MATH 116-023 QUIZ 3 / 26 Jan 2006 Name:________________________

1. If $F(x)$ is an antiderivative of $f(x) = x^2\cos(1-x^2)$ with $F(0) = 3$, **a.** give an expression for $F(x)$, and **b.** find $\frac{d}{dx}F(\sin(x))$. (3 points)

Solution: $F(x) = \int_0^x t^2\cos(1-t^2)\,dt + 3$, and $\frac{d}{dx}F(\sin(x)) = F'(\sin(x))\cdot\cos(x) = \sin^2(x)\cos(1-\sin^2(x))\cdot\cos(x)$. If we really wanted, we could simplify this with the trigonometric identity $\cos^2(x)+\sin^2(x) = 1$, to $\frac{d}{dx}F(\sin(x)) = \sin^2(x)\cos(\cos^2(x))\cdot\cos(x)$.

2. An astute calculus student invests P dollars in a bank in an account that accrues interest at a continuous rate of $r\%$ a year. Find an expression for the average amount in the student's account in the first T years. (4 points)

Solution: Note that if r gives a percent growth rate, then $r/100$ gives the decimal equivalent to this. Then, if the money is invested at a continuous rate, we know that the amount in the account after t years is $A(t) = Pe^{rt/100}$. Therefore the average amount in the first T years is

$$\begin{aligned}\text{avg} &= \frac{1}{T-0}\int_0^T Pe^{rt/100}\,dt \\ &= \frac{100P}{rT}\left(e^{rT/100}-1\right).\end{aligned}$$

3. Find exactly, using the Fundamental Theorem of Calculus: $\int_0^3 x(x^2+1)^{-1}\ln(x^2+1)\,dx$. (3 points)

Solution: We use substitution with $w = x^2+1$. Then $dw = 2x\,dx$, or $\frac{1}{2}dw = dx$, $w(0) = 1$, and $w(3) = 10$, so

$$\int_0^3 x(x^2+1)^{-1}\ln(x^2+1)\,dx = \int_1^{10}\frac{1}{2}\frac{\ln(w)}{w}\,dw.$$

This is easily found using a second substitution: let $\overline{w} = \ln(w)$; then $d\overline{w} = \frac{1}{w}\,dw$, and (noting that $\overline{w}(1) = 0$ and $\overline{w}(10) = \ln(10)$)

$$\begin{aligned}\int_1^{10}\frac{1}{2}\frac{\ln(w)}{w}\,dw &= \int_0^{\ln(10)}\frac{1}{2}\overline{w}\,d\overline{w} \\ &= \frac{1}{4}\overline{w}^2\Big|_0^{\ln(10)} = \frac{1}{4}(\ln(10))^2.\end{aligned}$$

We can also do this with a single substitution if we notice that $w = \ln(x^2+1)$ is a good substitution. Then $dw = 2x(x^2+1)^{-1}\,dx$, $w(0) = 0$, $w(3) = \ln(10)$, and we end up immediately with the integral in $\overline{w}$ that we found above.