mathcal{S}_n$ and the elements $[ij]$, subject to (1) and the additional relations

$$w\,[ij] = [w(i)w(j)]w, \quad w \in \mathcal{S}_n. \tag{5}$$

Thus $\tilde{\mathcal{E}}_n$ can be defined by (1)–(5), together with some presentation of $\mathcal{S}_n$.

We will denote by (ij) (or sometimes by s_{ij}) the transposition of i and j. The usual convention will be that (ij) stands for an element of $\mathcal{S}_n$ viewed inside $\tilde{\mathcal{F}}_n$, $\tilde{\mathcal{E}}_n$, or $\tilde{\mathcal{G}}_n$, while s_{ij} will denote the corresponding automorphisms of these algebras. If we choose the set of all transpositions as a generating set for $\mathcal{S}_n$, then all relations will be (non-homogeneous) quadratic:

$$(ij)^2 = 1,$$

$$(ij)(jk) = (jk)(ki) = (ki)(ij),$$

$$(ij)(kl) - (kl)(ij) = 0, \quad \text{all } i, j, k, l \text{ distinct}.$$

A non-faithful representation of $\tilde{\mathcal{E}}_n$ in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ can be constructed as follows. The symmetric group naturally acts on this ring by permuting variables. The generators $[ij]$ are then represented by the divided difference operators ∂_{ij} defined by

$$\partial_{ij} f = \frac{f - s_{ij} f}{x_i - x_j}$$

(cf. [FK, Sect. 3.2]). The commutation relations in question are easily verified.

HOPF ALGEBRAS

We will now define a Hopf algebra structure on each of $\tilde{\mathcal{F}}_n$, $\tilde{\mathcal{E}}_n$, and $\tilde{\mathcal{G}}_n$, extending the natural Hopf algebra structure of the group algebra of $\mathcal{S}_n$. The reader is referred to [A] or [DCP, Sect. 1] for general Hopf algebra background.

The coproduct Δ, the antipode S, and the counit ϵ are defined on the generators $[ij]$ and on group elements $w \in \mathcal{S}_n$ as follows:

$$\begin{aligned} \Delta([ij]) &= [ij] \otimes 1 + (ij) \otimes [ij], \qquad & \Delta(w) &= w \otimes w, \\ S([ij]) &= (ij)[ji], & S(w) &= w^{-1}, \\ \epsilon([ij]) &= 0. & \epsilon(w) &= 1. \end{aligned} \tag{6}$$

We then extend Δ, S, and ϵ to $\tilde{\mathcal{F}}_n$ so that Δ and ϵ are homomorphisms, and S an anti-homomorphism. Since $\mathcal{F}_n$ is a free algebra, we only have to check the compatibility of our definition with the twisted group algebra structure (i.e., with the relation (5)); this is straightforward.

PROPOSITION 1. *With the coproduct, antipode, and counit defined as above, $\tilde{F}_n$ is a Hopf algebra. This Hopf algebra structure passes to the quotients $\tilde{\mathcal{E}}_n$ and $\tilde{\mathcal{G}}_n$.*

Proof. Let us verify the remaining Hopf algebra axioms. Coassociativity:

$$\begin{aligned} (\Delta \otimes 1)(\Delta([ij])) &= (\Delta \otimes 1)\big([ij] \otimes 1 + (ij) \otimes [ij]\big) \\ &= [ij] \otimes 1 \otimes 1 + (ij) \otimes [ij] \otimes 1 + (ij) \otimes (ij) \otimes [ij] \\ &= (1 \otimes \Delta)\big([ij] \otimes 1 + (ij) \otimes [ij]\big) \\ &= (1 \otimes \Delta)\big(\Delta([ij])\big). \end{aligned}$$

The antipode axioms (μ: $\tilde{\mathcal{F}}_n \otimes \tilde{\mathcal{F}}_n \to \tilde{\mathcal{F}}_n$ denotes the multiplication map):

$$\begin{aligned} \mu(1 \otimes S)(\Delta([ij])) &= \mu(1 \otimes S)\big([ij] \otimes 1 + (ij) \otimes [ij]\big) \\ &= [ij] + (ij)(ij)[ji] = 0 = \epsilon([ij]), \\ \mu(S \otimes 1)(\Delta([ij])) &= \mu(S \otimes 1)\big([ij] \otimes 1 + (ij) \otimes [ij]\big) \\ &= (ij)[ji] + (ij)[ij] = 0 = \epsilon([ij]). \end{aligned}$$

