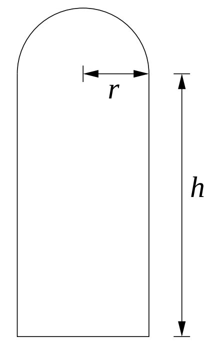
MATH 115-018 QUIZ 11 / 7 Apr 2005 Name:

1. (3 points) A stained glass window is to be made in the shape shown to the right, with a rectangular section surmounted by a semi-circular top. If P ft of border material are available, what should the dimensions of the window be to maximize its area? (You may assume that $P > 2$. The circumference of a circle is $C = 2\pi r$.)



Solution: The area of the window is the sum of the area of the rectangular section and the area of the semi-circular top, so $A = 2rh + \frac{1}{2}\pi r^2$. To get this to a single variable, use the constraint that the perimeter is P ft long: $P = 2h + 2r + \pi r = 2h + (2+\pi)r$, so, solving for h, $h = \frac{P}{2} - (1+\frac{\pi}{2})r$. Plugging this in to A,

$$\begin{aligned} A &= 2r(\frac{P}{2} - (1+\frac{\pi}{2})r) + \frac{1}{2}\pi r^2 \\ &= Pr - (2+\pi)r^2 + \frac{1}{2}\pi r^2 \\ &= Pr - (2+\frac{1}{2}\pi)r^2. \end{aligned}$$

This will be maximized at the end points ($r = 0$, where $h = \frac{P}{2}$, or $r = \frac{P}{\pi}$, where $h = 0$), or at critical points. Critical point(s) are where $A'(r) = P - (4+\pi)r = 0$, or $r = \frac{P}{4+\pi}$. Here $A''(r) = -(4+\pi)$, so we know that $A(r)$ is concave down, and therefore, because this is the only critical point, we know this will give the maximum value for A. For this r, $h = \frac{P}{2} - (1+\frac{\pi}{2})(\frac{P}{4+\pi})$.

What's with the $P > 2$? It's a red herring.

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2. (2 points) Suppose that the velocity of an orange-and-chartreuse-clothed math professor gradually increases in the course of a class period, and is given (in meters/second) by $v(t) = 2\,e^{t^2}$ (where t is in hours). Use a Riemann sum with $\Delta t = 0.5$ hr to estimate the total distance travelled by the professor during an hour and a half class period.

Solution: Note that the velocity is in m/s, while time is given in hours (3600 sec). We know that the distance travelled is

$$\begin{aligned} D = \int_0^{1.5} v(t)\,dt &\approx (0.5)(v(0) + v(0.5) + v(1)) \\ &= (0.5)(2 + 2.57 + 5.44) = 5.01 \text{ m/s} \cdot \text{hr}. \end{aligned}$$

Converting hours to seconds, this is $5.01 \cdot 3600 = 18,036$ m. We could, of course, use a right-hand sum instead, to get $D \approx (0.5)(v(0.5) + v(1) + v(1.5)) = 13.50$ ($\times 3600$) m.

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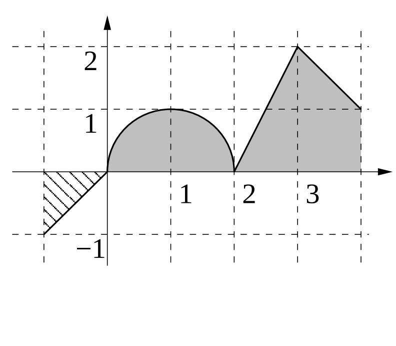
3. (3 points) Find the average value of the function shown to the right, for the domain shown. The arc in the figure is a semi-circle. Be sure it is clear how you obtain your answer.



Solution: The average value of the function, which we'll call $f(x)$, is $\frac{1}{b-a}\int_a^b f(x)\,dx$. Here $a = -1$ and $b = 4$, and the value of the integral is equal to the area under the curve $y = f(x)$ (treating area below the x-axis as negative). This area is shown in the figure, with negative area hatched instead of shaded. Thus, taking each geometrically distinct section of f in turn, $\int_{-1}^4 f(x)\,dx = -\frac{1}{2} + \frac{1}{2}\pi + 1 + \frac{3}{2} = 2 + \frac{\pi}{2}$. The average value is therefore $\frac{2}{5} + \frac{\pi}{10}$.