MATH 116-023 QUIZ 5 / 14 Feb 2006 Name:________________

1. The integral $\int_0^1 \frac{\pi}{\sqrt{x}}\,dx$ gives the volume of the solid created when the curve $y = \frac{1}{\sqrt[4]{x}}$, for $0 < x \le 1$, is rotated around the x-axis. Find analytically (by hand) the volume of this object. (3 points)

Solution: This integral is improper because the integrand, $\frac{1}{\sqrt{x}}$, is undefined at the left endpoint of the interval, $x = 0$. So we proceed by integrating to a point bounded away from zero and then taking the limit as that goes to zero.

$$\begin{aligned}\int_0^1 \frac{\pi}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \pi x^{-1/2}\,dx \\ &= \lim_{a\to 0^+} 2\pi x^{1/2}\Big|_a^1 \\ &= \lim_{a\to 0^+} 2\pi(1-\sqrt{a}).\end{aligned}$$

As $a \to 0^+$, this clearly converges to 2π.

---

2. Have you passed the integral gateway? (Check one.) ☐ yes; ☐ no. If no, when will you be going to the lab to take it? ________________. (1 point)

Solution: Answers will vary.

---

3. Carefully explain, without working out the integral, whether $\int_1^\infty \frac{e^x}{1+e^x}\,dx$ converges. (3 points)

Solution: We note that for $x \ge 0$, $\frac{e^x}{1+e^x} > \frac{e^x}{e^x+e^x} = \frac{1}{2}$. Clearly $\int_1^\infty \frac{1}{2}\,dx = \lim_{b\to\infty}\frac{1}{2}(b-1)$ diverges, and therefore $\int_1^\infty \frac{e^x}{1+e^x}\,dx$ must, being larger, also diverge.

---

4. An overly enthusiastic math professor moves along a path given by $x(t) = t\cos(t)$, $y(t) = t\sin(t)$. Is the professor ever at the point (1,0) (if so, when)? Is the professor's speed increasing or decreasing? At an increasing or decreasing rate? (3 points)

Solution: The professor will be at the point (1,0) if there is a t for which $t\cos(t) = 1$ and $t\sin(t) = 0$. The second of these requires that $t = 0$ or $t = n\pi$, n an integer. If $t = 0$ the first equation is clearly not satisfied. If $t = n\pi$, we get $(n\pi)(\pm 1) = 1$, which is again clearly not true. Thus the professor is never at the point (1,0).

The professor's speed is given by

$$\begin{aligned}|v| &= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2} \\ &= \sqrt{\cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t) + \sin^2(t) + 2t\cos(t)\sin(t) + t^2\cos^2(t)}.\end{aligned}$$

Using the identity $\cos^2(t) + \sin^2(t) = 1$, this is $|v| = \sqrt{1+t^2}$, which is clearly an increasing function. In addition, $|v|' = \frac{t}{\sqrt{1+t^2}}$ and

$$|v|'' = \frac{\sqrt{1+t^2} - \frac{t^2}{\sqrt{1+t^2}}}{1+t^2} = \frac{1+t^2-t^2}{(1+t^2)^{3/2}} = \frac{1}{(1+t^2)^{3/2}} > 0,$$

so it is increasing at an increasing rate.