MATH 116-009 QUIZ 4 / 6 Oct 2006 Name:____________________________

Possibly useful: $\int_1^\infty \frac{1}{x^p}\,dx$ *converges for* $p > 1$ *and diverges for* $p \le 1$. $\int_0^1 \frac{1}{x^p}\,dx$ *converges for* $p < 1$ *and diverges for* $p \ge 1$, *and* $\int_0^\infty e^{-ax}\,dx$ *converges for* $a > 0$.

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1. Find the area under the curve $y = x^2 e^{-x^3}$ for $x \ge 3$. (3 points)

Solution: The area is given by $\int_3^\infty x^2 e^{-x^3}\,dx$. With $w = x^3$, this is

$$\int_{27}^\infty \frac{1}{3}e^{-w}\,dw = \lim_{b\to\infty} -\frac{1}{3}e^{-w}\Big|_{27}^{b} = \lim_{b\to\infty} -\frac{1}{3}e^{-b} + \frac{1}{3}e^{-27} = \frac{1}{3e^{27}}.$$

Which is a very small number (about 10^{-13}).

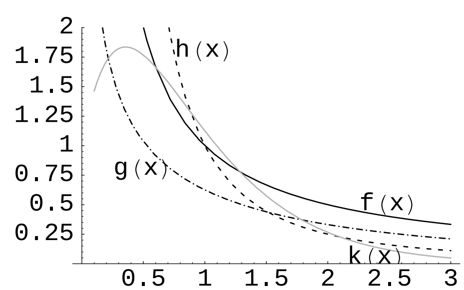
2. Carefully determine whether each of the following converge

(a) $\int_0^1 \frac{5x}{\sqrt{x^3+x^4}}\,dx$ (b) $\int_1^\infty \frac{1}{\sqrt{x^3-x}}\,dx$

(4 points)

Solution: (a) We note that for very small x, the denominator of the integrand looks like $\sqrt{x^3}$ (because x^4 is very much smaller than x^3), so that the integrand looks like $\frac{5x}{\sqrt{x^3}} = \frac{5}{\sqrt{x}}$. The integral of $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, so we expect this integral will also. In fact, $\frac{5x}{\sqrt{x^3+x^4}} < \frac{5x}{\sqrt{x^3}} = \frac{5}{\sqrt{x}}$. Thus $\int_0^1 \frac{5x}{\sqrt{x^3+x^4}}\,dx < 5\int_0^1 \frac{1}{\sqrt{x}}\,dx$. The latter integral converges, so the former must also.
(b) For large x the integrand looks like $\frac{1}{\sqrt{x^3}} = x^{-3/2}$, which is convergent. Thus we expect this to converge. Then $\frac{1}{\sqrt{x^3-x}} < \frac{1}{\sqrt{x^3-\frac{1}{2}x^3}} = \frac{2}{x^{3/2}}$, so $\int_1^\infty \frac{1}{\sqrt{x^3-x}}\,dx < 2\int_1^\infty \frac{1}{x^{3/2}}\,dx$. This latter integral converges, so the former must also.

3. Consider the four functions shown in the graph to the right. (In order from top to bottom at the right side of the graph they are $f(x)$, $g(x)$, $h(x)$ and $k(x)$.) Assume that the relative behaviors shown for these four functions continues for $x > 3$. $f(x) = \frac{1}{x}$ and $h(x) = \frac{1}{x^2}$. Which of the four integrals $\int_{0.5}^\infty f(x)\,dx$, $\int_{0.5}^\infty g(x)\,dx$, $\int_{0.5}^\infty h(x)\,dx$, and $\int_{0.5}^\infty k(x)\,dx$ converge? Which diverge? Are there any for which you cannot tell? Explain briefly (no calculations are necessary). (3 points)



Solution: We know that $\int_{0.5}^\infty f(x)\,dx$ diverges and $\int_{0.5}^\infty h(x)\,dx$ converges (these are known improper integrals). $k(x) < h(x)$ for larger x, so by comparison $\int_{0.5}^\infty k(x)\,dx$ must converge. $f(x) > g(x) > h(x)$, so we don't know what $\int_{0.5}^\infty g(x)\,dx$ does; it may diverge (albeit not as fast as $\int_{0.5}^\infty f(x)\,dx$), or converge.