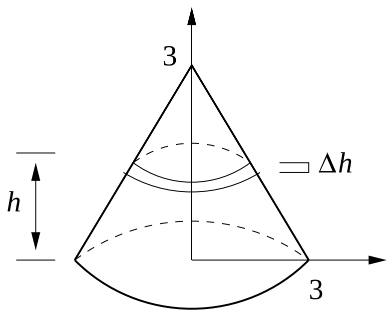
MATH 116-009 QUIZ 5 / 24 Oct 2006 Name:____________________

1. Consider the integral $\int_0^3 \pi(3-h)^2\,dh$, which gives the volume of a region in space. Sketch the region, showing all relevant variables and lengths and the slice used to write the integral. (3 points)

Solution: The region is shown to the right. We arbitrarily choose h to be the vertical distance measured up the figure. Then the slice used to generate the integral has volume $\pi(3-h)^2\,\Delta h$, so that it has a surface area of $\pi(3-h)^2$. We recognize this as the area of a circle with radius $r = 3-h$, so the radius of the region is 3 when $h = 0$ and 0 when $h = 3$. Thus we must be finding the volume of the cone shown, which has height and base radius three.



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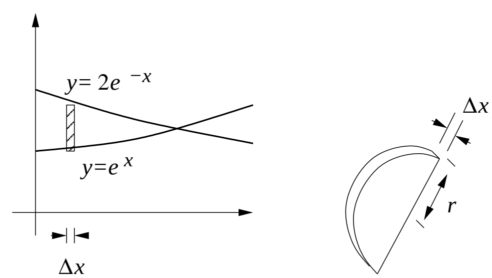
2. Suppose that you are considering a career as a modern mathematical artist specializing in three-dimensional sculpture. Your latest work is to be a piece set on a base described as the region bounded by $y = e^x$, $y = 2e^{-x}$, and the y-axis. The cross-sections of the piece perpendicular to the x-axis are semi-circular. Sketch a representative slice and set up an integral to find the volume of the region. For a bonus point, evaluate the integral analytically (that is, by hand). But do #3 first. (3 points)

Solution: The region and a representative slice are shown to the right. Each slice has a cross-section that is half of a circle, the diameter of which is $d = 2e^{-x} - e^x$. Thus the surface area of the slice is $\frac{1}{2}\pi((\frac{1}{2})(2e^{-x} - e^x))^2$, and the volume of the slice is $V_{sl} = \frac{1}{2}\pi((\frac{1}{2})(2e^{-x} - e^x))^2\,\Delta x$. Note that the curves that define the base intersect when $e^x = 2e^{-x}$, or when $e^{2x} = 2$, so $x = \frac{1}{2}\ln 2$. Thus the total volume is given by $\int_0^{\ln(2)/2} \frac{1}{2}\pi((\frac{1}{2})(2e^{-x} - e^x))^2\,dx$. To evaluate this, we can expand the square, finding the volume to be $\frac{\pi}{8}\int_0^{\ln(2)/2} 4e^{-2x} - 4 + e^{2x}\,dx = \frac{\pi}{8}(-2e^{-2x} - 4x + \frac{1}{2}e^{2x})\Big|_0^{\ln(2)/2} = \frac{\pi}{8}(-1 + 2 - 2\ln(2) + 1 - \frac{1}{2}) = \frac{\pi}{16}(3 - 4\ln(2))$.



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3. Find the area of the region between $r = \cos(\frac{\theta}{2})$ and $r = \sin(\frac{\theta}{2})$ that lies in the first quadrant. (4 points)

Solution: The region in question is shown to the right, with the solid curve being $r = \cos(\frac{\theta}{2})$ and the dashed curve $r = \sin(\frac{\theta}{2})$. The region extends over $0 \le \theta \le \frac{\pi}{2}$, which is the range of values for θ used to produce the graph. A polar slice of a region $r = f(\theta)$ is given to be $\Delta A = \frac{1}{2}(f(\theta))^2\,\Delta\theta$, so the area within $r = \cos(\frac{\theta}{2})$ is $\int_0^{\pi/2} \frac{1}{2}\cos^2(\frac{\theta}{2})\,d\theta = \frac{1}{4} + \frac{\pi}{8}$. Similarly, the area inside $r = \sin(\frac{\theta}{2})$ is $\int_0^{\pi/2} \frac{1}{2}\sin^2(\frac{\theta}{2})\,d\theta = \frac{\pi}{8} - \frac{1}{4}$. Thus the area between the two is $\frac{1}{2}$.