MATH 116-028 QUIZ 9 / 1 Apr 2008 Name:

1. Find the first three non-zero terms, and the general term, of the Taylor series of $f(x) = \frac{1}{1+3x}$ around $x = 1$ *(not around $x = 0$)*. (3 points)

Solution: We have $f'(x) = \frac{-3}{(1+3x)^2}$, $f''(x) = \frac{3^2 \cdot 2}{(1+3x)^3}$, $f'''(x) = \frac{-3^3 \cdot 2 \cdot 3}{(1+3x)^4}$, and so on, so that $f^{(n)}(x) = \frac{(-3)^n \cdot n!}{(1+3x)^n}$. Evaluating at $x = 1$, we have $f(1) = \frac{1}{4}$, $f'(1) = \frac{-3}{4^2}$, $f''(1) = \frac{3^2 \cdot 2}{4^3}$, and so on, with $f^{(n)}(1) = \frac{(-3)^n \cdot n!}{4^{n+1}}$. Thus the Taylor series is

$$\begin{aligned}\frac{1}{1+3x} &= \frac{1}{4} + \frac{-3}{16}(x-1) + \frac{9}{64}(x-1)^2 - \cdots + \frac{(-3)^n}{4^{n+1}}(x-1)^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-3)^n}{4^{n+1}}(x-1)^n\end{aligned}$$

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2. Recall that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$. Use the Taylor series for $\frac{1}{1-x}$ (around $x = 0$) to find the Taylor series for $\arctan(x)$. Without calculating it, what would you expect the radius of convergence of your new series to be? (4 points)

Solution: Because $\frac{1}{1-x} = 1 + x + x^2 + \cdots$, we have $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots$. Integrating, we find $\arctan(x) = C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + \frac{(-1)^n}{2n+1}x^{2n+1} + \cdots$. Then $\arctan(0) = 0$, so $C = 0$, and we have

$$\begin{aligned}\arctan(x) &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + \frac{(-1)^n}{2n+1}x^{2n+1} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}\end{aligned}$$

Because the radius of convergence of $\frac{1}{1-x}$ is one, and because when $-1 < x < 1$ we know $0 < x^2 < 1$, we expect the radius of convergence of the expansion for $\arctan(x)$ to be one as well.

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3. For what values of n, if any, is $y = x^n$ a solution to the differential equation $x^2\frac{d^2y}{dx^2} - 2y = 0$? (3 points)

Solution: If $y = x^n$, then $\frac{dy}{dx} = n\,x^{n-1}$ and $\frac{d^2y}{dx^2} = n(n-1)\,x^{n-2}$. Plugging in, we therefore have

$$x^2(n(n-1)\,x^{n-2}) - 2x^n = (n^2 - n)\,x^n - 2x^n = (n-2)(n+1)x^n = 0.$$

This is true for all x if $n = 2$ or $n = -1$. Thus $y = x^2$ and $y = x^{-1}$ are solutions to the differential equation.