The counit axioms:

$$\begin{aligned} (\epsilon \otimes 1)(\Delta([ij])) &= (\epsilon \otimes 1)\big([ij] \otimes 1 + (ij) \otimes [ij]\big) = [ij], \\ (1 \otimes \epsilon)(\Delta([ij])) &= (1 \otimes \epsilon)\big([ij] \otimes 1 + (ij) \otimes [ij]\big) = [ij]. \end{aligned}$$

Let us prove the second part of the proposition. In what follows, we will use the notation

$$R[i, j, k] \stackrel{\text{def}}{=} [ij][jk] + [jk][ki] + [ki][ij] \tag{7}$$

for the left-hand side of (3); we will also denote by $(ijk) \in \mathcal{S}_n$ the cycle $i \to j \to k \to i$. Then the computations

$$\begin{aligned} \Delta([ij]^2) &= [ij]^2 \otimes 1 + 1 \otimes [ij]^2, \\ S([ij]^2) &= -[ij]^2, \\ \epsilon([ij]^2) &= 0, \\ \Delta(R[i,j,k]) &= R[i,j,k] \otimes 1 + (ijk) \otimes R[i,j,k], \\ S(R[i,j,k]) &= -(ikj)R[i,j,k], \\ \epsilon(R[i,j,k]) &= 0 \end{aligned}$$

show that the Hopf algebra operations in $\tilde{\mathcal{F}}_n$ preserve the defining ideals of $\tilde{\mathcal{E}}_n$ and $\tilde{\mathcal{G}}_n$. ∎

ADJOINT ACTION AND TWISTED DERIVATIONS

Let us recall two important constructions.

Adjoint Action

Given a Hopf algebra A, the homomorphism

$$(1 \otimes S)\,\Delta : A \to A \otimes A^{\mathrm{op}},$$

where A^{op} is the opposite (Hopf) algebra, induces an action of A on itself, called the *adjoint* action (see, e.g., [DCP, Sect. 1.7]). Thus $a \in A$ acts on A by

$$x \mapsto \sum a_{(1)} x S(a_{(2)}),$$

where $\Delta a = \sum a_{(1)} a_{(2)}$.

In particular, consider the Hopf algebra $\tilde{\mathcal{F}}_n$. Then the elements $w \in \mathcal{S}_n \subset \tilde{\mathcal{F}}_n$ act by conjugation

$$w(x) = x \mapsto wxw^{-1},$$

while $[ij]$ acts by

$$D_{[ij]} : x \mapsto [ij]x + (ij)x(ij)[ji] = [ij]x - s_{ij}(x)[ij]. \tag{8}$$

It follows that $a(x) \in \mathcal{F}_n$ for any $x \in \mathcal{F}_n$ and $a \in \tilde{\mathcal{F}}_n$.

Twisted Derivations

Recall [DCP, Sect. 5.1] that, given an algebra A with a distinguished automorphism w, a *twisted derivation relative to* w is a linear map $D: A \to A$ satisfying

$$D(ab) = D(a)b + w(a)D(b). \tag{9}$$

For any $a \in A$, the map $D_a: A \to A$ defined by

$$D_a: x \mapsto ax - w(x)a$$

is a twisted derivation (an *inner* twisted derivation). For instance, the map $D_{[ij]}$ defined in (8) is an inner twisted derivation, with $a = [ij]$ and $w = s_{ij}$. (So our notation is consistent.)

Note that all of the above considerations apply to the Hopf algebras $\tilde{\mathcal{G}}_n$ and $\tilde{\mathcal{E}}_n$ as well. Let us summarize.

