Some Notes on Improper Integrals

(see also <http://www.math.lsa.umich.edu/%7eglarose/classes/calcII/>)

This isn't a general consideration of improper integrals; for that, see the web page noted above. Here we look a bit more carefully at integrals of functions that have vertical asymptotes, partly to see how to set them up and evaluate them, and partly to see some of the subtlety that can show up there. In particular, let's consider the integral

$$\int_0^2 \frac{x}{x^2-1}\,dx.$$

We know from the graph, shown to the figure to the right below, that this is an improper integral, because the integrand (the function $\frac{x}{x^2-1}$) has a singularity at $x = 1$ (that is, there's a vertical asymptote there).

Before considering the integral that jumps over the singularity, let's stop to make sure that we're on top of the more tractible integral $\int_1^2 \frac{x}{x^2-1}\,dx$. We note that the left endpoint of the domain $[1, 2]$ is problematic, so we can evaluate this by rewriting the integral with the smaller domain $[b, 2]$ (with $1 < b < 2$) and then seeing what happens as we approach the range that we want (that is, as $b \to 1^+$). That is, we have

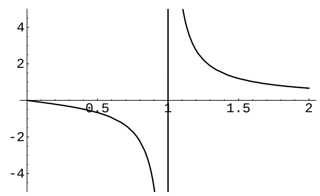

Figure 1: graph of $\frac{x}{x^2-1}$

$$\int_1^2 \frac{x}{x^2-1}\,dx = \lim_{b\to 1^+}\int_b^2 \frac{x}{x^2-1}\,dx = \lim_{b\to 1^+}\frac{1}{2}\ln|x^2-1|\Big|_b^2 = \frac{1}{2}\lim_{b\to 1^+}\left(\ln(3) - \ln|b^2-1|\right).$$

The original integral exists if this limit is well-defined, but it isn't! As $b \to 1^+$, $b^2 - 1 \to 0$, and the natural log function isn't defined at zero. Thus the limit is not defined, and we say that the original integral diverges.

One more diversion before we consider the original integral we had above: suppose we had been finding $\int_0^2 |\frac{x}{x^2-1}|\,dx$. The function $|\frac{x}{x^2-1}|$ is what we'd expect, shown in figure 2, to the right. Then it's clear that if we were trying to evaluate the integral we'd have to break it into pieces:

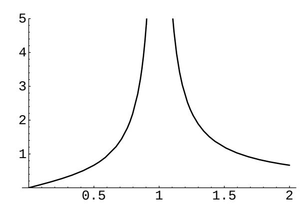

Figure 2: graph of $\left|\frac{x}{x^2-1}\right|$

$$\begin{aligned}\int_0^2 |\frac{x}{x^2-1}|\,dx &= \int_0^1 |\frac{x}{x^2-1}|\,dx + \int_1^2 |\frac{x}{x^2-1}|\,dx \\ &= -\int_0^1 \frac{x}{x^2-1}\,dx + \int_1^2 \frac{x}{x^2-1}\,dx.\end{aligned}$$ [1]

Then we can evaluate both of these as we did before, and, knowing that the second diverges (and, if we worked out the first, knowing that it diverges as well) we can clearly say that the integral as a whole diverges.

Finally, what about the integral $\int_0^2 \frac{x}{x^2-1}\,dx$? The first thing we note from figure 2 above is that the function $\frac{x}{x^2-1}$ actually *isn't* symmetric about $x = 1$! Thus, the problem is tricky: we expect that some of the area below the x-axis (as seen in figure 1) will cancel that above the x-axis, but it's not necessarily clear if that will be enough to make the integral converge!

To get some insight on this, let's first do some numerical calculations. Integrating with a calculator (or other tool), we find $\int_0^{0.99} \frac{x}{x^2-1}\,dx + \int_{1.01}^2 \frac{x}{x^2-1}\,dx = 0.544306$, $\int_0^{0.9999} \frac{x}{x^2-1}\,dx + \int_{1.0001}^2 \frac{x}{x^2-1}\,dx = 0.549256$, and $\int_0^{0.999999} \frac{x}{x^2-1}\,dx + \int_{1.000001}^2 \frac{x}{x^2-1}\,dx = 0.549306$. This looks promising: these sums appear to be converging to 0.549306 (which is actually strikingly like $\frac{\ln 3}{2}$). But there's a big "if" here: they seem to be

[1]: Note that we can drop the absolute values for the second integral because we know for x larger than one the integrand is always positive; similarly, for the first integral we know that by multiplying it by negative one we get a positive value and can then drop the absolute values.

converging if we make sure that at every step the boundaries creep toward $x = 1$ at the same rate. What happens if they don't?

Let's look at a different set of integrals to see: $\int_0^{0.99} \frac{x}{x^2-1}\,dx + \int_{1.001}^2 \frac{x}{x^2-1}\,dx = 1.69784$, $\int_0^{0.999} \frac{x}{x^2-1}\,dx + \int_{1.00001}^2 \frac{x}{x^2-1}\,dx = 2.85159$, and $\int_0^{0.9999} \frac{x}{x^2-1}\,dx + \int_{1.0000001}^2 \frac{x}{x^2-1}\,dx = 3.99819$. These certainly don't look as if they're approaching anything. So the magic of making sure that the boundaries move together clearly matters.

Let's try to explore this more rigorously. We know

$$\int_0^2 \frac{x}{x^2-1}\,dx = \int_0^1 \frac{x}{x^2-1}\,dx + \int_1^2 \frac{x}{x^2-1}\,dx = \lim_{b\to 1^-} \int_0^b \frac{x}{x^2-1}\,dx + \lim_{c\to 1^+} \int_c^2 \frac{x}{x^2-1}\,dx.$$

Note that we've picked two different parameters for the two limits, because we want to be sure that we're not making any accidental assumptions about how the areas cancel. Continuing to evaluate the integrals, we get

$$\begin{aligned}&\lim_{b\to 1^-} \int_0^b \frac{x}{x^2-1}\,dx + \lim_{c\to 1^+} \int_c^2 \frac{x}{x^2-1}\,dx \\ &\qquad = \frac{1}{2}\lim_{b\to 1^-} \left(\ln\left|\frac{b-1}{b+1}\right| - \ln|1|\right) + \frac{1}{2}\lim_{c\to 1^+} \left(\ln(3) - \ln\left|\frac{c-1}{c+1}\right|\right).\end{aligned}$$

So what's happening? If we're really careful and make sure that b and c are doing exactly the same thing, then the two limit terms cancel and we're left with $\frac{\ln 3}{2}$. **But** if this **isn't** the case, all bets are off! We have two limits that are sailing off to $-\infty$ and ∞, respectively, and the sum of these values could be anything. Because of this, we **can't** say that this integral converges.

We can, however, say that this is more complicated than anything that we'd be likely to see on a midterm. Be sure that you can explain and evaluate $\int_1^2 \frac{x}{x^2-1}\,dx$, $\int_0^1 \frac{x}{x^2-1}\,dx$, or $\int_0^2 \left|\frac{x}{x^2-1}\right|dx$, however.