MATH 116-028 QUIZ 3 / 22 Jan 2008 Name:______________________________

1. Find an explicit formula for the function $A(t)$ giving the area under the curve $y = \frac{\cos\sqrt{x}}{\sqrt{x}} + 1$ between $x = 1$ and $x = t$. Is $A(t)$ concave up or down on the interval $1 \le t \le 6$? (4 points)

Solution: The area under the curve is given by $\int_1^t \frac{\cos\sqrt{x}}{\sqrt{x}} + 1\,dx = 2\sin\sqrt{x} + x\Big|_1^t = 2\sin(\sqrt{t}) + t - 2\sin(1) - 1$. We know that $A'(t) = \frac{\cos\sqrt{t}}{\sqrt{t}}$, so $A''(t) = -\frac{\sin\sqrt{t}}{2t} - \frac{\cos\sqrt{t}}{2t^{3/2}} < 0$ for $1 \le t \le 6$, so $A(t)$ is concave down on this interval.

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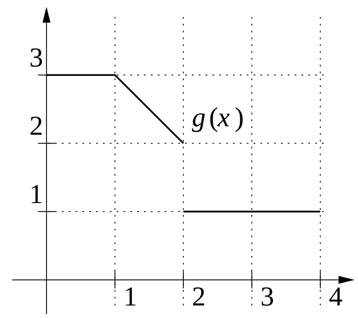
2. Suppose that $g(x)$ is given by the graph to the right. Find $\int_0^4 x\,g'(x)\,dx$. (3 points)



Solution: One way of solving this is to note that $g'(x) = 0$ everywhere except $1 < x < 2$, where $g'(x) = -1$. Therefore $\int_0^4 x\,g'(x)\,dx = \int_1^2 x\,(-1)\,dx = -\frac{1}{2}x^2\Big|_1^2 = -2 + \frac{1}{2} = -\frac{3}{2}$.

We might also try integration by parts on this: with $u = x$ and $v' = g'(x)$ (so that $u' = 1$ and $v = g(x)$), $\int x\,g'(x)\,dx = x \cdot g(x) - \int g(x)\,dx$. If we evaluate this over $0 \le x \le 4$, we get $x \cdot g(x)\Big|_0^4 - \int_0^4 g(x)\,dx = 4\,g(4) - 0\,g(0) -$ area under $g(x)$. We see $g(4) = 1$ and the area under $g(x)$ is 7.5, so that the integral gives $4(1) - 7.5 = -3.5$. However, this seems suspect—we got a different value than before! It turns out that the reason why this doesn't work is a bit subtle, so the quiz was marked to give full credit for this answer.

So, what went wrong with our integration by parts solution? The difficulty comes from the discontinuity in $g(x)$. To see this, suppose that we use integration by parts to find $\int_2^2 x\,g'(x)\,dx$: doing the same calculation as we did above, we get $\int_2^2 x\,g'(x)\,dx = x\,g(x)\Big|_2^2 - \int_2^2 g(x)\,dx$. The area under a point is zero, so the integral is zero, but what happens to $g(x)$ at $x = 2$ isn't so clear: immediately to the left of $x = 2$, g is 2, and immediately to the right it's 1. So if we think about moving across the discontinuity there's a drop of one in $g(x)$—that results in a drop in $x\,g(x)$ of two—and it's that factor of two that is the difference between the two calculations above. Therefore to get the integral over the full interval from $x = 0$ to $x = 4$ to work out correctly we have to work it out for the interval $0 \le x < 2$, and then separately in the interval $2 < x \le 4$, using the values of $g(x)$ on either side of the discontinuity and making sure that the integrals match appropriately.

Obviously, we didn't intend for this problem to be so interesting! It would have been better if the function $g(x)$ had been 2 for $2 < x < 4$, so that this subtlety didn't show up and integration by parts gives the same value as the first calculation. (Work it out to see that the two calculations do give the same value in this case!)

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3. Suppose that the rate at which happy students flock to calculus II is given by $r(t) = t^2 e^{t/10}$ students/week, where t is measured in weeks since the start of the registration period. How many students added calculus II between the fifth and eight weeks after the start of registration? Be sure that all of the steps in your calculation are clear. (3 points)

Solution: We're given a rate, so the number of students who add is just the integral $\int_5^8 t^2\,e^{t/10}\,dt$. Integrating by parts with $u = t^2$ and $v' = e^{t/10}$ (so that $u' = 2t$ and $v = 10e^{t/10}$), we get $\int_5^8 t^2\,e^{t/10}\,dt = 10t^2 e^{t/10}\Big|_5^8 - \int_5^8 20\,t\,e^{t/10}\,dt = 10(64e^{4/5} - 25e^{1/2}) - 200\,t\,e^{t/10}\Big|_5^8 + \int_5^8 2000\,e^{t/10}\,dt = 10(64e^{4/5} - 25e^{1/2}) - 200(8t^{4/5} - 5e^{1/2}) + 2000(e^{4/5} - e^{1/2}) \approx 253.7$. (Where we also found $\int t\,e^{t/10}\,dt$ by integrating by parts.)