PROPOSITION 2. *The Hopf algebra $\mathcal{F}_n$ (resp. $\mathcal{G}_n$, $\mathcal{E}_n$) is invariant under the adjoint action of $\tilde{\mathcal{F}}_n$ (resp. $\tilde{\mathcal{G}}_n$, $\tilde{\mathcal{E}}_n$). The elements $w \in \mathcal{S}_n$ act by the automorphisms $w([ij]) = [w(i)w(j)]$. The element $[ij]$ acts by the corresponding inner twisted derivation $D_{[ij]}$ (for the automorphism s_{ij}).*

It is easy to check that the inner twisted derivations $D_{[ij]}$, $i, j < n$, in the algebra $\tilde{\mathcal{G}}_n$ act on the elements of the form $[kn]$ by

$$\begin{aligned} D_{[ij]}([kn]) &= 0 \quad \text{if } k \notin \{i, j\}, \\ D_{[ij]}([in]) &= -[in][jn], \\ D_{[ij]}([jn]) &= [jn][in]. \end{aligned} \tag{10}$$

TENSOR PRODUCT DECOMPOSITION

Let us now study the subalgebra $\mathcal{U}_{n-1}$ of $\mathcal{F}_n$ that is (freely) generated by the elements

$$x_i \stackrel{\text{def}}{=} [i\,n],$$

for $i = 1, \ldots, n-1$. The symmetric group $\mathcal{S}_{n-1}$ acts on $\mathcal{U}_{n-1}$ by

$$w(x_i) = x_{w(i)}. \tag{11}$$

Motivated by (10), let us define, for $i, j \leq n-1$, the twisted derivations D_{ij} (for the automorphisms s_{ij}) that act on the generators of $\mathcal{U}_{n-1}$ by

$$\begin{aligned} D_{ij}(x_k) &= 0 \quad \text{if } k \notin \{i, j\}, \\ D_{ij}(x_i) &= -x_i x_j, \\ D_{ij}(x_j) &= x_j x_i. \end{aligned} \tag{12}$$

PROPOSITION 3. *Formulas* (11) *and* (12) *determine a well-defined action of the twisted group algebra* $\tilde{\mathcal{F}}_{n-1}$ *on the subalgebra* $\mathcal{U}_{n-1}$.

Proof. We only have to check consistency with (1) and (5). Both are straightforward. ∎

Recall the notation $R[i, j, k]$ introduced in (7).

PROPOSITION 4. *The action of the elements* $R[i, j, k] \in \tilde{\mathcal{F}}_{n-1}$ *on* $\mathcal{U}_{n-1}$ *is trivial, i.e., they act by* $R[i, j, k](x) = 0$. *Thus* $\mathcal{U}_{n-1}$ *is naturally a* $\tilde{\mathcal{G}}_{n-1}$-*module*.

Proof. Induction on the degree of a monomial x in the variables x_i. Note that $R[i, j, k](1) = 0$. Let $x = x_h N$. If $h \notin \{i, j, k\}$, then we have $R[i, j, k](x_h N) = x_h R[i, j, k](N) = 0$. Otherwise it suffices to check the case $h = i$ (by symmetry). Using (9) and (12), we obtain

$$\begin{aligned}
&R[i, j, k](x_i N)\\
&\quad = \big(D_{ij}D_{jk} + D_{jk}D_{ki} + D_{ki}D_{ij}\big)(x_i N)\\
&\quad = D_{ij}(x_i D_{jk}(N)) + D_{jk}\big(x_i x_k N + x_k D_{ki}(N)\big)\\
&\qquad + D_{ki}\big(-x_i x_j N + x_j D_{ij}(N)\big)\\
&\quad = -x_i x_j D_{jk}(N) + x_j D_{ij} D_{jk}(N) + x_i x_k x_j N + x_i x_j D_{jk}(N)\\
&\qquad + x_k x_j D_{ki}(N)\\
&\qquad + x_j D_{jk} D_{ki}(N) - x_i x_k x_j N - x_k x_j D_{ki}(N) + x_j D_{ki} D_{ij}(N)\\
&\quad = x_j R[i, j, k](N),
\end{aligned}$$

and the claim follows. ∎

Let $\mathcal{J}_{n-1}$ be the minimal ideal of the free algebra $\mathcal{U}_{n-1}$ that contains the elements x_i^2 and is stable under the twisted derivations D_{ij}. Denote $\mathcal{C}_{n-1} = \mathcal{U}_{n-1}/\mathcal{J}_{n-1}$. To illustrate, consider the special case $n = 3$. The algebra $\mathcal{C}_2$ is generated by x_1 and x_2, subject to the relations

$$x_1^2 = x_2^2 = 0, \qquad x_1 x_2 x_1 + x_2 x_1 x_2 = 0. \tag{13}$$

We note that, in view of (10), the map $x_i \mapsto [in]$ extends to an algebra homomorphism $\mathcal{C}_{n-1} \to \mathcal{E}_n$.

