MATH 116-009 QUIZ 9 / 1 Dec 2006 Name: ____________________

It may or may not be useful to note that:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$

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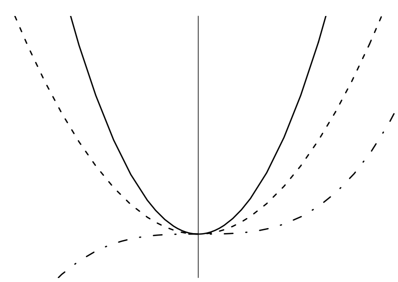
1. What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{3^n x^n}{n+2}$? (3 points)

Solution: We find the radius of convergence by using the ratio test. We need $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ for convergence. This limit is, in this case, $\lim_{n\to\infty} \left|\frac{3^{n+1}x^{n+1}}{n+3} \cdot \frac{n+2}{3^n x^n}\right| = \lim_{n\to\infty} \left|3x \cdot \frac{n+2}{n+3}\right|$. As $n \to \infty$, the ratio $\frac{n+2}{n+3} \to 1$, so the limit is $3|x|$. We therefore need $3|x| < 1$, and the radius of convergence is $R = \frac{1}{3}$.

2. Suppose that the Taylor series for a function $f(x)$ is given to be $f(x) = 2x + \frac{8x^3}{2!} + \frac{32x^5}{4!} + \frac{128x^7}{6!} + \cdots$. What are $f(0)$? $f'''(0)$? $f^{(19)}(0)$? (3 points)

Solution: The Taylor series around $x = 0$ for any function $f(x)$ is given by $f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \ldots$. Thus $f(0) = 0$ and $f'''(0) = 3!\left(\frac{8}{2!}\right) = 24$. The nth term in the series is $\frac{2^n x^n}{(n-1)!}$, so the 19th derivative is given by $f^{(19)}(0) = 19!\left(\frac{2^{19}}{18!}\right) = 19 \cdot 2^{19}$.

3. A wandering polar weasel meditates for 2.718 minutes and then sketches the graph to the right, which shows three functions for values of x near 0. Astonishingly, one of these turns out to be exactly $\frac{1}{1-x^2}$, one $2 - \cos(x)$, while the third is another function that remains anonymous to protect its identity. Which of the graphs correspond to each of the two functions specified? (4 points)



Solution: We know that the geometric series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$, so $\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots$. Similarly, $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$, so that $2 - \cos(x) = 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots$.

We note that both of these functions are even, so the dash-dotted graph, which is odd, is the anonymous function. Further, looking at the x^2 terms of both of these, we see that $\frac{1}{1-x^2} \approx 1 + x^2 > 1 + \frac{x^2}{2} \approx 2 - \cos(x)$, so near $x = 0$ we know that $\frac{1}{1-x^2} > 2 - \cos(x)$. Thus the solid curve must be $\frac{1}{1-x^2}$ and the dashed one $2 - \cos(x)$.