MATH 116-028 QUIZ 8 / 25 Mar 2008 Name:______________________________

1. Find the interval of convergence of the series (4 points)

$$1 + \frac{x}{\ln(2)+1} + \frac{4x^2}{\ln(3)+1} + \frac{9x^3}{\ln(4)+1} + \frac{16x^4}{\ln(5)+1} + \frac{25x^5}{\ln(6)+1} + \cdots$$

Solution: The nth term (after $n=0$)† in the series is $a_n = \frac{n^2 x^n}{\ln(n+1)+1}$, so the ratio test gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2\,x^{n+1}}{\ln(n+2)+1}\cdot\frac{\ln(n+1)+1}{n^2\,x^n}\right| = \lim_{n\to\infty}|x|\cdot\frac{(n+1)^2\,(\ln(n+2)+1)}{n^2\,(\ln(n+1)+1)} = |x|.$$

For convergence, the ratio test requires that this be less than one, so $|x|<1$ (the radius of convergence is one), and the interval of convergence is at least $-1<x<1$. At $x=\pm 1$, the series is $\sum \frac{(\pm 1)^n n^2}{\ln(n+1)+1}$, which clearly diverges because the terms do not go to zero. Thus the interval of convergence is $-1<x<1$.

2. Find the Taylor polynomial of degree 3 for the function $f(x)=e^{2x}$ near the point $a=1$. (3 points)

Solution: We have $f(x)=e^{2x}$, so $f^{(n)}(x)=2^n e^{2x}$. Thus $f(1)=e^2$, and $f^{(n)}(1)=2^n\,e^2$. The Taylor polynomial is $P_3(x)=f(1)+f'(1)\cdot(x-1)+\frac{1}{2!}\,f''(1)\cdot(x-1)^2+\frac{1}{3!}\,f'''(1)\cdot(x-1)^3$, so

$$P_3(x) = e^2 + 2e^2\,(x-1) + 2e^2\,(x-1)^2 + \frac{4}{3}\,e^2\,(x-1)^3.$$

3. If a function $g(x)$ is approximated near zero by the fourth degree Taylor polynomial $P_4(x)=3-2x^2-x^4$, what are $g(0)$, $g'''(0)$ and $g^{(4)}(0)$? (3 points)

Solution: We know that $P_4(x)=g(0)+g'(0)\cdot x+\frac{1}{2!}\,g''(0)\cdot x^2+\frac{1}{3!}\,g'''(0)\cdot x^3+\frac{1}{4!}\,g^{(4)}(0)\cdot x^4$, so

$$g(0)=3, \qquad g'''(0)=0, \quad \text{and} \quad g^{(4)}(0)=-4!=-24.$$

† Note that because convergence is unaffected by a finite number of leading terms we can ignore the zeroth term when finding convergence.