LEMMA. *The module action of* $\tilde{\mathcal{F}}_{n-1}$ *on* $\mathcal{U}_{n-1}$ *factors to a module action of* $\tilde{\mathcal{E}}_{n-1}$ *on* $\mathcal{C}_{n-1}$.

Proof. We have to show that, modulo $\mathcal{J}_{n-1}$, the elements D_{ij} satisfy the relations (2)–(4). We already checked (3), and (4) is trivial. Let us prove $D_{ij}^2(M) \in \mathcal{J}_{n-1}$ by induction on the degree of a monomial M in the variables x_i.

Let $M = x_h N$. If $h \notin \{i, j\}$, then $D_{ij}^2(M) = x_h D_{ij}^2(N) \in \mathcal{J}_{n-1}$. Otherwise

$$\begin{aligned} D_{ij}^2(x_i N) &= D_{ij}\big(-x_i x_j N + x_j D_{ij}(N)\big) \\ &= \big(x_i x_j^2 - x_j^2 x_i\big)N - x_j x_i D_{ij}(N) + x_j x_i D_{ij}(N) + x_i D_{ij}^2 N \in \mathcal{J}_{n-1} \end{aligned}$$

and, similarly, $D_{ij}^2(x_j N) \in \mathcal{J}_{n-1}$. ∎

Action on the Tensor Product

We will now consider $\mathcal{C}_{n-1} \otimes \mathcal{E}_{n-1}$ as a module under $\tilde{\mathcal{E}}_{n-1}$, using the Hopf algebra structure on $\tilde{\mathcal{E}}_{n-1}$ and the two module structures on the factors:

$$a(x \otimes y) = \sum a_{(1)}(x) \otimes a_{(2)}(y).$$

For example, the element $[ij]$, for $i, j \le n-1$, acts by an operator $\{ij\}$ given by

$$\{ij\}(x \otimes y) = D_{ij} x \otimes y + s_{ij}(x) \otimes [ij] y.$$

In the same way, $\mathcal{U}_{n-1} \otimes \mathcal{G}_{n-1}$ is a module under $\tilde{\mathcal{G}}_{n-1}$.

Let $\{in\}\colon \mathcal{U}_{n-1} \otimes \mathcal{G}_{n-1} \to \mathcal{U}_{n-1} \otimes \mathcal{G}_{n-1}$ be the operator of multiplication by x_i in the left factor: $\{in\}(x \otimes y) = x_i x \otimes y$. We also denote $\{ni\} = -\{in\}$.

PROPOSITION 5. *The operators $\{ij\}$ satisfy the basic relations* (3).

Proof. The only nontrivial thing to check is the identity $R[i, j, n] = 0$,

$$\begin{aligned} &\big(\{ij\}\{jn\} + \{jn\}\{ni\} + \{ni\}\{ij\}\big)(x \otimes y) \\ &\quad = \{ij\}(x_j x \otimes y) - \{jn\}(x_i x \otimes y) + \{ni\}\big(D_{ij}(x) \otimes y + s_{ij}(x) \otimes [ij] y\big) \\ &\quad = x_j x_i x \otimes y + x_i D_{ij}(x) \otimes y + x_i s_{ij}(x) \otimes [ij] y \\ &\qquad - x_j x_i x \otimes y - x_i D_{ij}(x) \otimes y - x_i s_{ij}(x) \otimes [ij] y \\ &\quad = 0. \end{aligned}$$

∎

Thus $\mathcal{U}_{n-1} \otimes \mathcal{G}_{n-1}$ acquires a $\mathcal{G}_n$-module structure, which clearly factors to an $\mathcal{E}_n$-module structure on $\mathcal{C}_{n-1} \otimes \mathcal{E}_{n-1}$.

Consider the additive homomorphism $\pi\colon \mathcal{E}_n \to \mathcal{C}_{n-1} \otimes \mathcal{E}_{n-1}$ defined by $a \mapsto a(1 \otimes 1)$; in other words, we apply the module action that has just been introduced to the identity element. (There is also a similar map $\mathcal{G}_n \to \mathcal{U}_{n-1} \otimes \mathcal{G}_{n-1}$.)

