MATH 116-023 QUIZ 10 / 6 Apr 2006 Name:______________________________

Possibly useful formulae (all series around $x = 0$): $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$; $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$; $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$; and $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$.

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1. Use Taylor series to determine which of $f(z) = \sin(z^2)$ or $g(z) = z(e^z - 1)$ is smaller near $z = 0$ (consider $z > 0$). (3 points)

Solution: We know that $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$, so $\sin(z^2) = z^2 - \frac{z^6}{3!} + \cdots$. Similarly, $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$, so $z(e^z - 1) = z^2 + \frac{z^3}{2!} + \cdots$. For small z, therefore, both of these look like the parabola z^2. However, looking at higher order terms, the exponential $g(z)$ is increased by $\frac{z^3}{2!}$ while $f(z)$ is decreased by $\frac{z^6}{3!}$. Thus the latter is smaller.

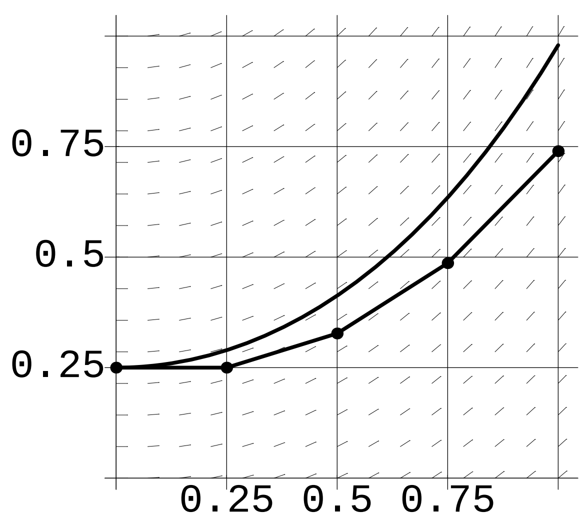
2. Find a value of r for which $y = x^r$ is a solution to the differential equation $x^2y'' - 2y = 0$. (3 points)

Solution: With $y = x^r$, we know that $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Thus, plugging y'' into the differential equation, we must have

$$(x^2)(r(r-1)x^{r-2}) - 2x^r = 0, \text{or}$$
$$r(r-1)x^r - 2x^r = 0.$$

This is $(r^2 - r - 2)x^r = 0$, so, if this is to work for all values of x, $r^2 - r - 2 = 0$. This factors as $(r-2)(r+1) = 0$, so $r = 2$ or $r = -1$ work. Solutions are $y = x^2$ or $y = x^{-1}$.

3. On the slope field for the differential equation $\frac{dy}{dx} = (y+1)\sin(x)$ shown below, **(a)** sketch the solution to the differential equation that passes through $(0, 0.25)$, and **(b)** sketch the Euler solution to this differential equation, starting at $y(0) = 0.25$ and using a step size of $\Delta x = 0.25$, that finds $y(1)$. *Note that you do not need to actually calculate the Euler method solution for (b)!* (4 points)



Solution: Shown (the smooth curve is the solution, piecewise linear, Euler).