MATH 116-023 QUIZ 4 / 2 Feb 2006 Name:______________________________

The following may or may not be useful things to know.

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + C,\ a \neq 0$ $\int \frac{bx+c}{x^2+a^2}\,dx = \frac{b}{2}\ln|x^2+a^2| + \frac{c}{a}\arctan(\frac{x}{a}) + C,\ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a}) + C,\ a \neq 0$ $\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{b-a}(\ln|x-a| - \ln|x-b|) + C,\ a \neq b$

$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln|x+\sqrt{x^2\pm a^2}| + C$ $\int \sqrt{a^2 \pm x^2}\,dx = \frac{1}{2}(x\sqrt{a^2\pm x^2} + a^2\int \frac{1}{a^2\pm x^2}\,dx) + C$

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1. An alert University of Michigan squirrel (major: undecided, but leaning towards math) notices that the rate at which students in calculus II pass a gateway test is given by $r(t) = \frac{2ke^{kt}}{(1+e^{kt})^2}$ (in %/day after the test opens), where k is a positive constant that depends on what section of the course the students are in. Find the percent of students who have passed after t days. What happens to this as $t \to \infty$? (3 points)

Solution: The total number of students passing the gateway is the integral of the rate, so $N = \int_0^t \frac{2ke^{kx}}{(1+e^{kx})^2}\,dx$. With $w = 1+e^{kx}$, $dw = ke^{kx}\,dx$, this is

$$N = \int_0^t \frac{2ke^{kx}}{(1+e^{kx})^2}\,dx = \int_2^{1+e^{kt}} \frac{2}{w^2}\,dw = -\frac{2}{w}\bigg|_2^{1+e^{kt}} = 1 - \frac{2}{1+e^{kt}}.$$

Thus, as $t \to \infty$, $N \to 1$, which is reassuring—100% of students eventually pass, given enough time!

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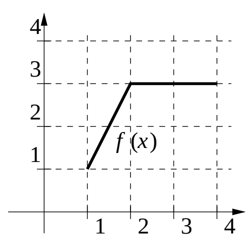
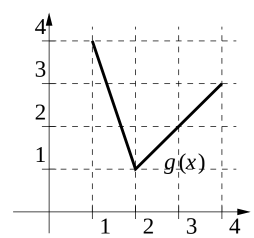
2. The second cousin of our mathematically inclined squirrel wants to get a piece of the action. "Given the graphs of $f(x)$ and $g(x)$, below," she says, "you can find $\int_1^4 f(x)\cdot g'(x)\,dx$ in at least two different ways."





Find the value of this integral in two different ways. (4 points)

Solution: Method 1: Because $g(x)$ is piecewise linear, we can find $g'(x)$ for each of the regions $1 < x < 2$ and $2 < x < 4$. In the first, $g'(x) = -3$, and in the second $g'(x) = 2$. Thus the integral is the same as $\int_1^2 -3f(x)\,dx + \int_2^4 2f(x)\,dx$. The area under $f(x)$ for $1 < x < 2$ is 2, and for $2 < x < 4$ is 6, so $\int_1^2 f(x)\cdot g'(x)\,dx = -3(2) + 1(6) = 0$.

Method 2: Using Integration by Parts, we have $\int_1^4 f(x)\cdot g'(x)\,dx = f(4)g(4) - f(1)g(1) - \int_1^4 f'(x)\cdot g(x)\,dx$. This gives $\int_1^4 f(x)\cdot g'(x)\,dx = (3)(3) - (1)(4) - \int_1^4 f'(x)\cdot g(x)\,dx$. Noting that $f'(x) = 2$ for $1 < x < 2$ and $f'(x) = 0$ for $2 < x < 4$, this becomes $\int_1^4 f(x)\cdot g'(x)\,dx = 5 - \int_1^2 2g(x)\,dx = 5 - 2(2.5) = 0$.

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3. Find $\int \frac{1}{x^2+5x+6}\,dx$ by using partial fractions. (3 points)

Solution: First, we rewrite the integrand $\frac{1}{x^2+5x+6}$ $(= \frac{1}{(x+2)(x+3)})$ as a sum of two terms: $\frac{1}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$. Multiplying through by x^2+5x+6, we get $1 = (A+B)x + (3A+2B)$, so that $A+B = 0$ and $3A+2B = 1$. Thus $A = 1$ and $B = -1$, so that $\int \frac{1}{x^2+5x+6}\,dx = \int \frac{1}{x+2} - \frac{1}{x+3}\,dx = \ln|x+2| - \ln|x+3| + C$.