Going in the opposite direction, consider the map p: $\mathcal{C}_{n-1} \otimes \mathcal{E}_{n-1} \to \mathcal{E}_n$ given in the following natural way. We have an algebra map p_1: $\mathcal{C}_{n-1} \to \mathcal{E}_n$ given by $x_i \mapsto [in]$, and the embedding p_2: $\mathcal{E}_{n-1} \to \mathcal{E}_n$ given by $[ij] \mapsto [ij]$. Let us then define $p(a \otimes b) = p_1(a)p_2(b)$.

THEOREM 1. *The maps p and π are inverse to each other, and establish an isomorphism of the graded modules $\mathcal{E}_n$ and $\mathcal{C}_{n-1} \otimes \mathcal{E}_{n-1}$. Analogously, we obtain an isomorphism between $\mathcal{G}_n$ and $\mathcal{U}_{n-1} \otimes \mathcal{G}_{n-1}$.*

Proof. From the explicit definitions of the module structures it is easily seen that the composition $\pi \circ p$ is the identity map. On the other hand, the commutation relations (3)–(4) can be used in a "straightening" procedure that rewrites any element in $\mathcal{E}_n$ as a linear combination of elements of the form $[i_1 n] \cdots [i_l n]x$, with $x \in \mathcal{E}_{n-1}$. This implies that p is surjective, and the claim follows. ∎

COROLLARY. *The algebras $\mathcal{E}_n$ and $\mathcal{G}_n$ have tensor product decompositions (as graded modules)*

$$\mathcal{E}_n \cong \mathcal{C}_{n-1} \otimes \mathcal{C}_{n-2} \otimes \cdots \otimes \mathcal{C}_2 \otimes \mathcal{C}_1,$$

$$\mathcal{G}_n \cong \mathcal{U}_{n-1} \otimes \mathcal{U}_{n-2} \otimes \cdots \otimes \mathcal{U}_2 \otimes \mathcal{U}_1.$$

As a corollary, we obtain the corresponding factorization of the Hilbert series that was conjectured in [Ki, Conjecture 8.6].

To illustrate, let $n = 3$. Let x_1, x_2 be the generators of $\mathcal{C}_3$, and let y_1 be the generator of $\mathcal{C}_2$. Recall that $\mathcal{C}_2$ is given by (13), and therefore its Hilbert polynomial is equal to $1 + 2q + 2q^2 + q^3 = (1+q)(1+q+q^2)$. The only relation in $\mathcal{C}_1$ is $y_1^2 = 0$, which gives the Hilbert polynomial $1 + q$. We conclude that the Hilbert polynomial for $\mathcal{E}_3$ equals $(1+q)^2(1+q+q^2)$, matching [FK, (2.8)].

We have in fact proved the following more precise statement.

THEOREM 2. *The subalgebra $\mathcal{A}_k$ of $\mathcal{E}_n$ generated by the elements $[ik]$, for $i < k$, is canonically isomorphic to $\mathcal{C}_k$. The multiplication map*

$$\mathcal{A}_{n-1} \otimes \mathcal{A}_{n-2} \otimes \cdots \otimes \mathcal{A}_2 \otimes \mathcal{A}_1 \to \mathcal{E}_n,$$

$$a_{n-1} \otimes a_{n-2} \otimes \cdots \otimes a_2 \otimes a_1 \mapsto a_{n-1}a_{n-2} \cdots a_2 a_1$$

is a linear isomorphism.

The next natural step would be to obtain a more explicit description of the ideal $\mathcal{J}_{n-1}$ that defines the algebra $\mathcal{C}_{n-1}$. At present, we do not know such a description. Applying the twisted derivations $D_{12}, D_{23}, \ldots$ (in this order) to the relation $x_1^2 = 0$, we obtain the "cyclic" relations

$$x_1 x_2 \cdots x_{n-1} x_1 + x_2 x_3 \cdots x_{n-1} x_1 x_2 + \cdots + x_{n-1} x_1 \cdots x_{n-2} x_{n-1} = 0. \quad (14)$$

We can then apply an arbitrary permutation of indices to (14). Even then, these relations (which are just another form of [FK, Lemma 7.2]) do *not* determine the ideal $\mathcal{J}_{n-1}$. The simplest instance of this appears in $\mathscr{C}_4$: applying the twisted derivation D_{13} to the cyclic identity

$$x_1x_2x_3x_4x_1 + x_2x_3x_4x_1x_2 + x_3x_4x_1x_2x_3 + x_4x_1x_2x_3x_4 = 0$$

results in a new 10-term relation in degree 6 (see [Ki, (8.10)]).

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